RESEARCH
LABORATORIES OF THE
GENERAL
ELECTRIC
COMPANY LTD

535.1

THE LIBRARY

RESEARCH LABORATORIES
THE GENERAL ELECTRIC CO., LTD.
WEMBLEY

27. OCT. 1960	-5. JAN. 1966	24th July 1971
21. NOV. 1960	10. JAN. 1967	21.10.71
25. NOV. 1960	-7. MAY 1968	25. NOV. 1971
-1. JUN. 1961	13. FEB. 1969	21.10.71
-2. [illegible] 1962	28. OCT. 1969	21-4-72
30. [illegible] 1962	29. JUN. 1970	24.8.72
26. OCT. 1962	-3. AUG. 1970	5 JUL. 1973
18. APR. 1963	-7. OCT. 1970	
16. JUL. 1964	? Jan.	

This book is due back on the last date stamped above.

PHYSICAL OPTICS

1916.

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
SAN FRANCISCO

MACMILLAN & CO., LIMITED
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN CO. OF CANADA, LTD.
TORONTO

PHYSICAL OPTICS

BY

ROBERT W. WOOD, LL.D.

PROFESSOR OF EXPERIMENTAL PHYSICS IN THE JOHNS HOPKINS UNIVERSITY

NEW AND REVISED EDITION

New York
THE MACMILLAN COMPANY
1921

All rights reserved

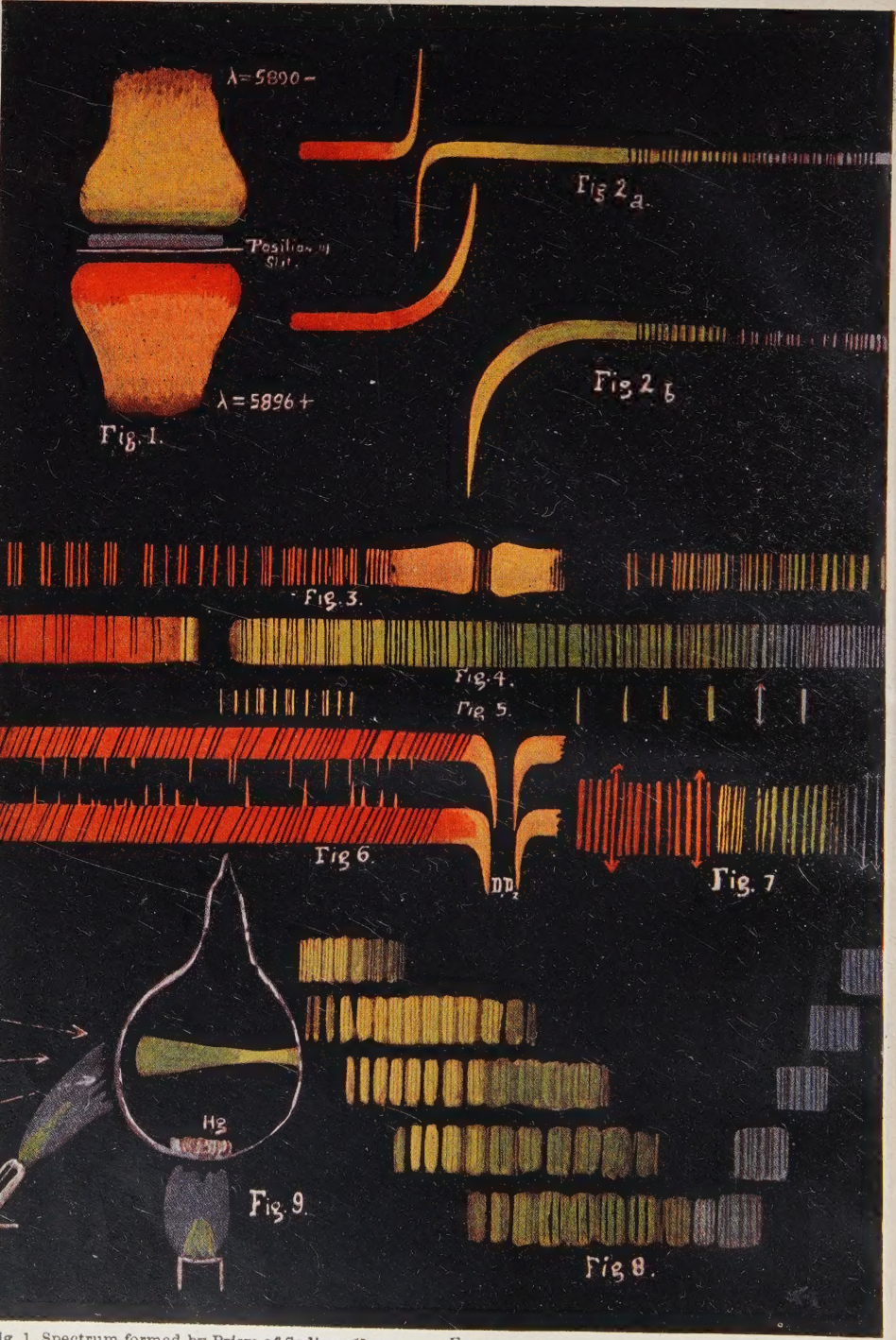


Fig. 1. Spectrum formed by Prism of Sodium Vapor.
Fig. 2. Anomalous Dispersion of Sodium Vapor.
Fig. 3. Magnetic Rotation of Sodium Vapor.
Fig. 4. Fluorescence of Sodium Vapor. White light excitation.
Fig. 5. Fluorescence of Sodium Excitation by Bismuth Arc.
Fig. 7. Fluorescence of Sodium Excitation by Lithium Arc.

Fig. 6. Positive and Negative Magnetic Rotation by Sodium Vapor. (Black absorption lines should be vertical—engraver's error.)
Fig. 8. Fluorescence of Sodium Excited by Different Spectral Regions.
Fig. 9. Destruction of Mercury Vapor Fluorescence by High Temperature

PREFACE TO THE SECOND EDITION

THE rapid advances made in the science of Physical Optics since the appearance of the first edition of this book in 1905, have made additions necessary in practically every chapter. The numerous typographical errors which marred the first edition, have been corrected, and certain sections of small interest or importance have been removed bodily, to make more room for new material. Even with these removals, the new edition will be found increased in size by about 150 pages, and nearly 100 new illustrations. Three new chapters have been added, dealing with the subjects of Meteorological Optics, Electro-Optics, and the Principle of Relativity. The numerous additions and enlargements deal almost exclusively with the experimental side of the subject, and the larger part of the matter removed is mathematical, the loss of which, it is believed, will not be felt. I am under great obligation to Professor Frost, who furnished a large number of illustrations which have appeared in the Astrophysical Journal, and to Mr. Francis, of the Philosophical Magazine, for the same courtesy. I am also under obligation to Professor Zeeman for his kindness in reading the proof of the chapter on Magneto-Optics, and supplying me with some of his most recent results which otherwise could not have been included.

R. W. WOOD.

GENEVA, February, 1911.

Digitized by the Internet Archive
in 2025

https://archive.org/details/bwb_KV-543-846

PREFACE TO FIRST EDITION

The present volume was commenced at a time when Preston's *Theory of Light* was practically the only advanced textbook on the subject in English suitable for general class work. This work, while excellent in every respect, could scarcely be said to represent our present knowledge of the subject. Anomalous dispersion and the relation existing between absorption and dispersion was barely mentioned, and of course the recent remarkable discoveries in the field of magneto-optics were not recorded. In the meantime, two very excellent books have appeared, the English translation of Drude's *Lehrbuch der Optik*, which cannot be surpassed, and Schuster's *Theory of Optics*, which, while extremely interesting and suggestive, omits all mention of the Laws of Radiation, Fluorescence, and Phosphorescence, and the whole subject of the "Optics of moving media."

In the present volume especial stress has been laid on the experimental side, and it is the author's hope that the perhaps too frequent references to experiments with which he has been more or less directly associated will not be taken as an indication of a lack of perspective.

No pretence at originality in the mathematical treatment is made: the work has been compiled to a great extent from lecture notes, and many plagiarisms doubtless occur. The excellent theoretical treatment, based upon the electro-magnetic theory given by Drude, has been followed very closely, and it is hoped that this acknowledgment may serve in place of the numerous quotation marks which would otherwise be necessary. Various other standard textbooks have been drawn upon freely, especially the very comprehensive work of Verdet.

Too much space has perhaps been given to the theory of dispersion, and the incorporation of the somewhat lengthy development of the dispersion formula by elementary methods, based upon the elastic-solid theory, may appear superfluous. The advantage of this treatment lies in the fact that it does not

involve the use of imaginary quantities, which are always a little troublesome to the student at first; in addition to this, it appears to be a little more intelligible, the reciprocal actions between the vibrating atom and the ether being more readily grasped by the mind than the somewhat vaguer conception of displacement currents in the ether and their action upon charged electrons. The electro-magnetic treatment follows.

The illustration of the book has been greatly facilitated through the courtesy of Mr. William Francis, who has furnished blocks of many plates and figures from the *Philosophical Magazine*.

I am under very great obligation to my friend, Professor J. S. Ames, who has made many valuable suggestions from time to time, cleared up many doubtful points, and read the manuscript during the process of its preparation.

R. W. WOOD.

BALTIMORE, May 2nd, 1905.

CONTENTS

CHAPTER I

The Nature of Light

PAGES

Older Theories — The Electro-Magnetic Theory — Simple Periodic Motion — Wave-Motion — Absence of Back-Wave — Wave-Front — Frequency and Wave-Length — Sources of Light for Experimental Purposes — Color Screens or Ray Filters — Velocity of Light — Fizeau's Method — Foucault's Method — Michelson's Experiments — Newcomb's Experiments — Group Velocity — The Doppler-Fizeau Principle . 1–27

CHAPTER II

Rectilinear Propagation of Light

Huygens's Principle — Rectilinear Propagation — Effect of a Plane-Wave on an Exterior Point — Zone-Plate — Huygens's Principle as applied to Solitary Waves — Law of Regular Reflection deduced from the Principle of Interference of Elementary Waves — Reflection and Refraction by Unpolished Surfaces 28–45

CHAPTER III

The Reflection of Light from Plane and Curved Surfaces

Reflection of a Plane-Wave from a Plane-Mirror — Reflection of Sound-Waves — Reflection by Ellipsoidal Mirror — Reflection from a Parabolic Mirror — Reflection by Hyperboloid — Reflecting Telescopes — Reflection from Spherical Surfaces — Reflection from Convex Spherical Surfaces — Reflection of Plane-Waves from Concave Spherical Mirrors — Formation of Images — Focal Lines — Fermat's Principle . 46–63

CHAPTER IV

Refraction of Light

Refraction of Plane-Wave at Plane Surface — Total Reflection — Effect of Refraction on the Width of the Beam — The Ophthalmometer — Refraction of a Spherical Wave at a Plane Surface — Refractive Index of a Plate measured by the Microscope — Fermat's Law — Refraction by a Prism — Refractive Index of a Prism — Magnifying Power of Prisms — Refraction by a Lens — Refraction by Sphere — Microscope Objective — Focal Length of a Lens — Refraction of Light in Non-Homogeneous Media — Astronomical Refraction — Schmidt's Theory of the Sun — Mirage — Non-Homogeneous Cylinders as Pseudo-Lenses — Curved Light Rays — Scintillation — The Method of Striae — Invisibility of Objects 63–100

CHAPTER V

Dispersion

PAGES

Achromatism — Partial Achromatization by a Grating — Direct Vision Prisms — Achromatic Lenses — Calculation of Achromatic and Direct Vision Prisms — Resolving Power of Prisms — Christiansen's Experiment — Determination of the Dispersion of a Substance in the Form of a Powder — Anomalous Dispersion — Anomalous Dispersion in its bearing on Solar Phenomena 101–118

CHAPTER VI

Interference of Light

Average Kinetic Energy of a Vibrating Particle — Composition of Vibrations — Distribution of Illumination — Resultant of a Large Number of Disturbances of Arbitrary Phase — Interference of Light — The Flow of Energy in a System of Interference Fringes — Lloyd's Single Mirror — Fresnel's Bi-Prism — Bi-Prism of Variable Angle — The Corresponding Points of the Sources — Limit to the Number of Fringes — Shift of the Fringes by Introduction of Thin Transparent Plate — Interference Fringes with White Light — A Simple Interference Refractometer — Application of Interference Phenomena to Astronomical Observations — Interference of Waves of Different Lengths; Light-Beats — Achromatic Interference Fringes — Introduction of Thin Transparent Plate — Distribution of Phase over Small Area illuminated by Source of Finite Size — Interference Fringes when the Sources are in Line — Interference Fringes along Caustics — Interference of Polarized Light — The Fresnel-Arago Laws — The Colors of Thin Plates — Newton's Rings — Newton's Rings in White Light — Refractive Index and Dispersion of a Thin Plate — Influence of Multiple Reflections — Curves of Equal Thickness and Equal Inclination; Haidinger's Fringes — Testing Glass Plates for Flatness and Plane-Parallelism — Colors of Iridescent Crystals and Opals — Effect of a Prism upon Newton's Rings — Achromatization of the Fringes formed by a Thin Reflecting Lamina — On the Polarized Fringes produced by the Interference of Two Streams of Light polarized at Right Angles — Preparation of Films for the Exhibition of Newton's Colors — Colors of Frilled Transparent Films on Metallic Surfaces — Stationary Light-Waves — Lippmann's Color Photographs 119–180

CHAPTER VII

The Diffraction of Light

Straight Edge — Circular Disk and Circular Aperture — Small Circular Aperture — Fraunhofer Class of Diffraction Phenomena — Narrow Slit — Two Parallel Slits — Transition from Fresnel to Fraunhofer Class — Graphical Method of Solution of Diffraction Problems — Cornu's Spiral — Mathematical Treatment of Fraunhofer Diffraction Phenomena — Diffraction by a Rectangular Aperture — Introduction of Angular Measure — Diffraction by a Single Slit — Diffraction by

PAGES

Any Number of Parallel Equidistant Slits (Diffraction Grating) — Diffraction by Two Parallel Slits — Number of Spectra and Intensity Distribution: Absent Spectra — Verification of Results — The Laminary Grating — The Plane Grating — Overlapping Spectra — Resolving Power of Gratings — Focal Properties of Gratings: Zone Plates — Intensity of Grating Spectra — Intensifying Glass Gratings — False Lines due to Periodic Errors or "Ghosts" — The Nature of Optical Images — Abbé's Diffraction Theory of Microscopic Vision — Concentration of Light into a Single Spectrum — The Echelette Grating, for the Infra-Red — Spectral Intensity and Form of Grooves — Concave Gratings — Theory of the Concave Grating — Diffraction by a Circular Aperture — Resolving Power of Telescope — Babinet's Principle — Diffraction by Two Small Apertures — Diffraction by a Large Number of Irregularly Arranged Circular Apertures or Disks — Halos — Young's Eriometer — Effect of Moving One of the Two Apertures in the Direction of the Source — Diffraction by Small Particles on the Surface of a Mirror — Fresnel Diffraction Phenomena — Cornu's Spiral — Diffraction by a Straight Edge — Diffraction by Thin Laminæ — The Colors of Mixed Plates — Talbot's Bands — Anomalous Propagation of Waves in the Vicinity of Foci 181–264

CHAPTER VIII

Interference of Spectroscopes and the Resolution of Spectral Lines

Michelson's Interferometer — Use of the Interferometer — Adjustment — Determination of Refractive Index and Dispersion with the Interferometer — Light-Waves as Standards of Length — The Visibility Curves — The Echelon Grating — Theory of Echelon — The Interferometer of Fabry and Perot — Silvering the Plates — The Interferometer of Lummer and Gehrcke — Interference Points — Resolving Power and Spectral Range 265–287

CHAPTER IX

Polarization of Light

Discovery of Polarization — Polarization by Reflection — Angle of Polarization; Brewster's Law — Plane of Polarization — Nörrenberg's Reflecting Polariscope — Polarization by Refraction — Law of Malus — Polarization by Double Refraction — The Nicol Prism — Angular Aperture of Polarizing Prisms — Detection of Polarized Light — Determination of the Percentage of Polarized Light — The Eye as an Analyzer — Haidinger's Brush — Polarization by Oblique Emission — Stationary Polarized Waves — Landolt's Fringe 288–302

CHAPTER X

Double Refraction

Physical Explanation of Double Refraction — Wave-Surface in Uniaxal Crystals — Huygens's Construction — Verification of Huygens's Construction — Fresnel's Theory of Double Refraction — The Normal Velocity Surface — The Wave-Surface — The Optic Axes or Axes of

PAGES

Single Wave-Velocity — Internal Conical Refraction — Axes of Single Ray Velocity — External Conical Refraction — Crystal Plates in Convergent or Divergent Polarized Light — Colors of Thin Crystal Plates — Colors of Crystal Plates in Convergent and Divergent Polarized Light — Uniaxal Crystal Cut Perpendicular to Axis — Isochromatic Surfaces — Position of Axes as a Function of Temperature — Phenomena exhibited by Twin Crystals — Convergent Circular Light — Double Refraction in Non-Crystal Media 302–325

CHAPTER XI

Circular and Elliptical Polarization

Plane Polarization of the Emergent Light — Circular Polarization of the Emergent Light — Intensity of Circularly Polarized Light — Production and Properties of Circular Light — Circular Polarization by Total Reflection — Fresnel's Rhomb — Elliptical Polarization of the Emergent Light — Production and Properties of Elliptical Light — Determination of the Constants of Elliptical Polarization — Position of the Axes — Ratio of the Axes — Elliptical Polarization by Reflection — Direction of Revolution in Circularly Polarized Light — Direction of Revolution in the Case of Fresnel's Rhomb — Natural and Partially Polarized Light — Light-Beats 326–341

CHAPTER XII

Meteorological Optics

The Rainbow — Halos, Mock Suns, and Related Phenomena . . 341–349

CHAPTER XIII

Theory of Reflection and Refraction

Derivation of Maxwell's Equations — Displacement Currents in Free Ether — Isotropic Dielectrics — Boundary Conditions — Velocity of the Wave — Deduction of the Laws of Reflection and Refraction for Transparent Media — Polarizing Power of Oblique Plates — Perpendicular Incidence — Change of Phase by Passage through the Polarizing Angle — Elliptical Polarization; Surface Films or Strains? — Total Reflection — Penetration of the Disturbances into the Second Medium 350–374

CHAPTER XIV

The Theory of Dispersion

Helmholtz's Mechanical Theory of Dispersion — Application to Transparent Media — Calculation of the Position of the Absorption Bands of Transparent Media — Radiometric Observations on Quartz — Dispersion and Absorption of Various Substances in the Infra-Red — Application to Absorbing Media and Selective Dispersion — Determination of the Extinction Coefficient — Dispersion of Nitroso-dimethyl Aniline — Optical Properties of Stibnite — Electro-Magnetic Theory

PAGES

of Dispersion — Normal Dispersion — Selective Reflection in Absorption Region; Residual Rays — Residual Rays from Powders — Interferometer Investigation of Long Heat-Waves — The Longest Heat-Waves; Focal Isolation — Selective Reflection in the Ultra-Violet — Selective Dispersion of an Absorbing Gas — The Refraction and Dispersion of Sodium Vapor of Great Density — Application of the Results to the Dispersion Formula — Selective Reflection by an Absorbing Gas — Absorption and Selective Dispersion of Hydrogen 375–435

CHAPTER XV

Absorption of Light

Laws of Absorption: Lambert's and Beer's — Coefficient of Transmission: Dichromatism — Body Color and Surface Color — Influence of Temperature on Absorption Bands — Influence of Solvent on Position of Absorption Bands — Absorption of Light by Gases and Vapors — Absorption by Sodium Vapor — Effect of Density on Absorption Bands — Absorption by Hydrogen Gas — Form of Absorption Bands: Influence of Other Media — Absorption by Porous Surfaces — Absorption by Metals — Absorption Spectra of the Rare Earths — Theories of Absorption 436–455

CHAPTER XVI

Optical Properties of Metals

Electro-Magnetic Theory of Absorption — Metallic Reflection — Determination of Principal Incidence and Azimuth — Determination of the Change of Phase by Reflection — Absorption and Reflection by Metals — Effect of Thickness of Film — Optical Constants of the Metals — The Dispersion of Metals — Relations between Optical and Electrical Properties of Metals — Relation between the Emission Coefficients of Metals and the Temperature 456–475

CHAPTER XVII

Rotary Polarization

Rotatory Dispersion — Fresnel's Explanation of the Rotation — Babinet's Experiment — Unequal Absorption of the Circular Components — Other Rotatory Crystals — Rotatory Polarization in Convergent Light — Rotatory Polarization of Liquids — Rotatory Power and Change of State — Rotatory Dispersion of Liquids — Molecular Rotation and the Influence of the Solvent on the Rotatory Power — Anomalous Rotatory Dispersion — Double Refraction of Active Liquids — Theory of Rotatory Polarization — Physical Explanation of the Rotation — Real and Spurious Anomalous Rotatory Dispersion — Rotatory Dispersion — Rotatory Dispersion of Absorbing Media — Elliptical Polarization produced by Absorbing Active Media — Possible Production of an Optically Active Substance from an Inactive, by Circularly Polarized Light — Rotatory Dispersion in Absorbing Media . . 476–495

CHAPTER XVIII

Magneto-Optics

PAGES

The Faraday Effect: Magnetic Rotation of the Plane of Polarization — Explanation of the Magnetic Rotation — Resolution into Circular Components — Direction of the Magnetic Rotation — Relation between the Field Strength and Rotation — Time required for the Faraday Effect: Rotation by Oscillatory Discharge — Magnetic Rotation and the Second Law of Thermodynamics — The Kerr Effect — Magnetic Rotatory Dispersion — The Zeeman Effect — Complicated Types of the Zeeman Effect — Asymmetrical Separation of the Triplet — Shift of the Middle Component — The Zeeman Effect and Spectral Series — Magnitude of the Separation — Study of the Zeeman Effect without a Spectroscope — The Inverse Zeeman Effect — The Zeeman Effect on the Sun — Partial Polarization of the Light emitted by a Flame in a Magnetic Field — Abnormal Zeeman Effect in Band Spectra — Theory of the Zeeman Effect — Theory of Magnetic Rotation — Hypothesis of the Hall Effect — Light-Rays Parallel to the Magnetic Field — Proof of the Rotatory Dispersion Formula — The Bright-Line Rotation Spectrum of Absorbing Vapors — Positive and Negative Rotation by Absorption Lines — Magnetic Rotation within an Absorption Band: Experiments of Zeeman — Magnetic Rotation at Low Temperatures — Rays perpendicular to the Magnetic Field — Double Refraction near the Components of Absorption Lines magnetically split into Several Components 496–547

CHAPTER XIX

Electro-Optics

The Kerr Electro-Optic Effect in Liquids — The Method of Abraham and Lemoine — Electro-Optics of Nitro-Benzol — The Electro-Optic Analogy of the Zeeman Effect — The Photo-Electric Effect — Photo-Electric Effect and Wave Length — Polarization of the Light — The Normal and the Selective Photo-Electric Effect — Photo-Electric Effect in Gases — Theory of the Photo-Electric Effect 548–555

CHAPTER XX

Transformation of Absorbed Radiation. Fluorescence, Phosphorescence, and Resonance-Radiation

Fluorescence — Methods of investigating Fluorescence — Polarized Fluorescence — Variation of the Intensity with the Angle of Emission — Fluorescence Absorption — Phosphorescence — Duration of the Phosphorescence — Phosphoroscope — Influence of Temperature — Spectra of Phosphorescent Light — Theories of Fluorescence — Effect of the Solvent on the Intensity of Fluorescence — Fluorescence of Vapors — The Fluorescence of Sodium Vapor — Resonance Spectra of Sodium Vapor — Fluorescence of Mercury Vapor — The Fluorescence of Iodine and Bromine Vapor — Destruction of Fluorescence by

PAGES

Other Gases — Emission of Polarized Light by Fluorescent Gases — Resonance Spectrum of Iodine Vapor in Helium: Effect of Molecular Collisions — Molecular Changes — Chemical Changes — Chemical Combination — Transformation of Luminous into Electrical Energy . 556–590

CHAPTER XXI

LAWS OF RADIATION

The Relation between Emission and Absorption — Ritchie's Experiment — Kirchhoff's Law — Absorption by Flames — Temperature Radiation of Gases — Temperature Radiation of Solids and Liquids — Hydraulic Analogy of a Radiating Body — Emission of Substances having Absorption Bands — Selective Emission and Absorption of Incandescent Salts — Emission of Polarized Light — Deduction of Kirchhoff's Law — The Perfect Black Radiator — Equilibrium between Radiation and Material Bodies — Pressure of Radiation — Experiments of Nichols and Hull — Tangential Component of Radiation Pressure — Pressure of Radiation on an Absorbing Gas — The Stefan-Boltzmann Law — Proof of Stefan's Law — Optical Pyrometers — Temperature of the Sun — Change in the Spectrum of a Black Body with the Temperature; Wien's Laws — Complete Radiation Formulae — Energy Distribution in the Spark Spectrum 591–623

CHAPTER XXII

SCATTERING OF LIGHT BY SMALL OBSTACLES. OPTICAL RESONANCE

Scattering of Light by Small Particles — The Residual Blue — Abnormal Polarization and Color of Light scattered by Small Absorbing Particles — Scattering of Light by Metal Spheres — Optical Resonance — Electrical Resonance — Resonance for Heat-Waves — Polarizing Action of Gratings — The Color of Light diffracted by Screens, showing Selective Absorption — Effect of Changing the Surrounding Medium — Changes produced by Oxidation — Colors of Granular Films of Gold and Silver — Anomalous Dispersion of the Films — The Scattering of Light by Sodium and Potassium Fogs — Colors of Lippmann Photographic Plates — Colors in Metal Glasses and Metallic Films — Polarization Effects of Ultra-Microscopic Particles — Colors of Granular Metallic Films 624–647

CHAPTER XXIII

THE NATURE OF WHITE LIGHT

Theory of Damped Vibrations — Type of the Impulse constituting White Light — Interference Experiments in the Light of the Pulse Hypothesis — Analysis of White Light by a Grating — Analysis by a Prism; Origin of Prismatic Colors — Lord Rayleigh's Treatment . . 648–666

CHAPTER XXIV

The Relative Motion of Ether and Matter

PAGES

Aberration of Light — Airy's Experiment — Fizeau's Experiment — Retardation by a Moving Plate — The Michelson-Morley Experiment — Lodge's Experiment — Influence of the Earth's Motion on Rotatory Polarization — Effect of the Earth's Motion upon the Intensity of Terrestrial Sources — Optics of Moving Media; Lorentz's Treatment — Velocity of Light in Moving Medium 667–683

CHAPTER XXV

The Principle of Relativity 684–695

Index 697–705

PHYSICAL OPTICS

CHAPTER I

THE NATURE OF LIGHT

Older Theories. — The foundations of our present knowledge respecting the nature of light were laid during the latter part of the 17th century, although the modern wave-theory did not take definite form until over a century later. The important discoveries which may be said to mark the beginning of the science of optics may be summed up in a few words.

In 1666 Sir Isaac Newton effected the prismatic decomposition of white light into its component colors, and proved that no further color change resulted from subsequent refractions. He moreover recombined the spectrum colors, and formed from them white light. This was a great step in advance in one way, for it had been thought previously that color was produced by refraction, manufactured by the prism so to speak, whereas Newton showed that the colors were originally present in the white light, the function of the prism being merely to separate them or sort them out, which it accomplished in virtue of its power of deviating rays of different colors through different angles. Curiously enough this discovery, which we are taking as marking the beginning of our definite knowledge about light, is one which we shall demolish in the last chapter of this book, for our present idea regarding the action of the prism more nearly resembles the idea held previous to Newton's classical experiments: we now believe that the prism actually manufactures the colored light, and what is more to the point, we have a pretty definite idea regarding the manner in which it manufactures it, in which respect we may consider ourselves in advance of Newton's contemporaries.

The importance of Newton's discovery is not to be underestimated on this account, and his conception of the nature of white light will be held to throughout the greater part of this book, for it represents perfectly all of the experimental facts with which we are acquainted, and the treatments of nearly all of the optical phenomena which we are to study are greatly simplified by its use.

Newton elaborated what is known as the corpuscular theory of light, and clung to it tenaciously to the last, the weight of his opinion retarding in no small degree the development of the wave-theory, which was first clearly expressed in 1678. On the corpuscular theory light was regarded as a flight of material particles

emitted by the source, the sensation of sight being produced by their mechanical action upon the retina. The rectilinear propagation followed at once from the second law of motion, whereas the early supporters of the wave-theory were unable to account for it, as every known form of wave motion bent freely around the edges of obstacles.

In 1676 it was demonstrated by Römer, a Danish astronomer, that light required a finite time for its propagation, travelling across space with a velocity which he estimated at 192,000 miles per second. Now the impact of corpuscles moving at such a speed might well be expected to exert a pressure, and attempts were at once made to establish the materiality of light by detecting this pressure, all of which were failures however. At the present time we know that light does exert a pressure, though a very small one, but this pressure can be shown to be the necessary consequence of the impact of waves, so that it is as strong evidence of the truth of the wave-theory, as it would have been of the emission-theory had it been discovered in the days of Newton.

A wave-theory of light was first expressed in definite form by Huygens in 1678, and twelve years later he satisfactorily explained reflection, refraction, and the phenomenon of double refraction in uniaxal crystals, which was discovered by Bartholinus in 1670. Although he discovered the phenomenon of polarization, which would have practically been the death-blow to the emission-theory, had its nature been understood, he was wholly unable to account for it. We must remember, however, that he had longitudinal waves in mind, *i.e.* waves in which the direction of the vibration was parallel to the direction of propagation, and polarization would be as difficult to account for by such a theory as by the corpuscular one. He was moreover unable to offer any satisfactory explanation of the rectilinear propagation of light, or the formation of shadows, and his theory fell into disrepute.

Strangely enough Newton himself made the discovery which, if handled in the proper manner, would have established almost beyond a doubt the validity of the wave-theory.

He devised a method for studying the dependence of the colors of thin films, first observed by Boyle and Hooke, upon the thickness of the film. These colors, however, he sought to explain on the emission hypothesis. Grimaldi in 1665 was engaged with the study of diffraction, or the bending of light around the edges of obstacles. Admitting sunlight through two small apertures into a darkened room, he observed what he thought to be a darker region at the point where the two diverging beams overlapped. As he was merely looking for evidence of the non-materiality of light, he regarded his experiment as conclusive and pursued the subject no further. The apparent destructive interference of light, which Grimaldi thought that he had observed, was without doubt an effect due to contrast.

True interference was first observed by Dr. Young at the beginning of the 19th century nearly 150 years later, whose justly celebrated experiments established almost beyond question the validity of the wave-theory.

Young, however, at first regarded the waves as longitudinal, which assumption, though erroneous, did not affect the validity of his reasoning concerning the formation of interference fringes and the colors of thin plates. Fresnel commenced his optical studies in 1814 and introduced, for the first time, the conception of transverse waves, a conception which he found necessary for an explanation of polarization. Rectilinear propagation he accounted for by a most ingenious method of dividing the wave front up into zones, often wrongly attributed to Huygens, and showing that the disturbances coming from the collective zones, produced zero illumination within the shadow according to the well-known principles of interference. This was a very bold hypothesis, for it necessitated an ether having the properties of an elastic solid, a condition difficult to reconcile with the free and unobstructed motion of the planets through it. This "elastic solid" theory, however, came to be generally accepted, and can still be used to advantage in treating many optical phenomena, for it is more easily intelligible than the modern electro-magnetic theory. Light, on this theory, is regarded as a transverse displacement of a medium called ether, having properties similar to those of an elastic solid, the displacement being propagated from point to point, according to the well-known laws which govern wave motion. There are many objections, one of which is the difficulty regarding the longitudinal disturbance, which always accompanies the transverse one, in the case of a solid. No existence of any such longitudinal disturbance in the ether has ever been found.

Various hypotheses have been made to get around the difficulty. The phenomena of light cannot well be reconciled with the presence of any longitudinal disturbance which is propagated with finite velocity. It has been gotten rid of in the theory by considering the ether as incompressible, which gives to the longitudinal disturbance an infinite velocity. Lord Kelvin made a still bolder suggestion in 1888; he showed that if a "contractile-ether" be assumed, the velocity of the longitudinal wave is infinitely small. In a solid if ϵ is the elasticity or resistance opposed to a shearing strain, k the resistance to compression, and d the density, it can be shown that the velocity of the transverse wave is $\sqrt{\frac{\epsilon}{d}}$, while that of the longitudinal is $\sqrt{\frac{k+\frac{4}{3}\epsilon}{d}}$. In an incompressible fluid k would be infinitely large, and we should have an infinite velocity for the longitudinal impulse. To give us zero velocity for this disturbance, $k + \frac{4}{3}\epsilon$ must equal zero, that is k must be negative, or there must be a negative resistance to compression. This can only be true in a medium in which the conditions are such that it would shrink if left to itself, and it is hard to imagine a stable ether endowed with such properties. Lord Kelvin gets over the difficulty by showing that the instability disappears if we regard the ether as rigidly supported at its boundaries. The condition may be illustrated by considering the case of a mass of small soap bubbles, such as is formed by blowing into a soap solution. The mass in this condition offers a resistance to compression in virtue of

the enclosed air. Suppose we could spirit the air away: the mass would then contract, owing to surface tension, until it vanished into a small liquid drop. At the moment at which the air disappeared it would offer a negative resistance to compression. If the mass of bubbles were, however, blown within a hollow spherical vessel, with a continuous surface, spiriting the air away would cause no change, as the mass would be everywhere supported by the walls.

Lord Kelvin showed that on the assumption of an ether of this nature, the relation between the intensities of the incident and reflected light, in the case of transparent bodies, as expressed by Fresnel's tangent formula, could be accounted for, and Glazebrook showed that it led as well to Fresnel's wave surface in the case of double refraction, both of which phenomena had not previously been well explained on the electro-magnetic theory.

Lord Kelvin probably did not mean to imply that the ether really was finite and supported by a rigid shell, though some have imagined that the ether may stop somewhere. Such a condition of affairs would prevent the escape of energy from the universe by radiation to infinity, for the waves would be reflected back at the boundary. Such speculations belong rather to metaphysics, and have no place in the present volume.

The Electro-Magnetic Theory. — This theory assumes light to be identical with the electro-magnetic disturbances, which are radiated from bodies in which electrical oscillations are taking place. The fundamental equations we owe to Maxwell, who predicted the existence of the waves which were discovered and studied by Hertz. The periodic disturbances, which are supposed to constitute these waves, were called displacement currents by Maxwell, and these displacement currents can occur in the free ether or in a dielectric, *i.e.* in a non-conductor of electricity. A medium, to be capable of propagating vibrations, must possess two qualities. There must be a force of restitution which pulls a displaced particle back into its original position the moment it is released, and the medium must in addition possess inertia, or something corresponding to it, otherwise it would not swing past its position of equilibrium in opposition to the elastic forces which oppose its motion. The electrical behavior of the dielectric in a condenser furnishes abundant evidence of a force closely related to elasticity, and the oscillatory discharge of a Leyden jar points to the fact that electricity in motion has a tendency to continue in motion. Maxwell's theory does not tell us anything about the nature of this electric displacement, so that in one sense our ideas about the real nature of the luminous disturbances are much vaguer than they were fifty years ago, when the elastic solid theory was generally accepted, for in the motion of a solid we are dealing with perfectly definite physical processes. As Schuster remarks in the preface of his recent work on Optics, "So long as the character of the displacements which constitute the waves remains undefined, we cannot pretend to have established a theory of light." The fundamental equations of the electro-magnetic theory will be developed in the chapter on The Theory of Reflection and Refraction, and we shall have occasion to make frequent use of them.

Simple Periodic Motion. — If a particle moves along a straight line in such a way that its distance y from a fixed point satisfies the equation

$$y = a \sin (\omega t - \alpha),$$

in which t is the time and a and ω are constants, its motion may be defined as simple harmonic, or, as Schuster prefers to call it, simple periodic motion. A particle which is acted upon by a force which varies directly with its distance from a fixed point will, if displaced and released, execute a motion represented by the above equation, if no other forces, such as friction for example, come into play. Forces of this nature are assumed to be called into play by the displacements, mechanical or electrical, which constitute light, and we will accordingly begin by establishing the above equation and interpreting its meaning.

Let the force corresponding to displacement y be

$$\psi = -py,$$

in which p is a constant, viz. the force corresponding to unit displacement; the minus sign is given since the direction of the force is opposed to that of the displacement. If m is the mass of the particle, we have for the acceleration,

$$-\frac{py}{m} = -k^2 y \left(\text{writing } k^2 = \frac{p}{m}\right); \quad \therefore \frac{dv}{dt} = -k^2 y.$$

Now $v = \frac{dy}{dt}$ and we therefore have $\frac{dv}{dt} = \frac{d^2y}{dt^2} = -k^2 y$, the integral of which is $y = a \sin (kt - \alpha)$.

This can also be shown in the following way.

The work done on a mass is measured by its kinetic energy and is represented by the product of the force and the distance through which it acts. Since the force varies in the present case, we have for the work done on the particle, displaced a distance a, and moved back by the force to position y,

$$-\int_a^y pydy,$$

the minus sign being given because the path traversed is in the negative direction of y.

Equating this to the expression for kinetic energy gives us

$$\frac{1}{2}mv^2 = -\int_a^y pydy = -\frac{1}{2}p\int_a^y 2\,ydy = \frac{p}{2}(a^2 - y^2);$$

$$\therefore v^2 = \frac{p}{m}(a^2 - y^2), \quad v - \pm k\sqrt{a^2 - y^2};$$

$$\therefore \frac{dy}{dt} = k\sqrt{a^2 - y^2}, \quad \frac{dy}{\sqrt{a^2 - y^2}} = kdt,$$

whence $$kt = \int_0^y \frac{dy}{a\sqrt{1-\frac{y^2}{a^2}}} = \sin^{-1}\frac{y}{a},$$

or $$y = a \sin kt. \qquad (1)$$

Suppose the time to increase from $t=0$ to $kt=\frac{\pi}{2}$, then $y=a$, the maximum value which it can attain, or in other words the amplitude of the vibration.

$$\text{For } kt=\frac{\pi}{2}, \quad y=a, \qquad \text{For } kt=\pi, \quad y=0,$$

$$\text{For } kt=\frac{3\pi}{2}, \quad y=-a, \qquad \text{For } kt=2\pi, \quad y=0,$$

the particle having performed one complete vibration.

If the time occupied is called T, we have $kT=2\pi$, or

$$k=\frac{2\pi}{T}, \quad T=\frac{2\pi}{k}=2\pi\sqrt{\frac{1}{\frac{p}{m}}}=2\pi\sqrt{\frac{m}{p}}.$$

Substituting for k in equation (1) gives us

$$y = a \sin \frac{2\pi}{T} t.$$

Wave-Motion. — In the above discussion we have investigated the motion of a single particle. We will now consider what happens when the particle is bound to other particles by forces which tend to keep the particles at a fixed distance, such, for example, as an attractive force and a repulsive force, the latter increasing more rapidly than the former as the particles approach. Such a medium would be capable of transmitting transverse waves, and we can imagine a sort of atomic ether consisting of extremely minute particles bound together by forces as above specified. This conception need not be taken as expressing our views regarding the constitution of the ether by any means, but as we shall make use of a medium of this nature in the elementary deduction of the dispersion formula, we may as well take it for our type of medium in the preliminary study of wave-motion.

Suppose our particles to be arranged in a row (Fig. 1) and held at a fixed distance apart, say by a spiral spring in which they are imbedded. If one of them is displaced, it is drawn back by the vertical component of the forces T and T' due to the increased tension of the spring. Suppose the particles displaced as shown in Fig. 1, the radius of curvature at A being R. The force acting on A along y will be $2\,T \sin \alpha$, where α is the angle subtended at the centre of curvature by the element ds of the

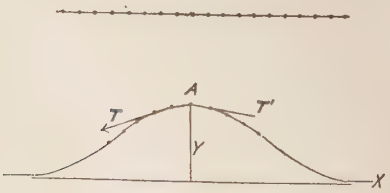


Fig. 1.

medium, *i.e.* the distance between the two particles adjoining A; $2T$ represents the sum of the tangential forces T and T', which are assumed equal for small values of α.

We have then $F = 2T \sin\alpha = T \cdot 2\alpha = \frac{T}{R} ds$ $\left(\text{since } 2\alpha = \frac{ds}{R}\right)$.

Now the curvature $\frac{1}{R} = -\frac{d^2y}{dx^2}$ for small displacements, and if ρ is the density per unit length, ρds is the mass, and we have

$$\rho ds \frac{d^2y}{dt^2} = -F = T\frac{d^2y}{dx^2} ds, \text{ or } \frac{d^2y}{dt^2} = \frac{T}{\rho}\frac{d^2y}{dx^2}.$$

This equation has for its solution

$$y = f\left(x - \sqrt{\frac{T}{\rho}}t\right) + f'\left(x + \sqrt{\frac{T}{\rho}}t\right),$$

or

$$y = f\left\{t - \frac{x}{\sqrt{\frac{T}{\rho}}}\right\} + f'\left\{t + \frac{x}{\sqrt{\frac{T}{\rho}}}\right\},$$

in which f and f' are arbitrary functions.

In the case which we are considering, if the row of particles is displaced as figured, and released, a wave-motion will spread out in both directions, with a velocity equal to $\sqrt{\frac{T}{\rho}} = V$.

If now the particle A vibrates in simple periodic motion, $y = a \sin 2\pi \frac{t}{T}$, we have one of our wave disturbances represented by

$$y = a \sin \frac{2\pi}{T}\left(t - \frac{x}{V}\right), \text{ or } y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right), \text{ since } VT = \lambda.$$

This equation represents as well a plane-wave travelling along the x-axis; its amplitude is a, its periodic time T, and its wave-length λ.

We may get the form of the wave by giving to t any fixed value, for example $t = 0$, when our equation becomes

$$y = a \sin 2\pi \frac{x}{\lambda}.$$

We can plot the curve in the following way.

We will plot the ordinates (y) for values of x equal to multiples of $\frac{\lambda}{12}$ (Fig. 2). Divide the circumference of a circle into 12 equal parts,

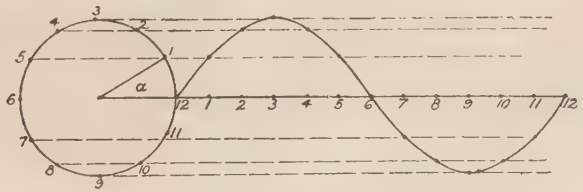


FIG. 2.

and call the radius unity. For $x = 1 = \frac{\lambda}{12}$, the cos of α is the ordinate of point 1 on the circumference of the circle. The same holds for the other points, therefore we have only to draw lines parallel to x through the points on the circle and mark their intersection with ordinates erected at 1, 2, 3, etc. The points thus determined lie on the wave.

Absence of Back-Wave. — If a point in a medium is made to vibrate in simple periodic motion, it sends out waves in both the positive and negative direction. Now when a wave meets a point in a medium, the point is made to execute periodic motion, and the wave beyond the point can be regarded as due to its motion. In this case, however, the moving point only sends out a disturbance in one direction, though its motion is identical with that of the point sending out waves in both directions. As we shall in the next chapter make use of this conception of a point thrown into vibration by a wave as a source of other waves, it is of some importance to distinguish between a secondary source of this nature and an actual source of light.

Let the curved line in Fig. 3 represent a wave travelling towards the right. We know that this wave will be propagated with its type unchanged, and that the medium behind it will come to rest the moment the wave has passed. If, however, we distort the medium into the shape figured, and then release it, we shall have a wave travelling in both directions. The difference between the two cases will become at once apparent if we consider the velocities as well as the displacements of the particles. Consider the first case, that of the moving wave: the particle at A is acted on by a force drawing it downward, and being at rest initially it moves in consequence. The particle at B is acted upon in the opposite direction by an equal force. It, however, is *not at rest*, for it is moving in a downward direction with a velocity represented by the dotted arrow, for the wave has just passed by it, and it is returning to its position of equilibrium: this velocity just compensates the force due to the distortion of the medium and the particle comes to rest. In the second case both A and B are at rest initially, and both move the moment the restraint is removed, and we have a wave moving in both directions. We can in the same way see how the vibration of A by the passage of the wave through it fails to give a back-wave. It moves let us say to A', which it will do in time $\frac{T}{4}$. In the meantime the point C has returned to C', and its velocity just compensates the force due to the displacement of A, which in a medium initially at rest would result in a back-wave.

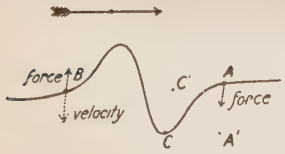


FIG. 3.

Wave-Front. — We may define the wave-front as the continuous locus of the points of the medium which are about to be disturbed. Thus defined the wave-front marks the limit which the disturbance has reached at the instant considered. A more general definition,

however, and one which we shall find more useful is the following. *The wave-front is the continuous locus of points which are in the same phase of vibration, or a surface of equal phase.* If this surface is plane, we speak of the waves as plane-waves, and since in isotropic media the rays are perpendicular to the wave-front, the rays are in this case parallel. The waves coming from sources of light situated at infinity (*e.g.* the stars) are plane.

If the source is at a finite distance, the wave-fronts are spherical, if the velocity of propagation is independent of the direction, as is the case in isotropic media. By means of mirrors or lenses it is possible to transform a spherical wave-front into a plane one, but we possess no means of starting a plane-wave directly. We can perhaps get a better case of what this would involve in the following way.

Consider a vibrating particle attached to an elastic string: waves will run along the string and the wave-front will be a point (Fig. 4*a*). Attach a number of strings to a rod vibrating in a direction parallel to its length (Fig. 4*b*), and the wave-front will be a straight line if we regard the strings as forming a continuous medium (Fig. 4*b*).

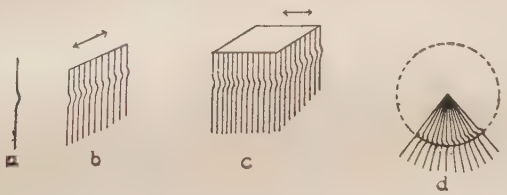


Fig. 4.

There is no such thing in nature as a linear wave of light, for the reason that such waves can only occur in a medium of two dimensions. The conception of such a wave is often made use of in elementary treatments of diffraction, as the problems are much simplified by restricting the disturbance to two dimensions.

If now our strings are attached to a vibrating plane, the continuous locus of equal phase is obviously a plane, parallel to the moving plane, since the waves all start at the same instant, and travel with equal velocities. To realize this condition in optics it would be necessary to arrange a plane source of light, over the surface of which the vibration was uniform, *i.e.* the phases of all the vibrating particles would have to be the same, a condition which obviously cannot be realized. By attaching the strings to a vibrating point and arranging them so that they stretch out in all directions, we represent roughly the conditions under which we obtain a spherical wave. It should be observed, however, that in the case of a to-and-fro motion of the point, there are two directions in which transverse waves will not be given out, these directions coinciding with the direction of motion of the point. We have this circumstance occurring in certain optical phenomena, as we shall see later on (certain facts connected with the Zeeman effect, for example).

Frequency and Wave-Length. — The length of the light wave depends as we have seen upon two factors, the velocity and the frequency or time of vibration. Since the velocity in refracting media is usually less than the velocity in ether, the wave-length is reduced when the disturbance enters such a medium, for the frequency remains the same. The wave-length and frequency obviously depend upon the nature of the source. Flames colored by metallic salts may emit light of definite frequencies, such as the sodium flame, the light of which consists chiefly of two yellow radiations, commonly designated as the *D* lines.

Light in which we have but a single wave-length is said to be monochromatic. It must be remembered, however, that strictly monochromatic light involves an infinite train of waves, such as would emanate from a particle the vibrations of which were subject to no sudden or gradual changes of phase. Absolutely homogeneous or monochromatic light is something that has no actual existence, though we are accustomed to speak of light which the spectroscope shows as a single narrow line, as monochromatic.

The color depends upon the wave-length, but the color cannot always be taken as an indication of wave-length, as certain colors can be imitated by the simultaneous action upon the retina of two trains of waves, either of which acting alone would give rise to a totally different color from that perceived when both act together.

For example, a yellow scarcely distinguishable from the yellow of the sodium flame can be produced by a mixture of red and green light in the proper proportions. A screen can be easily prepared which transmits red and green only and in about the right proportions to produce the sensation of " subjective yellow," as it is called.

Canada balsam, boiled down until it will solidify on cooling, is stained with " brilliant-green " and naphthalene yellow, in the same proportions used for making dichromatism prisms (see page 351) and a small quantity pressed out between two warm glass plates until the color of the transmitted light is yellow. Examination with a small spectroscope reveals the fact that in reality no yellow light is transmitted, only red and green. We have then the important distinction that while wave-length determines color, color does not necessarily determine wave-length.

Lord Rayleigh recommends a mixture of an alkaline solution of litmus with chromate of potash. If a window, backed by well-lighted clouds, is viewed through such a solution and a prism it presents a most splendid appearance, for the red and green images are widely separated, the region where they overlap being colored with the compound yellow. A screen capable of transmitting only the yellow region is difficult to prepare. A mixture of bichromate and permanganate of potash answers fairly well, and can be made to match the color of the first screen. A sodium flame is invisible through the first and easily visible through the second. Both together are practically opaque even with very intense white light.

The different radiations present in a source may be separated by a prism or diffraction grating, as we shall see, and we obtain in this way what is known as a spectrum of the source: not all of the radia-

tions in the spectrum affect the eye, for, as we know by experiment, there are regions beyond the red and violet which we cannot see. The longer waves in the infra-red spectrum can be recognized by their heating power, or by their action on phosphorescent substances; the ultra-violet or short waves can be detected by photography or by their action in causing fluorescence.

The length of the light wave can be measured with great precision by methods which will be described later on. The shortest ultra-violet light waves were discovered by Schumann and were subsequently more accurately measured by Lyman: they are so strongly absorbed by air that they have to be investigated in vacuo, or in an atmosphere of hydrogen. The longest infra-red waves, found in 1910 by Rubens and Hollnagel, have a length of $\frac{1}{10}$ of a millimeter or are one thousand times as long as the Schumann ultra-violet waves. The wave-lengths in the different parts of the spectrum are given in the following table:

	Wave-Length
Schumann waves	.00010 mm.
Ultra-violet	.00020 mm. to .00035 mm.
Extreme violet	.00040 mm.
Blue	.00045 mm.
Green	.00050 mm.
Yellow	.00058 mm.
Red	.00065 mm.
Extreme red	.00072 mm.
Infra-red 1 μ	.001 mm.
Residual rays reflected from quartz, 8.5 μ	.0085 mm.
Residual rays from sylvite, 60 μ	.0600 mm.
Residual rays from iodide of potassium, 96 μ or nearly $\frac{1}{10}$ mm.	.096 mm.
Shortest electro-magnetic waves	6.0 mms.

The "residual rays" are obtained by reflecting the radiations of a Welsbach lamp from a number of surfaces of quartz or other material. To get a comparative idea of these wave-lengths we may take a metre stick as our scale. Calling the Schumann waves 1 mm., green light will be 5 mms., red light 7 mms., and the longest heat waves thus far found 96 cms., or practically the entire meter. Now consider this scale reduced in length to $\frac{1}{10}$ of a millimeter, and we have our waves as they actually are.

Sources of Light for Experimental Purposes. — It will perhaps be well in the introductory chapter to describe briefly a number of sources of light, which will be found useful in experimental work pertaining to the subject of Physical Optics. As a source of white light, the sun is to be preferred when great intensity is required. Next to this comes the electric arc, the type most suitable for experimental work being a lamp in which the positive carbon is horizontal. If great intensity is not necessary, the Nernst filament will be found very serviceable. It ranks next to the arc in intrinsic

intensity, requires no attention, and has the added advantage of narrowness. It may thus be used in many cases as a substitute for an illuminated slit. If an electric current is not available, a Welsbach lamp, surrounded by a sheet-iron chimney furnished with a small vertical slit, will be found an excellent substitute.

If a source of light giving a continuous spectrum in the ultra-violet is required, as in the study of absorption spectra in this region, nothing is superior to the cadmium spark, which in addition to its bright lines has a fairly strong continuous background. An induction coil, or better still a 10,000-volt transformer, with one or two large Leyden jars in the circuit, furnishes the best means of obtaining a suitable spark. An acetylene flame, traversed by the discharge of the high potential transformer, gives a continuous spectrum free from bright lines, which extends down into the ultra-violet region much farther than most sources of light, and can be used in cases where the bright lines of the cadmium spectrum are undesirable. The spark under water is another source of an ultra-violet spectrum, but its use is troublesome. The modern incandescent lamps, with filaments of tungsten, have an intensity even greater than that of the Nernst lamp, and it is probable that a lamp made with a tungsten filament half a millimeter broad and one centimeter in length would be extremely useful for experimental work.

As sources of monochromatic light we possess various colored flames and vacuum-tubes, from the spectrum of which we can pick out a monochromatic radiation by screening off the wave-lengths which are not desired. A simple form of apparatus for accomplishing this is described in Mann's *Manual of Optics*. It is easily constructed, not expensive, and can be made without the services of a skilled mechanician (Fig. 5). Light from a slit S, made parallel by a lens L, traverses a glass prism, after which it is reflected back through the prism and collimating lens, the convergent beam being then deviated to one side by a small right-angled prism, the focussed spectrum falling upon a screen provided with a vertical slit S'. By turning the mirror M by means of adjusting screws, any desired portion of the spectrum may be passed out through the side slit. It is possible with this instrument to obtain fairly monochromatic light from a source giving a continuous spectrum, or to pick out the highly homogeneous radiations which are emitted by metallic vapors, brought to a state of luminescence by the electrical discharge in vacuum tubes, or in the arc or spark discharge.

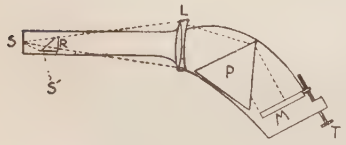


FIG. 5.

An instrument is made by R. Fuess, of Steglitz, near Berlin, which is a combined spectroscope and monochromatic illuminator. The author has found this instrument most useful. It is very easily calibrated, and will furnish a beam of approximately monochromatic light from sun or arc light, the width of the band being not much wider than the distance between the sodium lines. The transmitting

slit can be removed in an instant, and the insertion of the eye-piece in its place transforms the instrument into a very good spectroscope.

The sodium flame is the most generally used source of monochromatic light. Its radiations, however, consist of two wave-lengths, consequently its spectrum consists of two lines in the yellow very close together. To separate one of these from the other is a matter of considerable difficulty, and for most purposes the complete radiation will be found equally satisfactory. The most satisfactory flame can be obtained by winding a piece of asbestos paper around the top of a Bunsen burner (fastening it with wire) and saturating it with strong brine.

Monochromatic red light can be obtained by saturating an asbestos cylinder with a solution of chloride of lithium, and a satisfactory green by means of a small fragment of metallic thallium, fused to a loop of platinum wire. The bead should be mounted so as to barely touch the outer edge of the flame, otherwise it will rapidly evaporate. For long-continued work, however, the most satisfactory light is the mercury arc, from the radiation of which we can pick out by means of color screens, or the simple spectroscope described above, any one of the numerous bright lines.

A commercial mercury arc is the easiest to operate, and gives no trouble. It will be found immensely useful and should be installed in every laboratory. Its light is not as intense as that emitted by the lamps made of fused quartz by Heraeus of Hanau, Germany, and it does not give us much of the ultra-violet, but for most purposes it is most satisfactory.

A very simple and easily constructed mercury lamp has been described by A. H. Pfund. It consists of a piece of glass tubing 7 cms. in length and 1.2 cms. in diameter sealed to a larger tube 3 cms. in diameter and 18 cms. long, provided with a side tube for exhaustion. A platinum wire is sealed into the lower end of the small tube, which is half filled with mercury. The mercury forms the negative electrode, while a hollow sheet-iron ring forms the positive (Fig. 6). This electrode is made by cutting out a piece of sheet iron as shown in the smaller figure (*a*) and bending it into the form of a double ring (*b*).

The lower ring should be a little smaller than the upper, so as not to come into contact with the glass when the upper, or supporting, ring is pushed into the small tube. The ring is heated to a high temperature when the current passes, and invariably cracks the glass if it touches it. A thin iron wire is fastened to the ring and passes out through the top, through a sealing wax seal. It is better to use copper wire at the point where it passes through the wax, as the current heats the iron. The joint is made at *C*. The top is closed by a quartz plate sealed on with wax.

A properly made joint made with sealing wax will hold a cathode ray vacuum for days. The end of the tube should be warmed and the wax applied until a thick ring is formed. The plate should then be heated and pressed against the wax ring, after which the joint should be gone over with a small pointed gas flame, burning at the tip of a "drawn-down" glass tube. The lamp is exhausted

with a mercury pump, and joined to a 110-volt direct current circuit, with a resistance of about 30 ohms in series with it. It is started by warming the bottom with a Bunsen burner and tilting or shaking it until the circuit is made. It should be run for fifteen minutes while connected with the pump, to get rid of the occluded gases, after which it can be sealed off. If the light in the visible spectrum alone is required, it may be taken from the side. The ultra-violet radiations come out through the quartz plate. If only the visible and a part of the ultra-violet are required, the quartz plate can be dispensed with and a sealed-in platinum wire joined to the iron wire. After several days running, gases may be liberated and impair the vacuum, causing the arc to acquire a temperature sufficient to melt the glass. The lamp should be tested from time to time by holding a piece of paper against the glass. If it chars, it is time to reëxhaust the tube.

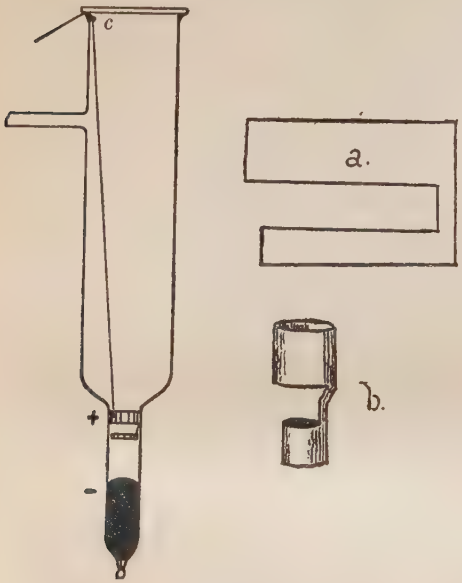


Fig. 6.

The mercury arc lamps made of fused quartz are most satisfactory in every respect, though rather expensive. The ultra-violet radiations are so intense that the air becomes immediately charged with ozone, and glasses should always be worn, as an exposure of only a minute or two of the naked eye to the light results in a very serious and painful inflammation. The glasses need not be dark, as the harmful rays are absorbed by ordinary transparent glass. Cadmium and zinc lamps can also be obtained of quartz. These, however, require a certain amount of preliminary heating.

During operation they should be kept in communication with the pump. After prolonged use (30 to 40 hours running), the quartz often becomes partially devitrified, resembling ground glass, and a black deposit sometimes forms on the inner walls. To remedy this, the tube should be removed from the pump and heated in an oxy-hydrogen flame, which restores the surface and burns off the black deposit.

Color Screens or Ray Filters. — The monochromatic constituents of a source of light such as the mercury arc can be separated by a spectroscope, but color screens are more convenient, and have the advantage that they permit us to utilize the entire source. There

are countless absorbing substances at our disposal, colored salts of metals and aniline dyes, solutions of which can be used in glass cells made by cementing glass plates to annular strips cut from heavy brass or glass tubing.

The following substances are most useful: cobalt chloride, copper chloride, nickel sulphate, potassium permanganate, chromium chloride, bichromate of potash, picric acid, sodium chromate, neodymium chloride, and praseodymium chloride. The last two are very useful when combined with other filters for sharpening the edges of absorption bands. We should have on hand also a good collection of aniline dyes and especially the very useful compound nitroso-dimethyl aniline, the numerous remarkable properties of which have been investigated by the author. A very complete list of the aniline dyes, with photographs of their absorption bands, will be found in the *Atlas of Absorption Spectra*, by Uhler and Wood, published by the Carnegie Institute, Washington, D.C., and a more recent one, dealing in particular with the absorption in the red region, by C. E. K. Mees, published by Longmans, Green, & Co.

For separating the radiations of the mercury arc the following solutions will be found suitable:

Bichromate of potash transmits the green and the two yellow lines. Addition of a neodymium salt removes the yellow lines, without reducing the intensity of the green line in the least. No other substance is as satisfactory as this. Cobalt glass + aesculin solution transmits the 4359 line. Guinea green B extra (Berlin) + chinin sulphate transmits 4916. Nickel sulphate is also useful. Chrysoidine + eosine transmits the yellow lines 5790. The chrysoidine should be dilute, and the eosine added until the green line disappears. A very thick cell with a saturated solution of bichromate of potash is perhaps as good. Methyl violet 4 R. (Berlin an. Fabrik) very dilute, and nitroso-dimethyl aniline transmits the ultra-violet line 3650. Methyl violet + chinin sulphate (separate solutions) transmits 4047 and 4078, also faintly 3984.

The absorption bands of a number of substances are shown in the form of a chart (Fig. 7), which will doubtless be found useful in preparing screens. A very useful set of ray filters are made by Wratten and Wainwright, Croyden, England.

A film of silver chemically deposited upon a quartz lens or plate is practically opaque to all radiations except the ultra-violet region 3160 3260. The silver film should be of such thickness that a window backed by a brilliantly lighted sky is barely visible through it. Directions for silvering will be found in the Chapter on Interference Spectroscopes. Lenses prepared in this way have been used by the author for photographing the moon, landscapes, and various objects in ultra-violet light. Very dense cobalt glass combined with a layer a centimeter or more in thickness of a saturated solution of bichromate of potash cuts off everything except the extreme red above wave-length 69. This screen was used by the author in making infra-red landscape photographs. A clear blue sky is nearly black through it, while sunlit foliage comes out very bright.

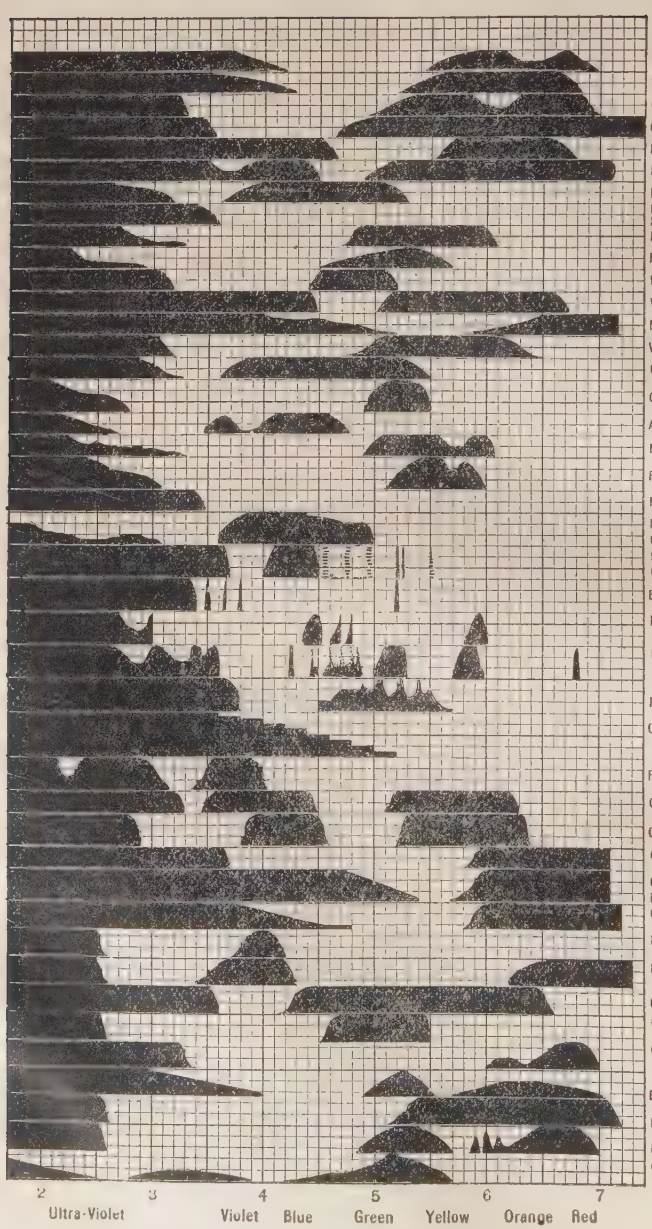

Methylene Blue
Turkish Blue
Cobalt Glass
Cuprammonium
Rapid Filter Green
Methyl Green
Methyl Orange
B.Naphtholdi-Sulphonic Acid
Heliotrope
Fuchsine
Uranine
Victoria Green
Naphthol Green
Victoria Blue
Chrysoidine
Cyanosine
Auramine
Methyl Violet
Rhodamine
Resorcin
Nitroso Di-methyl Aniline
Uranous Chloride
Erbium Chl.
Praseodymium Chloride
Neodymium Chloride
Potassium Permanganate
Chlorides of Calcium and Copper
Potassium Chromate
Chromium Nitrate
Chrom. Chloride
Copper Nitrate in Water
Copper Chloride in Ethyl Alcohol
Copper Chloride in Water
Nickel Acetate in Water
Nickel Chloride in Water
Cobalt Sulphocyanate
Cobalt Sulphate in Water
Cobalt Chloride in Acetone
Cobalt Chl. in Ethyl Alcohol + Water
Cobalt Chl. in Ethyl Alcohol + Ag.
Cobalt Chloride in Methyl Alcohol
Cobalt Chloride in Water
2
3
4
5
6
7
Ultra-Violet
Violet
Blue
Green
Yellow
Orange
Red

FIG. 7.

A saturated solution of iodine in carbon bisulphide is opaque to all visible radiations, and transmits freely the infra-red.

If, in addition to the substances enumerated above, we have a collection of scraps of colored glass, we shall be sufficiently well equipped in the way of ray filters for all ordinary optical and spectroscopic work.

Velocity of Light. — The first determination of the velocity of light was made by a Danish astronomer Römer in 1676. From observations made on the eclipses of Jupiter's satellites he showed that the inequalities noted in their times could be explained by the finite velocity of propagation of light. Since the time of rotation of the satellites around the planet is constant for each satellite, they will enter the shadow of the planet at regular intervals, and the times of the eclipses can be predicted with the greatest accuracy. Römer found, however, that the intervals between successive eclipses of a given satellite varied gradually if the observations extended over a year. The eclipses were found to occur earlier or later than the calculated time, according as the earth and Jupiter were on the same, or opposite sides, of the sun. The discrepancy was obviously due to the time taken by light to travel across the earth's orbit. Calculation showed that the velocity of light was about 192,000 miles per second.

The second determination was made in 1728 by Bradley, who discovered the phenomenon known as the aberration of light. He observed that the apparent position of the stars shifted slightly from time to time, and finally came to the conclusion that this small apparent motion could be explained by taking into account the earth's motion in its orbit, together with the fact that light is propagated with a finite velocity. The phenomenon of aberration will be more fully discussed in the chapter on the relative motion of matter and ether.

Fizeau's Method. — Galileo had made an unsuccessful attempt to determine the velocity of light, by placing two observers at a great distance apart, each furnished with a lamp. One observer uncovered his lamp and the second observer watched for the flash and removed the screen from his lamp at the moment it appeared. The first observer was to determine the velocity by noting the time elapsing between the uncovering of his own lamp and the appearance of the distant light.

This method failed obviously, owing to the enormous velocity of light. In 1849 Fizeau made an experimental determination of the

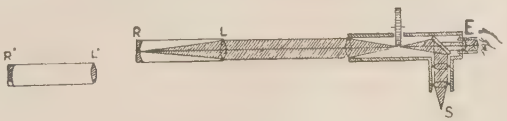


Fig. 8.

velocity of light by means of a revolving disk furnished with a toothed rim. The method is essentially as follows: A beam of light was introduced into the tube of a telescope by means of a collimator

fitted in to its side, and was focussed by means of a reflecting plate upon the rim of the toothed wheel (see Fig. 8). This point was at the principal focus of the object glass of the telescope; consequently the light, after passing between the teeth of the wheel, was made parallel by the objective.

After traversing a distance of three or four miles, it fell upon a second lens, which brought it to a focus upon a concave spherical mirror, the centre of curvature of which coincided with the centre of the lens. The light was thus returned as a parallel beam over the same path, and entered the eye-piece at E, passing through the reflecting plate. If the toothed wheel is rotated the beam of light will be made intermittent, and if the speed be great enough the light which passes through the space between the teeth will, upon its return, be cut off by the adjacent tooth, which in the meantime has advanced into the position previously occupied by the space. On looking into the telescope the observer sees at first a bright star, which diminishes in intensity as the speed of rotation is increased, finally disappearing entirely. Further increase in speed causes the reappearance of the star, the light passing through a given space, falling upon the next adjacent space upon its return. Fizeau experienced great difficulty in determining accurately the speed at the moment when the eclipses occurred. The image of the distant star was never bright, and the light reflected from the teeth of the wheel caused a general illumination of the whole field. To obviate this difficulty Young and Forbes, in repeating the experiment, bevelled the teeth so that the light reflected from them fell upon the blackened sides of the telescope. The teeth were also blackened so as to diminish their reflecting power as much as possible. In 1874 Cornu repeated the experiment with certain modifications. To avoid the difficulty of determining the exact moment at which the star was eclipsed, he made use of an electrical chronograph, arranged so as to record every hundred revolutions. Seconds were marked by a clock, and tenths of a second by means of a vibrating spring. By means of a key the observer could record any instant at which he wished to know the velocity. The speed and its rate of change could be determined at every instant from the record of the chronograph. Instead of attempting to determine the moment of complete extinction, Cornu compared the brilliancy of the image with a light of fixed intensity. On increasing the speed the intensity of the image sank, and the speed of the wheel was recorded at the moment at which it was equal to the intensity of the standard light. After extinction the star reappeared and the speed was recorded at the moment when it regained its former brightness. The speed corresponding to complete extinction was the mean of these two. Cornu's final result for the velocity was 300,330 Kms. per sec. in air, or 300,400 in vacuo.

Foucault's Method. — Wheatstone had suggested that a revolving mirror might be employed in the determination of the velocity of light, and his suggestion was taken up by Arago, but it remained for Foucault to carry out the experiment in a form capable of giving accurate results. The arrangement of his apparatus is shown in Fig. 9. Sunlight after transmission through an aperture at S and

an achromatic lens I falls upon a mirror R, which can be rotated at high speed. A concave mirror M fixed at a distance of several metres returns the light to the revolving mirror. If the mirror R is at rest, the light returned by it after reflection from the inclined plane mirror comes to a focus at a.

The axis of the mirror R is at the centre of curvature of the mirror M, consequently the cone of rays, which converges upon M, is returned over the same path, and the rotation of R will not affect the position of the image at a. This, however, is only true if the mirror is in the same position when the rays meet it a second time, as will be readily seen by considering the passage of a ray from S to a. If the mirror turns through an appreciable angle while the light is traversing the distance $2\,RM$, the image will be shifted to a point a'.

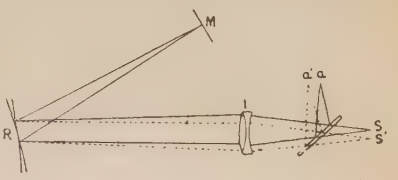


Fig. 9.

The revolving mirror was driven by an air turbine, the speed being determined by a stroboscopic method. The displacement of the image amounted to only .7 mm., which gave for the velocity of light 298,000,000 metres per second.

Michelson's Experiments. — Foucault's method was improved by Michelson, who placed the lens between the two mirrors (Fig. 10). The lens was 8 inches in diameter and had a focal length of 150 feet. The revolving mirror was placed 15 feet inside the principal focus, and the mirror M at a distance of 2000 feet. Deflections of the image amounting to 133 mms. were obtained, which made it possible to dispense with the oblique reflecting plate, and observe the image directly, with an eye-piece placed to one side of the slit. The speed of the mirror was determined by means of a tuning fork, one of the prongs of which carried a light mirror, which reflected the light from the revolving mirror into the eye-piece. When the fork vibrated, the spot of light was drawn out into a band, which broke up into a number of moving images as soon as the mirror was set in rotation. A single stationary image was obtained only when the mirror made as many turns per second as the frequency of the fork; this condition was easily secured by regulating the air pressure at the turbine. The mean result for the velocity of light (reduced to the velocity in vacuo) was 299,910 ± 50 kilometres per second. No indication of the phenomenon alleged to have been observed by Young and Forbes was observed. A difference of velocity between the red and blue radiations, as large as their experiment indicated, would have resulted in a drawing out of the image into a spectrum 10 mms. in length.

Fig. 10.

Experiments were also made on the velocity of light in bisulphide of carbon, a tube three metres in length being interposed between the

mirrors. The ratio of the velocity in air to the velocity in this fluid was found to be 1.758, while the ratio indicated by the refractive index is 1.64. This discrepancy will be explained in a subsequent chapter. Professor Michelson also experimented with lights of different colors, and found that red light travelled 1 or 2 per cent. faster than green light in the carbon bisulphide.

Newcomb's Experiments. — A series of experiments were made by Newcomb at Washington in 1880–82, with an apparatus of slightly different type. Sunlight entered the slit at S (Fig. 11), and, after reflection from a mirror at the elbow joint, passed through the telescope lens and fell upon the revolving mirror m, from which it was reflected along the line z to the distant mirror. The object glass of the receiving telescope was immediately below that of the sending telescope, the light entering it being received from the lower part of the revolving mirror. This consisted of a rectangular prism of steel

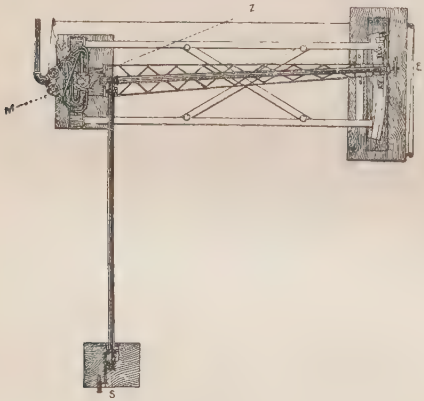


Fig. 11.

Fig. 12.

(Fig. 12), the surfaces of which were nickel plated, driven by means of an air blast. The speed was regulated by means of a slight counter blast directed against the lower fan wheel. By employing two lenses in the manner indicated, the diffused light from the strongly illuminated upper portion of the mirror did not enter the receiving telescope. The mirror could be driven in either direction, by interchanging the direct and counter blasts; the displacement measured could thus be doubled.

The quantity measured directly was the angular deviation of the return image, and not its linear displacement; this was accomplished by swinging the observing telescope, the eye-piece end moving along a graduated arc, the divisions of which were read by means of a pair of microscopes.

Newcomb's final result was, for the velocity in vacuo,

$$v = 299{,}860 \pm 30 \text{ kilometres.}$$

Group Velocity. — An important distinction exists between the velocity of a group of waves and the velocity of a single wave. We can get a very good idea of what is meant by group velocity

by throwing a stone into a quiet pond, and watching the circular waves which spread out. If the attention be fixed on a single wave-crest at the centre of the group, it will be seen presently to lead the group, the waves ahead of it appearing to die out, and in a few seconds its amplitude will become so small that the eye can no longer be kept on it. There are just as many waves in the group, however, as there were before, and a little further observation will reveal the fact that, as the waves in front die out, new ones appear in the rear. The group is obviously moving forward with a velocity less than that of the individual waves.

The explanation of the phenomenon was first given by Stókes, who regarded the group as formed by the superposition of two infinite trains of waves, of slightly different wave-length, which advanced in the same direction but with different velocities.

Lord Rayleigh was the first to draw attention to the bearing of group velocity upon optical problems. In his article on " The Velocity of Light " (*Nature*, 1881), he called attention to the fact that, in all experiments made for the purpose of determining the velocity of light, it is the group-velocity, and not the wave-velocity, which is actually measured. What is actually determined is the velocity with which some peculiarity impressed upon the wave-train moves forward. Since it is impossible in the case of light to pick out and watch a single wave, the best that we can do is to measure the speed with which a block, cut out of a wave-train, advances. If the medium is free from dispersion, *i.e.* if waves of all possible lengths are propagated with the same velocity, the group-velocity and wave-velocity will be the same, the group being propagated without alteration.

This will be made clear by reference to Fig. 13. In the lower diagram we have two superposed trains of waves, moving in the direction of the arrow. The resultant disturbance is indicated in the upper diagram. The longer waves (dotted line) are out of step with the shorter (solid line) at A and C, and the resultant is zero at these points. At B, where there is agreement of phase, the resultant amplitude is double that of the single waves. If now the velocities of the two sets of waves are equal, it is evident that the group shown in the upper diagram will move forward without alteration with the wave-velocity. If, however, the shorter waves move at the higher speed, it is evident that they will presently get out of step at B, and into step at C, which now becomes the centre of the group. The group thus advances with a velocity greater than that of the individual waves. If the reverse is the case, the amplitude to the left of B increases as the group advances, the amplitude to the right of B diminishing, A becoming eventually the centre of the group. In this case the group-velocity is less than the wave-velocity.

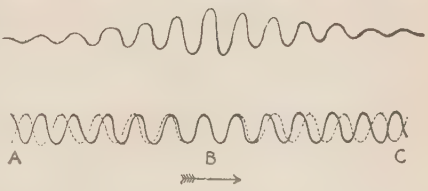


FIG. 13.

We will now derive an expression for the group-velocity. Let the longer wave λ' (dotted line in Fig. 14) move with a velocity $V' > V$, the velocity of the shorter wave λ. Let T be the time required for the point marked V' to overtake the point marked V. When this event has occurred, the centre of the group, defined as the point of maximum resultant amplitude, and originally at B, will have moved back a distance of one wave-length. Now the crest V' is approaching the crest V with a velocity $V' - V$, therefore $(V' - V)T =$ distance $V'V = \lambda' - \lambda$. If we write $dV = V' - V$ and $d\lambda = \lambda' - \lambda$, we have

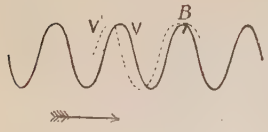


Fig. 14.

$$T = \frac{1}{\frac{dV}{d\lambda}}.$$

During the time T the λ-wave train moves forward a distance VT, therefore the centre of the group as defined above has moved a distance

$$X = VT - \lambda = \left(V - \lambda\frac{dV}{d\lambda}\right)T,$$

and the group-velocity is given by dividing this quantity by T. Calling U the group-velocity, we have

$$U = V - \lambda\frac{dV}{d\lambda}.$$

If the medium is free from dispersion $\frac{dV}{d\lambda} = 0$, and $U = V$.

We shall have occasion to use the formula for group-velocity in studying the action of a prism on white light, which will be taken up later.

The difference between U and V only comes into play in determinations of the velocity of light in strongly dispersive media, the correction to be applied amounting to 7.5% in the case of bisulphide of carbon.

Michelson, employing the revolving mirror method, which has been shown by Rayleigh to yield U and not V, found the velocity of light in air 1.758 times greater than in this fluid, while determinations made by measuring the refractive indices gave the value 1.64. If we increase 1.64 by 7.5% we obtain the value 1.76, which is in close agreement with the value observed by Michelson.

It is worthy of remark that determinations of the velocity of light by observations made on the aberration of light from the stars, give us V. Römer's method, however, yields U, and the close agreement between the values obtained by these two astronomical methods indicates that light is propagated across interstellar space without dispersion.

Independent evidence that waves of all lengths travel with the same velocity in the free ether, is furnished by the variable star Algol, which shows no color sequence when increasing in brightness, as would be the case if waves of different lengths travelled with different velocities.

The Doppler-Fizeau Principle. — Doppler, in 1842, called attention to the change in the pitch of a sound, which resulted when the source was moving towards or away from the observer, and applied the principle to luminous disturbances radiated from bodies in motion, explaining the colors exhibited by certain stars as due to their proper motion. The acoustical phenomenon is most frequently heard when travelling in a railroad train. If a whistling locomotive is passed, the drop in the pitch is very noticeable, especially if the locomotive is moving rapidly in the opposite direction. Doppler's application of the principle to stellar phenomena was unsound, and Fizeau appears to have been the first to show that the effect would manifest itself as a slight shift in the position of the bright or dark lines in the spectrum. If the source of light is moving towards the observer, the frequency of the disturbance as it passes the observer is increased, and the wave-length diminished: the spectrum lines are therefore shifted towards the violet: the reverse is true when the source is moving away in the line of sight. By photographing the spectrum of a star alongside of a comparison spectrum, it is possible to determine, not only whether the star is moving towards or away from us, but also the velocity with which it approaches or recedes. The principle has had wide applications in astro-physical research, and the rapidly accumulating data regarding stellar velocities will, at some future date, in all probability furnish the key to the solution of that greatest problem of astronomy, the nature of the motion of the multitude of suns which make up the universe.

Double stars have been discovered by the Doppler effect, the components of which no telescope will show separated, and their time of revolution about their common centre of gravity determined. Such stars are called spectroscopic binaries. The first was discovered at the Harvard Observatory by Pickering. Observations of a number of spectra of this star, taken at different times, showed that the lines became double at stated intervals, an effect which could only be accounted for by assuming the source of light to consist of two bodies which alternately approached and receded, in other words two bodies revolving around their common centre of gravity.

Keeler applied the principle to the study of the rings of Saturn, and showed that each portion of the ring was rotating at the speed which an isolated satellite would have at the same distance from the planet.

The effect was first obtained in the laboratory by Bélopolsky in 1901 (*Astro. Phys. J.* 13, pg. 15–24), who reflected a beam of light from a system of moving mirrors, subsequently analyzing the light with a spectroscope. The displacement of the spectrum lines was of the calculated order of magnitude, which was, however, an exceedingly small quantity. The minimum velocity capable of modifying

the wave-length to such a degree that the spectroscope will note the change is a kilometre or perhaps half a kilometre a second. The change of wave-length resulting from reflection from a moving mirror is double the change resulting from the motion of the source with the same velocity. Bélopolsky made use of multiple reflections from two systems of mirrors, mounted on the rims of a pair of opposed wheels, which could be revolved at high speed. In this way he was able to obtain a shift of the spectrum lines which, though small, was easily measurable.

The experiment was repeated in 1907 by Prince Galitzin and J. Wilip with Bélopolsky's apparatus. They employed an echelon spectroscope and the mercury arc, and obtained much larger shifts than those observed previously on account of the much greater power of the spectroscope. The mirror wheels rotated at a speed of 45 revolutions per second, which represented a linear velocity of the mirrors of 30 metres per second. Six reflections were used and the displacement amounted to $\frac{1}{32}$ of the distance between the spectra of different orders. This was a double displacement obtained by two exposures, with the direction of rotation of the mirrors reversed between them. It is a very small amount, as we shall see when we come to the study of the echelon, but the calculated velocity of the mirrors agreed well with the observed. For example, the velocity calculated from the line shift in one case was .405 Km. per second while that determined by measuring the speed of the wheels was .379.

More recently the Doppler effect has been found by Stark in the case of the light emitted by the canal rays in vacuum tubes. The canal rays occur where the cathode is perforated with small holes, and they are known to consist in all probability of the positively charged residue of the atom after the negative electron has been

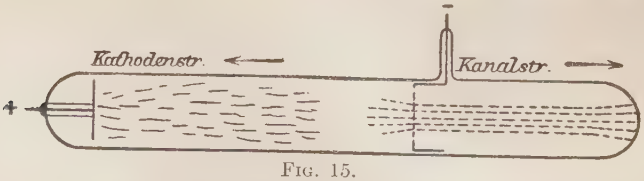


Fig. 15.

expelled. They are hurled down the tube with a prodigious velocity, and if the stream is pointed towards the spectroscope a line is observed shifted towards the violet. If the tube is oriented so that the stream is directed away from the instrument, the shift is in the opposite direction. If, however, the canal ray stream stands perpendicular to the collimator of the spectroscope, no shift is observed. The effect is not at all difficult to observe, and the canal ray tube can be made in a few minutes from some small pieces of glass tubing. The electrodes can be sealed in with sealing wax, if the discharge is prevented from reaching the wax joint. The construction is shown in Fig. 15. The cathode should be made from a piece of aluminum plate a trifle over a millimetre thick, perforated with numerous holes not over a millimetre in diameter. The anode is formed from a short

piece of aluminum wire. The rounded end of the tube should be placed against the slit of a two or three prism spectroscope, filled with hydrogen and connected to a mercury pump. The effect is best seen at the blue line of hydrogen, on account of the higher dispersion of the instrument in this region. At a pressure of a millimetre or two the line appears perfectly sharp and in its normal posi-

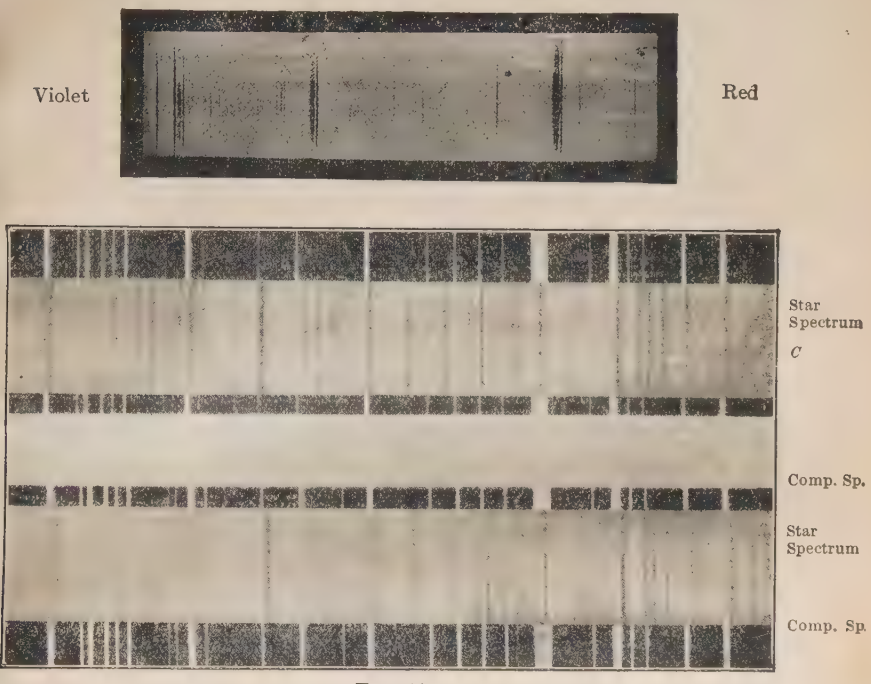


Fig. 16.

tion. As the exhaustion proceeds a wing appears on the violet side, which presently detaches itself from the line and creeps slowly away from it. This gradual movement is due to the fact that as the vacuum becomes higher the canal rays travel at a higher velocity, and the change in the wave-length becomes greater. There is always present the line in its normal position, which shows that there is an emission as well from hydrogen atoms which do not partake of the motion of the canal rays. The velocity of the rays can be determined by measuring the position of the shifted part of the line. A photograph by Stark of the hydrogen lines showing the effect, and the effect as shown in the spectrum of a double star, are reproduced in Fig. 16. The latter was made by Professor Frost, and the difference of shift between the dark lines of the star and the bright reference lines in the two photographs is plainly evident. In this case we have a star revolving about a dark body, its velocity in the line of sight varying with its position in the orbit.

The motions of the molecules of a luminous gas modify slightly the wave-length of the emitted light. Since the molecules are moving in all possible directions, with all sorts of velocities, the result is that the spectrum lines appear slightly broadened, the broadening increasing with the temperature. The subject has been fully treated by Lord Rayleigh (*Phil. Mag.* (5), 27, page 298, 1889).

The change in the period T, of the radiation coming from a source of light moving with a velocity v, is given by the equation

$$T' = T\left(1 \pm \frac{v}{c}\right),$$

in which T is the actual period of the vibration, T' the period of the radiation, and c the velocity of light. It is to be carefully observed that when the source is in motion, the frequency of the vibration in the source differs from the frequency with which the waves pass by the observer, the former being unaffected by the motion.

The above equation for the change in the frequency of the vibration applies to the case of a stationary source of light and a moving observer, as well as to a moving source. The two conditions are, however, represented by the same formula only when the velocity of translation is small in comparison with the velocity with which the waves travel. That this is true is evident from the following very elementary consideration. Let A be a source which emits ten waves per second, the waves travelling with a velocity of one metre per second. Let an observer B advance against this wave-train with a velocity of $\frac{1}{2}$ m. per second. It is evident that the waves will pass him at the rate of 15 per second. Though the actual wave-length remains unaltered, the frequency of the vibration so far as B is concerned has been increased from 10 to 15. The deviation of waves by a prism depends upon their frequency; consequently in the case of light waves we obtain the spectrum line in a shifted position when the spectroscope is moving with a high velocity towards the source. Suppose now that B remain at rest, and the source A to move towards him with a velocity of $\frac{1}{2}$ m. per second. During the time occupied by the source in moving $\frac{1}{2}$ m., it emits ten waves. These waves will be crowded together into a space of half a metre, that is, between the point occupied by the first wave of the train of ten waves, at the end of one second, and the point occupied by the source at the same time. In other words the wave-length has been halved. This wave-train will sweep by the observer with a velocity of 1 m. per second, or with a frequency 20. The Doppler effect is therefore greater for the case of a moving source than for a moving observer, when we are dealing with velocities comparable with the velocity with which the waves travel.

The number of waves of frequency N, coming from a fixed source, which in one second pass an observer moving towards the source with a velocity v, is $N + \frac{v}{\lambda}$ or $\frac{V + v}{\lambda}$, in which V = velocity of light.

If, however, the observer is fixed, and the source moves with a velocity v, the wave-length is changed from $\lambda = \frac{V}{N}$ to $\lambda = \frac{V - v}{N}$

and the number of waves which pass the observer per second is the velocity V divided by this number, or $\frac{V}{V-v}N$. If a source moves with a velocity equal to or faster than that of the radiation, no periodic waves are given out. There is, however, a single wave like the bow wave of a ship. The sharp click of a high velocity bullet as it flies past us is an illustration of this. Calculate the change of wave-length produced by reflection from a mirror moving towards the source with a velocity equal to one-half that of the radiation. The image in the mirror appears to move with a velocity equal to that of the radiation, but it will be seen that the effect upon the wave-train is not the same in the two cases.

CHAPTER II

RECTILINEAR PROPAGATION OF LIGHT

Huygens's Principle. — One of the objections which was first urged against the wave theory of light was its failure to account for the rectilinear propagation of luminous disturbances, and the formation of shadows. Waves of sound and water waves were observed to bend around the corners of obstacles, and it was perhaps naturally argued that if light consisted of a wave-motion, it should behave in a similar manner. The objection was partially answered by Huygens, though it remained for Fresnel to give the complete explanation.

Huygens's conception of the manner in which wave-motion was propagated was as follows: He regarded every vibrating point on the wave-front as the centre of a new disturbance: these secondary disturbances, travelling with equal velocity, are enveloped by a surface identical in its properties with the surface from which the secondary disturbances start, and this surface forms the new wave-front.

Fig. 17.

For example, in Fig. 17, consider O a luminous point, and AB a portion of the spherical wave-front. Adjoining points a, b, c, d, etc., on this wave-front are vibrating in unison and can be regarded as centres of new disturbances, which spread out around them as indicated by the dotted lines. It is evident that these secondary waves are enveloped by the spherical surface $A'B'$, and this surface is the new wave-front. If the luminous point is at a great distance, and we are dealing with a plane-wave, we have the condition shown in the lower figure.

This view of wave propagation is known as the Huygens principle. It can be applied to the calculation of the position of a reflected or refracted wave-front, by regarding the points on the reflecting or refracting surface, as they are collectively or successively struck by the incident wave, as individual centres of

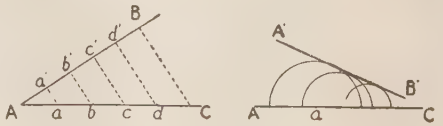


Fig. 18.

new disturbances. For example, consider a wave-front AB descending in an oblique direction on a reflecting surface AC. The points a, b, c, d of the surface will be struck in succession by the points a', b', c', d' of the wave-front, consequently they will become successively the centres of secondary disturbances, as indicated in Fig. 18, which are enveloped by the plane surface $A'B'$. This is the reflected wave-front, and we shall see later on that it makes the same angle with the reflecting surface as the incident wave.

Rectilinear Propagation. — Assuming Huygens's conception of the mechanism of wave-propagation to be correct, how are we to account for the rectilinear propagation of light? Suppose we have a luminous body at O (Fig. 19) and an opaque screen, a coin, for example, at A. We know that no light penetrates into the conical region behind the coin (neglecting for the present a phenomenon known as diffraction).

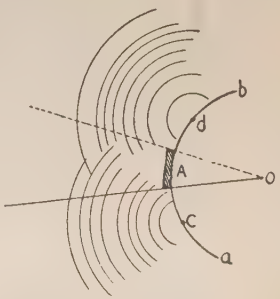


Fig. 19.

But if all points on the wave-front are acting as independent sources, we should naturally expect them to be effective in illuminating the region behind the obstacle; in other words, we should expect the luminous waves given off by the points c and d to have some effect in the space behind the coin. Why does not the entire wave appear luminous to an eye behind the obstacle if every point of it is giving off radiant energy in the form of secondary wavelets?

The answer given by Huygens was that these secondary waves produced no appreciable effect at a point unless they were *at that point* enveloped by a common tangent plane, or that the only effective portion of a secondary wavelet was the small point at its apex which touched the plane tangent to all of them. Huygens regarded the impulses as coming at irregular intervals, and his explanation of the rectilinear propagation of light amounted simply to the assumption that only one point on the secondary wavelet was effective in producing light.

Fresnel was the first to give a really satisfactory explanation. Making use of the principle of interference discovered by Young, by which two luminous vibrations may destroy one another, he arrived at the somewhat startling conclusion that the absence of light in the shadow of a body was due to destructive interference between the secondary wavelets. This explanation not only accounted for the darkness behind the obstacle, but explained perfectly the slight bending of the rays around the edges, a phenomenon known as diffraction, which had been previously explained by assuming the edge to exert a modifying action on the luminous rays which passed close to it.

It was no longer necessary to assume that only a minute portion of the secondary wave was operative in producing light, which as a matter of fact is contrary to experimental evidence, as can be shown by allowing a plane-wave to fall on an opaque screen perforated with

a very small aperture. The point on the wave-front not cut off by the screen acts as a centre of a disturbance, which spreads out into the space behind the screen as shown in Fig. 20, and a card placed in the position shown will be illuminated over an area many times greater than the aperture.

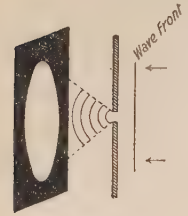


Fig. 20.

Waves of sound behave in a similar manner, and it is actually possible to photograph the secondary wavelet. The accompanying photograph is one of a series made by the author to illustrate certain features of wave-motion. The method by which the pictures were made will be discussed later on.[1]

For the present it will be merely necessary to state that in every case the sound photographed is the crack of an electric spark, which gives, of course, a single pulse, instead of a train of waves. The series shown in Fig. 21 was made to illustrate the principle of Huygens.

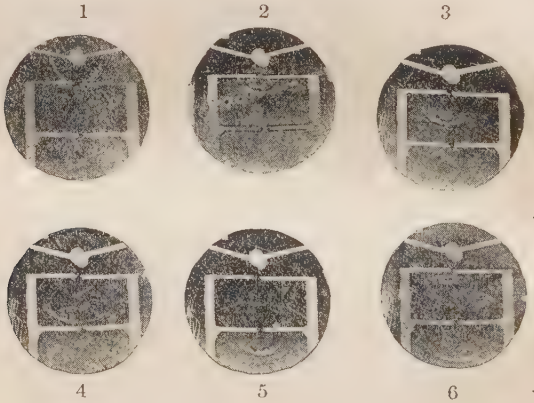


Fig. 21.

The spark which started the wave was arranged to snap directly above a screen provided with a narrow slit. A short distance below this first slit a second was mounted, and it will be seen that the two narrow apertures become in succession the centres of secondary waves which diverge precisely as if the source of the sound, that is to say the spark, were situated in the aperture itself. In No. 1 of the series, the wave, which started at the point A, has just encountered the first screen. The aperture, which appears in the photograph as a break in the horizontal white line, becomes the centre of a new hemispherical wave, the gradual development of which is shown in Nos. 2, 3, and 4. In No. 5 the secondary wave has collided with the second screen and been reflected, the aperture in this screen becoming in its turn the origin of a new secondary wavelet. These pictures show that if all but a small part of the original wave is screened off, this

[1] Wood, "Photography of Sound Waves," *Philosophical Magazine*, August, 1899.

small part becomes a complete wave, and again if a small portion of this secondary wave is allowed to pass through a small aperture, it becomes in turn a complete wave.

Before considering in detail Fresnel's explanation we must make an assumption regarding the nature of the secondary wavelet, which is based on the circumstance that no disturbance is radiated *backward*. An opaque screen which absorbs all of the energy falling on it has no effect whatever on the vibration of the medium between it and the luminous source.

From this we infer that the secondary wavelet is propagated only forward, and lies wholly in front of the plane tangent to the wave front at the centre of the wavelet. We are also justified both by theory and experimental evidence in assuming that the effect of the secondary wavelet is greatest on the line which is normal to the tangent plane at the point of tangency. This will be better understood by reference to Fig. 22, where AB is the wave-front, a the centre of any secondary wavelet, and xy the tangent plane behind which we assume that the secondary wavelet never spreads. The effect of the wavelet is greatest along the line, or in the direction ab, less along ac, and falls off continuously, having the value 0 in the direction ax. This may be summed up by saying that the effect of the secondary wavelet decreases with increasing obliquity. The reason for the absence of a back-wave has been given.

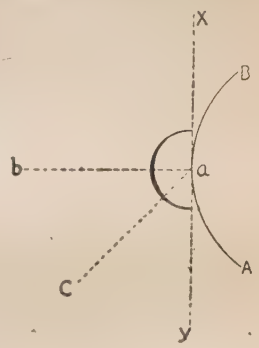


Fig. 22.

We will commence the investigation of Fresnel's treatment of the subject by examining the effect of a linear wave on a point P in front of it.

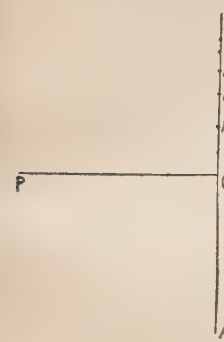


Fig. 23.

Let AB be the wave-front (Fig. 23) which we may consider moving as a whole up and down parallel to itself. Thus all the particles on AB move together, and the secondary waves leave them at the same moment.

Draw a perpendicular from P to the wave-front, meeting it at C, which point is called the pole of the wave with reference to the point P.

Lay off on AB points M_1, M_2, M_3, M_4, etc., so that the path from M_1 to P is half a wave-length longer than the path from C to P, and M_2 half a wave-length further from P than M_1, and so on. If secondary wavelets start simultaneously from these points and move with the same velocity, the disturbance from C will reach P first, since CP is the shortest path, and the wavelet from M_1 will reach P half a wave-length behind the one coming from C, since we have so located M_1 on the wave-front that the path PM_1 is half a wave-length longer than PC. This means that the crest of the wavelet from M_1 reaches P at

the same moment that the trough of the wavelet from C is passing through it, or the waves are in opposite phase and would destroy each other if both were equal. In the same way other wavelets coming from points lying between C and M_1 will reach P with phases opposite to those coming from corresponding points between M_1 and M_2. The same will be true for wavelets coming from points between M_2M_3 and M_3M_4.

To determine the effect of the whole wave at P we determine the total effect or resultant of all the secondary wavelets, paying attention to their phases as well as their amplitudes. The effect at P of each of the elementary arcs into which we have subdivided AB we consider as proportional to its length, and inversely proportional to its distance from P. As we recede from C the effect will also diminish on account of the increasing obliquity.

We will now determine the relative lengths of the arcs into which we have subdivided AB, Fig. 24.

Let the distance from the pole of the wave to P be b, then the distance of M_1 from P is $b+\frac{\lambda}{2}$ and the length of the arc CM_1 is

$$\sqrt{\left(b+\frac{\lambda}{2}\right)^2-b^2} \text{ or } \sqrt{b^2+2\frac{b\lambda}{2}+\frac{\lambda^2}{4}-b^2} \text{ or } \sqrt{b\lambda},$$

if we neglect $\frac{\lambda^2}{4}$, which is very small in comparison to $b\lambda$.

The path $PM_2=b+2\frac{\lambda}{2}$ or $b+\lambda$; therefore

$$CM_2=\sqrt{(b+\lambda)^2-b^2}$$
$$=\sqrt{b^2+2b\lambda+\lambda^2-b^2}=\sqrt{2b\lambda},$$

neglecting λ^2, which is small.

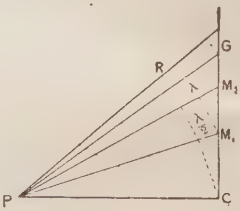


Fig. 24.

Therefore

$CM_1=\sqrt{b\lambda}$, and $CM_1=\sqrt{b\lambda}$,

$CM_2=\sqrt{2b\lambda}$, $M_1M_2=\sqrt{2b\lambda}-\sqrt{b\lambda}=\sqrt{b\lambda}(\sqrt{2}-\sqrt{1})$,

$CM_3=\sqrt{3b\lambda}$, $M_2M_3=\sqrt{3b\lambda}-\sqrt{2b\lambda}=\sqrt{b\lambda}(\sqrt{3}-\sqrt{2})$.

The arcs thus decrease rapidly in length in the neighborhood of the pole. The length of any arc at distance R (G in the diagram) is determined as follows, since the small right triangle at G is similar to the right triangle PCG.

$$G:\frac{\lambda}{2}=R:\sqrt{R-b^2} \text{ or } G=\frac{\lambda}{2}\frac{R}{\sqrt{R^2-b^2}},$$

a quantity which decreases with increasing R, approaching the value $\frac{\lambda}{2}$ as a limit. The arcs far removed from the pole decrease slowly in length, approaching the limiting value $\frac{\lambda}{2}$.

Besides decreasing in length, the elements become less and less effective owing to increasing distance and obliquity.

Remembering that adjoining elementary arcs send disturbances of opposite sign to P, we see that the effect of all is represented by a series of alternately positive and negative members, which at first decrease rapidly, then more slowly approaching 0 in value, since the very remote arcs are inoperative owing to their obliquity. Calling the effect due to the central arc l, and that due to the following ones m, m', m'', etc., the whole effect is represented by the series

$$l - m' + m'' - m''' + m^{iv},$$

the value of which is a fraction of the value of the first member.

Therefore the effect of the entire wave at P is less than that due to the first element acting alone. If we were to screen off all of the wave except the first element, the illumination at P would be greater than that due to the whole wave, a surprising conclusion which, as we shall soon see, can be verified by experiment.

The wave-length of light is so small that with P at a distance of 10 cms. from the wave-front M_1 would be scarcely more than .2 mms. from the pole of the wave. At a short distance from the pole the arcs would become very nearly equal and opposite in their effect, consequently the effective portion of the wave reduces itself to a comparatively small area around the pole; and if we screen off this region we shall have darkness at P owing to the destructive interference between the disturbances coming from the outlying elementary arcs, or a shadow will exist behind the screen.

Effect of a Plane-Wave on an Exterior Point. — Thus far we have been considering wave-motion in two dimensions only, a hypothetical case. Let us now find an analogous treatment for waves moving in space, which is the condition under which we observe them in our experiments.

Consider a plane-wave (Fig. 25) moving towards P, an exterior point: we require the effect at this point of all the secondary wavelets emanating from the wave-front. Draw a perpendicular from P to the wave-front, intersecting it at C, the pole of the wave with respect to P. Around C describe circles on the wave-front such that the first is half a wave-length further from P than C is, the second 2 half wave-lengths, etc. The rings thus formed on the wave-front will be analogous to the elementary arcs into which we divided the linear wave, that is to say, the secondary disturbances coming from any circle will reach P half a wave-length ahead of those coming from the circle encircling it.

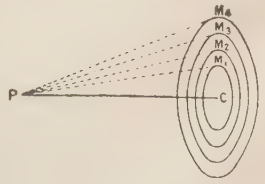


FIG. 25.

We regard the effect of the disturbances coming from each ring as proportional to its area and as decreasing with increasing distance and obliquity as before. Let us now investigate the areas of the rings.

The radii of the circles are obviously equal to the distances CM_1, CM_2 on the linear wave, namely $\sqrt{b\lambda}$, $\sqrt{2b\lambda}$, $\sqrt{3b\lambda}$, etc., and the

areas or πr^2

$$\pi b\lambda,\ 2\pi b\lambda,\ 3\pi b\lambda.$$

Neglecting the square of λ as we have done, we find the area of the central circle and each surrounding zone to be equal or $\pi b\lambda$.

For a zone at distance R from P we have its width given by $\frac{\lambda}{2}\frac{R}{\sqrt{R^2-b^2}}$, as in the case of the element of the linear wave.

Its circumference is $2\pi\sqrt{R^2-b^2}$, and its area, or the product of these two quantities, is $\pi\lambda R$.

The effect due to the disturbances coming from a single one of the zones will be proportional to its area and inversely proportional to its distance. The slight increase in the area of the zones as we recede from the centre of the system is compensated by the increased distance, so that, other things being equal, we could regard the successive zones as producing equal and opposite effects at the point. The zones, however, become less and less effective as we recede from the centre owing to the increased obliquity. We can therefore represent the resultant effect by a series of terms of alternate sign which decrease slowly at first, and then more rapidly, eventually becoming zero, thus:

$$S = m_1 - m_2 + m_3 - m_4, \text{ etc.}$$

The sum of this series is usually stated as being equal to one-half of the first term plus one-half of the last term; the method usually adopted to prove this consists in balancing the second term against half of the first and half of the third, and so on. Schuster has shown that this treatment is too arbitrary, no reason being given why the balancing is not effected in some other way, for example, by considering the second term balanced by three quarters of the first and one quarter of the third, which would make the resultant outstanding effect approximately equal to one quarter of that due to the first member acting alone. Schuster shows in what cases the addition of the series can be effected in the manner indicated. He first writes the series in the two following forms:

$$S = \frac{m_1}{2} + \left(\frac{m_1}{2} - m_2 + \frac{m_3}{2}\right) + \left(\frac{m_3}{2} - m_4 + \frac{m_5}{2}\right) + \left(\frac{m_{n-2}}{2} + m_{n-1} + \frac{m_n}{2}\right) + \frac{m_n}{2},$$

$$S = m_1 - \frac{m_2}{2} - \left[\left(\frac{m_2}{2} - m_3 + \frac{m_4}{2}\right) + \left(\frac{m_4}{2} - m_5 + \frac{m_6}{2}\right) + \left(\frac{m_{n-3}}{2} - m_{n-2} + \frac{m_{n-1}}{2}\right)\right] - \frac{m_{n-1}}{2} + m_n.$$

Suppose first that each term of the original series is greater than the arithmetical mean of the two adjacent terms. From the above equations we see that

$$m^1 - \frac{m_2}{2} + m_n - \frac{m_{n-1}}{2} < S < \frac{m_1}{2} + \frac{m_n}{2},$$

for in both equations the terms bracketed are all small negative quantities, and the value of S lies somewhere between the two quantities given above.

If m_1 is very nearly equal to m_2 and m_n nearly equal to m_{n-1}, the two limits lie close together, and we may write

$$S = \frac{m_1}{2} + \frac{m_n}{2}.$$

If the series is such that each term is *less* than the mean of its neighbors, S lies between the same limits (transposed).

If in the first p terms of the series each term has a greater value, and in the remaining part a smaller value than the arithmetical mean of the terms between which it stands, we may break up the series into two, and obtain the sum

$$S = \frac{m_1}{2} \pm \frac{m_p}{2} \mp \frac{m_{p+1}}{2} + \frac{m_n}{2}.$$

It is thus clear that the expression for S given above will be the correct summation only, if the series can be broken up into a small number of separate series for each of which the value of a term is either smaller or greater than the arithmetical mean of the terms between which it stands, so that the sum of all such values may be neglected.

The problem therefore reduces to a determination of the effect due to one-half of the central zone.

The secondary wavelets from this zone unite into a disturbance the phase of which is midway between those of the wavelets from the centre and rim, for we may divide the zone into a series of concentric rings of equal area, the effects of which at the point are equal in amplitude, and of phases ranging over half a complete period. These vibrations may be compounded as vectors by the method given on page 158. The resultant amplitude will be very nearly the diameter of a circle, the semi-circumference of which is made up of the vectors which represent the amplitudes contributed by the elementary zones into which we have divided the central circle. The direction of the diameter makes an angle of 90° with that of the first vector, consequently the phase of the resultant is a quarter of a period behind that due to the element at the centre. We must consequently consider that the secondary waves start with a phase one quarter of a period ahead of that of the primary wave. The amplitude of the resultant bears the same ratio to the amplitude which would be produced if all the disturbances arrived in the same phase, that the diameter bears to the semi-circumference, *i.e.* $2/\pi$. The matter of the acceleration of phase of the secondary wavelet of a quarter of a period, with respect to the phase of the primary wave, has sometimes been regarded as a sort of mathematical fiction. If the advance of phase really exists, a secondary wavelet, if isolated, would reach a distant point with a phase a little in advance of that of the primary wave which originated it. That this is actually so was shown by Gouy.[1]

[1] Gouy, "Sur la propagation anomale des ondes," *Comptes Rendus*, 1890.

The theory of this experiment will be taken up more in detail when we come to the subject of interference, and for the present it will serve our purpose to take note of the fact that the advance of a quarter of a wave-length has been demonstrated. Gouy formed a system of interference fringes with light reflected from a pair of Fresnel mirrors, and then introduced into the path of one of the interfering beams a screen perforated with a minute pin-hole, which became at once the source of secondary spherical wavelets. These interfered with the uninterrupted waves, and formed a system of fringes with a color distribution which showed that the disturbances from the pin-hole were advanced a quarter of a wave-length ahead of the primary wave which was intercepted by the screen.

Assume the amplitude on the wave front to be unity, and consider that the secondary wave from a small element of its surface produces a resultant effect represented by kds. If r is the radius of the zone, its resultant effect will be $\frac{2}{\pi} \cdot k\pi r^2$. Now $r^2 = b\lambda$, and the amplitude due to the whole zone is therefore $2kb\lambda$. The whole wave will produce an amplitude one-half as great, which we may equate to unity, since we have assumed unit amplitude on the wave-front, and a plane wave is propagated without loss of amplitude. From this we find that k, the factor which represents the effect of the secondary wave, is equal to $1/b\lambda$. That the amplitude due to the secondary wave should vary inversely as the distance b is to be expected, but it may not be at once obvious why it should vary inversely with the wave-length. There is no mystery about the matter, however. If we keep the distance b fixed and increase the wave-length, we are obliged to increase the size of the zone, if the conditions are to remain as before, that is, if the zone is to produce the same effect at the point. The secondary disturbances are now coming from a larger area, while only producing the same resultant effect, consequently the effect produced by any small element of surface ds will be proportionally smaller.

Let us now put the theory to experimental test.

Suppose we screen off all of the wave-front except the central circle of the zone series. There is now no encircling zone to partly neutralize it, and the illumination is greater than that due to the entire wave. This can be accomplished by placing a screen provided with a small circular aperture at such a distance from the point P that the area of the aperture is equal to the area of the central zone, when the amplitude at once becomes double, and the illumination four times that due to the unobstructed wave. It is of course apparent that the actual size of the zones on the wave-front in a given plane depends on the distance of the point P. As this distance increases the zones widen out. On a wave-front distant about 5 feet from the point, the zones would be of the size shown in Fig. 26, so that if our small circular aperture was of the size of the central circle in the figure, the illumination at a point on the normal 5 feet behind the

Fig. 26.

aperture would be greater than if the screen were not present. And now comes a very curious fact: suppose we increase the size of our aperture until it contains another zone. The disturbances coming from this ring will be out of phase with those coming from the central circle, and will entirely destroy them. Thus by increasing the size of the hole we can reduce the illumination to zero. The experiment can be performed with very simple apparatus, provided one has a dark room of sufficient length. A pin-hole in a piece of thin sheet metal illuminated with arc or sun light makes a suitable source of light. A first-class iris diaphragm, such as is provided with the best photographic objectives, furnishes us with an aperture the size of which can be varied at will. The diaphragm should contract to a diameter of 3 mms. or less, and the outline of the opening should be circular and not polygonal, as is the case with the cheaper kinds provided with but few wings. Suppose the smallest aperture to have a radius of 1.5 mms.: we require the distance of the point so situated that only the central circle of the zone system is exposed. The formula $CM_1 = \sqrt{b\lambda}$ shows us that, if we put $\lambda = .0005$ mm., the distance of the point is 4.5 metres. This is for plane-waves, or with our source at a great distance. For the condition of source and point at equal distances from the aperture we substitute $\frac{\lambda}{4}$ for $\frac{\lambda}{2}$ in the original formula, for now there will be a path difference on both sides of the screen; in other words, the vibration at the edge of the aperture will be slightly behind that at the centre. The distance now increases to 9 metres. Clearly we shall need a long room for our experiment, for the source must be 9 metres behind the screen, or our total optical path must be 18 metres. We can, however, reduce this by one-half by using a small reflector of silvered glass, an excellent arrangement being to so arrange things that the diaphragm and the illuminated point are close together. To accomplish this we place it at a distance of 9 metres from the source and mount our mirror 4.5 metres behind it, reflecting the light back to a point a little to one side. If we hold a sheet of paper here we shall see a little point of light. Put a little drop of white paint on a bit of glass, and mount it in such a position that it lies in the centre of the small spot of light. This forms our illuminated point. Now, increase very slowly the size of the diaphragm and the light gradually fades away, the drop of paint presently becoming invisible. Twice as much light comes through the hole as before, yet the point is in darkness. The law of the conservation of energy tells us, of course, that no light has been destroyed. It has simply gone somewhere else, and where it has gone does not concern us at present. The fact that it no longer manifests itself at the point in question is sufficient.

Let us now try the converse of this experiment by substituting for the aperture a small circular disk of the same diameter. According to Huygens's theory, if placed over the central zone, it should cut off the illumination at the point entirely. On the Fresnel theory we simply remove the first member of the series, and the effect is represented by the remainder of the series, namely, half the second member, or the

illumination is unaffected by the interposition of the circular disk, and this is precisely what we find to be the case. By increasing the size of the disk we cut off another zone, still without influencing the illumination, and this may be continued, not indefinitely, but until, owing to the increasing obliquity, the effect of the zones begins to diminish appreciably. We thus see that the centre of the shadow of a circular body may, under certain conditions, be as brightly illuminated as the surrounding field, a proposition due to Poisson.

Fresnel's memoir on diffraction was presented to the French Academy and reported on by Poisson, who raised the objection that if the treatment were applied to the case of a circular disk (a case which had not been treated by Fresnel), it would lead to the conclusion that the illumination along the axis of the disk would be the same as if the disk were absent, which was supposed to be a *reductio ad absurdum*. In this case it is clear that the illumination will be represented by the above-mentioned series, with as many members removed as there are zones covered by the disk, which will be, as before, one-half of the first exposed zone, and if we assume the zones to produce equal effects, the illumination should be the same as without the disk. As a matter of fact, the experiment had already been recorded by Deslisle, but it had been forgotten, and was rediscovered by Arago and Fresnel, who observed the bright spot in the centre of the shadow of a circular disc.

This experiment is easily performed with a small disk of metal, a copper cent, for example. If a coin is used, a new one should be selected, the edge of which is smooth and undented. It should be supported by means of three fine threads, which can be attached to the coin with wax. Using the same source of light as before, with the coin mounted at a distance of three or four metres, we shall find, if we explore the region behind the coin with a low-power eye-piece, that there is a brilliantly illuminated region along the axis of the geometrical shadow. The illumination is faint in the immediate vicinity of the coin, owing to the irregularities of the rim, as will be explained later, but at a distance of several metres behind the coin it is nearly as bright as if the coin were absent. If the eye is brought into coincidence with the luminous spot, it will be found that the light comes from the edge of the coin, which appears brilliantly illuminated. If the eye is moved a little to one side, the ring breaks up into two spots of light situated on opposite sides of the coin. We are now getting the light which bends into the shadow radially, without the great reënforcement due to agreement of phase.

Zone-Plate. — A remarkable verification of Fresnel's theory is furnished by what is known as a zone-plate. If we describe on a large sheet of white paper circles, the radii of which are proportional to the square roots of the natural numbers, we shall have very nearly an exact drawing of the zone system, the neglected terms containing the square of λ introducing a very slight error. If now we blacken the alternate rings with ink, and take a greatly reduced photograph of the whole on glass, we shall obtain a device which will enable us to screen off the alternate zones on the wave-front. Suppose we intercept a plane-wave with such a plate and consider the illumination at

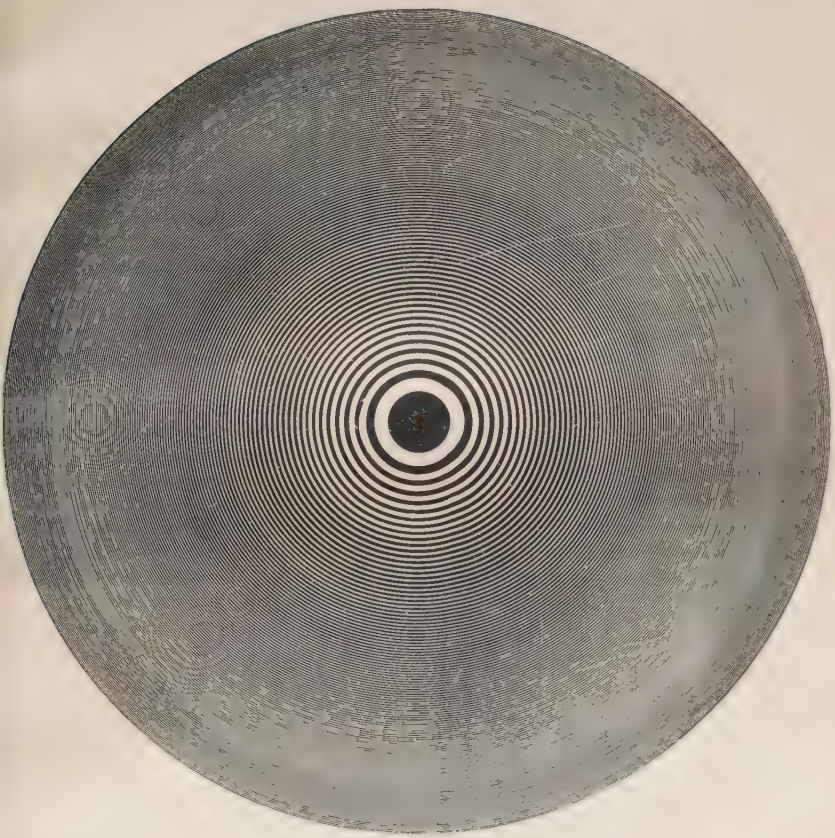


PLATE 1.
Zone-plate, from a drawing.

a point so situated behind the plate that the central circle of the plate corresponds in size and position to the first zone on the wave-front. The black rings stop all the secondary disturbances from the alternate or odd zones, which previously neutralized those coming from the even ones, consequently all the secondary disturbances coming from that portion of the wave-front covered by the plate reach the point in the same phase, and the illumination will be very intense. The whole surface of the zone-plate will send light to the point, the action being very similar to that of a convex lens. The distance of the illuminated point from the zone-plate we may speak of as its focus, and we readily see that the smaller the zones the shorter the focal length.

In the earlier edition of this book I gave a greatly reduced copy of such a drawing. A better method has recently been found and a second plate was prepared in the following way: A photograph was made of the circular system of interference rings obtained by passing the green light of the mercury arc through a combination formed by a large lens of long focus in contact with a flat glass plate. The two surfaces in contact were half silvered, which gives us a system of rings by transmitted as well as by reflected light. Photographs of these rings discovered by Sir Isaac Newton will be found in the Chapter on Interference, where it will be shown that the rings are on the same scale as the Huygens zones. A small scale photograph of Newton's rings will serve therefore as a zone-plate. As the number of zones obtained in this way was more than twice as great as the number in the original drawing, it appeared to be worth while to make a new plate. The photograph of the interference rings was mounted on the tool carriage of a lathe, and observed under a microscope provided with a cross-hair. Circles were now turned on a brass plate, the advance of the tool being controlled by watching the transit of the rings across the field of the microscope. Two hundred and fifty rings were cut in this way, and Plate 2 was printed from the resulting engraving. It is possible in this way to obtain lines much finer and sharper than those yielded by any photo-engraving process. (The omission of one ring was my fault and not Sir Isaac's!) Very good zone-plates can be made by making photographic reductions on glass of either plate, the perfection of which will depend upon the accuracy of the focus and the excellence of the lens used in making them. Lantern slide plates are fairly satisfactory, but the best results are obtained with a collodion emulsion on thin plate glass. (See Lippmann photographs for directions for preparing such emulsion.) The variation in the relative widths of the bright and dark rings will be found instructive when we come to the consideration of the zone-plate in the Chapter on Diffraction, in connection with the distribution of light in spectra of different orders. It is well to make several plates of different focal lengths. That they have properties similar to lenses may be well shown by holding one of suitable focus, say half a metre, between the eye and a distant lamp. If the central zone is brought over the flame, the whole plate fills up with light like a lens. By combining a zone-plate with a low-power

eye-piece we can form a telescope which will give a fairly sharp image of a brilliant object, such as an incandescent lamp.

Lord Rayleigh, in his article on Wave-Theory in the *Encyclopedia Britannica*, called attention to the fact that if it were possible to provide that the light stopped by the alternate zones could be allowed to pass, but with a reversal of phase, a four fold intensity in the illumination at the focus would result. In this case the secondary disturbances from all the zones, both odd and even, would reach the point in the same phase. This can be accomplished in two ways,[1] first by making the zones of a thin film of gelatine on glass, the thickness of the film being such as to retard the waves one-half wave-length. These were made by coating a glass plate with a thin film of gelatine containing a little bichromate of potash. Contact prints are made from the reduced photographs in sunlight, and washed for a few seconds in warm water. Still better plates have been recently made by the author by etching glass with hydrofluoric acid. The glass plate was coated with a thin film of wax, and mounted on a small turn-table revolved by a motor. The ruling needle was advanced by a dividing engine, and circles cut through the wax, the proper radii being secured by observing a photograph of Newton's rings through a microscope provided with a cross-hair, which was carried along with the ruling point. After etching the plate with the acid the wax is removed. Casts of these plates can easily be made in celluloid. Another method is to form the zones of metallic silver on the hypothenuse surface of a right-angle prism. In this case the light from the odd zones is reflected metallically from the silver, while that coming from the even zones has been reflected from the air surface (total internal reflection). Reflection under these two conditions introduces a phase change of almost exactly a half wave-length, and the light at the focus is quite as brilliant as with the gelatine plates.

The zone-plate has many peculiar properties. It has multiple foci and can act at the same time both as a convex lens and a concave lens; but these properties can be discussed to better advantage after we have studied diffraction.

Huygens's Principle as applied to Solitary Waves. — Objections have been raised from time to time as to the general applicability of Huygens's principle. For example, it is well known that a wave-disturbance, consisting of simply a "crest," unaccompanied by a "trough," is capable of propagation under certain conditions. Since rectilinear propagation depends upon interference, and as we can have no destructive interference between disturbances in which the displacements are all in one direction, it would seem as if the principle could not be applied in this case. Gouy has shown, in the paper previously alluded to, that we can have only plane-waves of this type: a spherical wave, even if started by a disturbance consisting of half an undulation, will develop the other part as soon as it breaks away, so to speak, from the source. This is proven for waves of sound: we cannot have a spherical sound wave which travels out as a condensation, even if we start it by the sudden expansion of

[1] Wood, "Phase Reversal Zone-Plates," *Phil. Mag.*, June, 1898.

PLATE 2.

Zone-plate engraved on lathe.

a small sphere unaccompanied by a contraction of the sphere. The wave will go out as a condensation followed by a rarefaction. Since our secondary wavelets are spherical in form, the same holds true for them. This peculiarity of spherical waves was shown by Stokes some time before the appearance of Gouy's paper, and is referred to by the author.

There appears also to be a feeling that Huygens's principle cannot be applied to a solitary wave. There certainly can be no interference between secondary wavelets which do not pass simultaneously through the given point, and the points on the wave front, from which the wavelets start, lie at different distances from the point in front of the wave at which we are determining the illumination.

We can form an idea of how the interference takes place in the case of a solitary wave in the following way.

Consider three points a, b, c, on the wave front each half a wave-length farther away from the exterior point than its neighbor. The crest of the wavelet from the middle point b will reach the exterior point at the same moment as the trough from a, and will be destroyed by it; the trough from b will, however, be destroyed by the crest from c; in other words the disturbance from b is destroyed in part by a disturbance from a, and in part by one from c. Another way of looking at the matter is to make use of the method of resolution employed by Stokes in his paper upon the nature of X-rays. In Fig. 143, we have a plane-wave with its crest represented by a dotted line, and its trough by a solid line, and we have to determine the portion of the wave effective in illuminating the point X. Describe around X spheres (circles in the diagram) of radii a, a', a'', also larger spheres of radii b, b', b''. It is clear that disturbances originating on the surface of any one of these spheres will pass simultaneously through X, since X is equidistant from every point on the surface of a given sphere. Disturbances on the solid and dotted line are of opposite sign and will destroy each other if they pass simultaneously through X. In the case of the spheres which cut the pole of the wave, the positive and negative disturbances pass through X in succession, and there is no destructive interference. In the case of disturbances from more remote parts of the wave, we have a positive disturbance from C and a negative one from B reaching X simultaneously and destroying each other, and the same for points D and E. We shall make use of this method again when we come to the subject of diffraction, in explaining the relation between the magnitude of diffraction and the wave-length. It is clear that the shorter the wave, *i.e.* the smaller the distance between the two lines, the more nearly is the effect at X due to a small region around the pole of the wave. The pole is intercepted by a screen in the figure.

Law of Regular Reflection deduced from the Principle of Interference of Elementary Waves. — The construction given by Huygens for the reflected and refracted waves is incomplete, just as was his construction for rectilinear propagation, for he was obliged to assume that only a single point on the secondary wavelet was operative in producing illumination. In applying the Fresnel principle of

interference to reflection from a plane surface, we regard each point on the surface of the mirror, as it is struck by the incident wave, as the centre of a secondary wavelet, and determine the collective effect of these wavelets at any point, just as we did when we regarded points on the wave-front as centres of disturbance. We know that the light, radiating from a point and reflected from a plane mirror, which is effective in illuminating a given point, comes from a point on the mirror so situated that lines joining it with the source of the light and the point illuminated, make equal angles with the normal. Let us see if we can construct a system of zones on the surface of the mirror in such a way that the effects coming from all will be essentially reduced to that due to a small area surrounding a point situated as described. The problem is somewhat more complicated than the one which we have just solved, for the centres of the secondary wavelets on the surface of the mirror are not vibrating in unison as are those on the wave-front. The reason of this is obvious, for different points on the mirror are struck by the wave-front at different times, and the secondary disturbances therefore do not start simultaneously, and will not be in agreement of phase except in the case of a plane-wave incident normally.

The calculation of the zone-system on a reflecting surface is rather tedious, and does not teach us very much. It will be found in the earlier edition of this book, where it is shown that the zones are ellipses, their eccentricity varying with the angle of incidence. At normal incidence the ellipses become circles.

The areas of the ellipses can be shown to be very nearly equal, forming a decreasing series similar to the circular zones on the plane-wave front. If we describe such a series of ellipses on a flat mirror and make the alternate zones opaque, the flat surface will concentrate parallel rays incident at the proper angle, much like a concave mirror. Such a plate can be made by photographing our drawing of the circular zone system at an angle of 45°. The negative when placed on a piece of silvered glass gives very sharp focal images for light incident at the same angle. If the elliptical zone system is printed on the hypotenuse surface of a right-angle prism, in metallic silver, our phase difference of half a period between the adjacent zone results from the reflection occurring under different conditions. Such prints can be made by a method given in the paper referred to in the previous section.

Reflection and Refraction by Unpolished Surfaces. — One of the most interesting and instructive applications of the Fresnel construction is to the diffuse reflection and refraction which occur when light strikes unpolished or matt surfaces, such as paper, plaster of Paris, or ground glass. We have explained regular reflection by showing that there is destructive interference between the wavelets arriving at any point from the surface of the mirror, and that the illumination is practically due to disturbances coming from a small region surrounding the point so situated that straight lines joining it to the source of light and the illuminated point make equal angles with the normal. An unpolished surface destroys all phase relation between the elements on the wave-front. The secondary wavelets

start from the elevated portions of the surface first, since these portions are struck first by the incident wave, and the reflected wave-front, instead of being plane, is pitted and corrugated in an irregular manner. It is impossible to arrange any zone system on such a surface, for there are all possible phase differences irregularly distributed over the reflected wave-front, consequently each point on the surface acts as an independent luminous source, sending light out in all directions. We can apply the Fresnel theory to reflection of this sort in the following way.

Suppose we have a plane surface $X\,Y$ (Fig. 27) and a luminous point S, and are considering the effect at P, which we will suppose to be the point to which a ray SA would be reflected. We have,

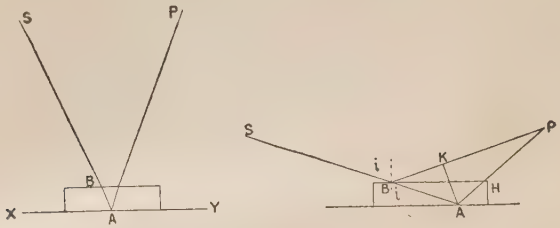


FIG. 27.

however, at A an elevation of height H, and the secondary wavelet will leave the point B sooner than it would have left the point A were the elevation absent. We can see that the effect at P will be the same in either event, provided the difference between the path SBP and SAP is small in comparison to the wave-length. At normal incidence it is obvious that this path difference will be $2H$, therefore a surface having elevations on it of such magnitude that twice their height is not small in comparison to the wave-length will not reflect regularly at normal incidence. With a given roughness long waves may be regularly reflected, and short waves irregularly. It can be seen from the right-hand figure that the difference of path becomes less as the angle of incidence increases, being in the case figured $BA - BK$, which is less than H, and that at grazing incidence it will become 0. It can be shown geometrically that the path difference is represented for all incidences by $2H\cos i$, the value of which must not exceed a small fraction of a wave-length if regular reflection is to occur.

$$SB = SA - AB - SA - \frac{H}{\cos i}.$$

and $$BP = AP + BK = AP + AB\cos(\pi - 2i) = AP - \frac{H}{\cos i}\cos 2i,$$

$$SA + AP - (SB + BP) = \frac{H}{\cos i}(1 + \cos 2i) = 2H\cos i.$$

Since the path difference decreases as the angle of incidence increases, it is obvious that for a given roughness we shall get regular reflection

when the incidence angle is so great that $\rho\lambda = 2H$ cosine i, where ρ is a small fraction; therefore if we gradually increase the incidence angle, the long waves will be reflected first, and then the shorter. Smoked glass, which at perpendicular incidence will show no image of a lamp at all, will at nearly grazing incidence give an image of surprising distinctness, which is at first reddish, becoming white as the angle increases.

Let us next consider the effect of a matt surface on refraction. Here the phase differences are due to retardations of the portions of the wave-front encountering the elevations, on those portions encountering the depressions. With a given degree of roughness the retardation will be greater when the substance has a high refractive index, or more accurately when the difference between the refractive indices of the media bounding the rough surface is large. When the retardation between two adjacent paths is larger than a small fraction of a wave length, we have diffuse transmission. If we take a sheet of ground glass and wet the surface, the glass transmits more direct light than it did before, since we have lessened the difference between the refractive indices of the bounding media. If we substitute benzole for water the glass becomes still more transparent, and by bringing up the refractive index of the benzole by an addition of Canada balsam, we can cause the ground surface to disappear entirely.

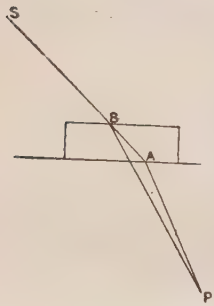


FIG. 28.

Let us now examine the effect of the angle of incidence on the transmission (Fig. 28).

The path difference expressed in time between two disturbances travelling over the paths SAP and SBP can be found as follows. The time occupied over the route SAP (which would be the one followed were there no elevation on the surface) will, if v and v' be the velocity of propagation in the upper and lower media, be $\frac{SA}{v} + \frac{AP}{v'}$, while the time over SBP or *via* the elevation will be $\frac{SB}{v} + \frac{BP}{v'}$ and the difference in time will be the difference between these two quantities.

The regularity of the transmission will not be affected if this time difference

$$\frac{SA}{v} + \frac{AP}{v'} - \left(\frac{SB}{v} + \frac{BP}{v'}\right)$$

is small in comparison to the time of a complete vibration. To change this time difference into a path difference, we substitute for $\frac{v}{v'}$ the quantity n, which is the relative refractive index between the two media, which gives $SA + nAP - (SB + nBP)$.

If this quantity is small in comparison to the wave-length, the regularity of the transmission will be unaffected.

This path difference can be shown to be

$$H\frac{n^2-1}{\sqrt{n^2-\sin^2 i}+\cos i},$$

where H is the height of the elevation and i the angle of incidence. This quantity has its smallest value when $i = 0$, when the path difference becomes $H(n-1)$, or the regularity of transmission decreases as the angle of incidence increases, the opposite of what we found in the case of reflection.

If the refractive index of the substance is 1.5, then $H(1.5-1)$ or $H/2$ must be small in comparison to the wave-length of light, if the light is to be regularly transmitted at perpendicular incidence. Inequalities can then exist, the heights of which are, say, not greater than $\frac{1}{4}\lambda$, which is four times as great a discrepancy as we could have on a reflecting surface.

Summing up, we have (for perpendicular incidence) for regular reflection, $2H=\rho\lambda$ and for transmission $H/2=\rho\lambda$.

If we procure a piece of ground glass, which will barely show the outline of a lamp flame by transmitted light, and thinly silver a portion of the ground surface, we shall have a reflecting and transmitting surface of the same degree of roughness. It will be found that by reflected light the outline of the flame is indistinguishable. Ground glass of this description may be made by grinding two pieces of ordinary ground glass together, with fine emery and water, the process being the first stage of polishing.

We thus see that a rough surface may regularly reflect the long waves while diffusing the shorter ones. Lord Rayleigh has make some interesting experiments upon the reflection of heat waves from ground-glass surfaces too rough to give any trace of regular reflection with visible light. The ground surface was silvered and the radiations of a Welsbach lamp reflected from it. In some cases two reflecting surfaces were used. It was found that the radiation, freed by the process from the shorter waves, was reflected almost as well by a third ground and silvered surface as by a polished silver mirror. The method is analogous to that originated by Rubens and Nichols for isolating long heat waves by repeated reflection from quartz or rock-salt surfaces.

A paper by Lord Rayleigh on "Polish" in the *Philosophical Magazine* will be found of interest.

CHAPTER III

THE REFLECTION OF LIGHT FROM PLANE AND CURVED SURFACES

When light strikes the boundary surface separating two media of different optical densities, some of the energy is reflected back into the first medium, and some crosses the boundary and is transmitted through, or absorbed by, the second medium. We have shown in the previous chapter that if the surface is smooth to within one-eighth ($\frac{1}{8}$) of a wave-length, we shall have regular reflection, and the law of reflection from a plane-mirror has been demonstrated by the Fresnel theory of destructive interference.

As a matter of fact, we are practically unable to make a surface so perfect that absolutely no light is diffused. Admit a ray of sunlight into a dark room and reflect it from the most perfect mirror attainable; were diffuse reflection not present the mirror itself would be invisible, which is never the case. The percentage of diffused light decreases as the angle of incidence increases, as has been shown in the previous chapter, regular reflection taking place even on matt surfaces at grazing incidence.

In studying the reflection of light from plane and curved surfaces we shall investigate not only the direction of the reflected rays, but also the form of the reflected wave-fronts.

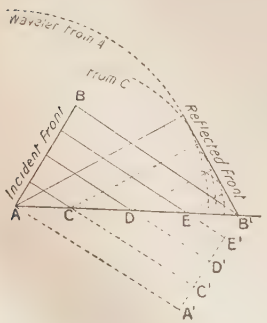


Fig. 29.

Reflection of a Plane-Wave from a Plane-Mirror. — Here the incident rays are parallel to one another and normal to the wave-front. We have seen that the reflected wave-front will be the surface enveloping the secondary wavelets given off by the various points on the mirror's surface as they are struck in succession by the incident wave. The general method of constructing the reflected wave-front is shown in Fig. 29. Let AB' be the surface of the mirror, and AB the incident-wave front, the rays being indicated by lines. At the moment figured, a secondary disturbance is about to leave the point A. This secondary disturbance will have spread out all around A, to a distance equal to BB' at the moment when the point B on the wave-front encounters the surface. The secondary wavelets from points C, D, E etc. in order to between A and B will have to be dealt with. I construct them from $A'E$ parallel to AB. This will give us subsequent position of the wave-front, assuming the mirror a

present. Join these two wave-fronts by perpendicular lines, which present rays, which cut the mirror's surface at C, D, E, etc. The wavelet around C must obviously have a radius equal to CC', while that around D has a smaller radius, DD', and so on for all the other points. If we describe these spheres (circles in the diagram) we shall find that they are enveloped by a plane surface, which makes the same angle with the mirror's surface as the incident wave. This can be proven by similar triangles, Rt. triangle ABB' = Rt. triangle AFB' = Rt. triangle $AB'A'$. (Hypotenuse in common and $AF = BB' = AA'$ by construction.) Therefore, their homologous angles are equal. The rays being normal to the wave-front will make equal angles with the normal to the surface of the mirror.

We can apply this same method to the construction of the wave-front after reflection from a surface of any form. In brief, we draw the wave-front before it encounters the reflecting surface and also in some subsequent position, behind the mirror, which it would occupy at a later moment were the mirror not present. Join these two fronts by normal lines (rays) and describe around the points at which they cut the reflecting surface circles whose radii are equal to the respective distances of the points from the wave-front in its second or imaginary position. The envelope of these circles shows us the position of the reflected wave-front, at the time at which the incident wave would have reached its imaginary position were the mirror absent.

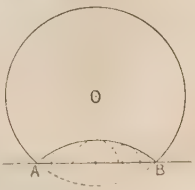


Fig. 30.

Let us now apply this method to the construction of the reflected wave-front, when a spherical wave encounters a plane-mirror. Let O be the luminous point around which we construct the circular section of the spherical wave intersecting the mirror at A and B (Fig. 30). Completing the wave-front below the surface of the mirror, describe around points on the mirror's surface circles whose radii are equal to their distances from the wave-front below the mirror measured in a normal direction, — that is, along the rays or radii of the original wave. The envelope of these circles is itself a circle of the same radius as the original wave, with its centre of curvature at the same distance below the mirror as the luminous point is above. The reflected wave is thus a portion of a sphere with its centre below the mirror. This can be proven as follows:

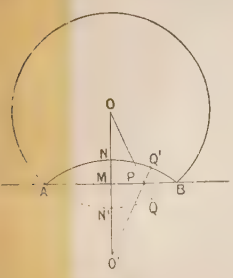


Fig. 31.

Draw OO' perpendicular to the mirror's surface (Fig. 31), making $OM = O'M$, and with O' as a centre draw the arc ANB representing the reflected wave-front. Draw rays OPQ and $O'PQ'$. $OP = O'P$ (homologous sides of equal Rt. triangles), $\therefore PQ' = PQ$, since $OQ = O'Q'$ (radii of equal circles).

This shows us that a wavelet drawn around P with a radius PQ touches the spherical arc ANP at Q', and since this is true for all

points on the mirror's surface, it follows that a spherical surface of the same radius as that of the incident wave is the envelope of all the secondary wavelets. The rays being normal to the wave-front, it follows that after reflection they come apparently from the point O', which we call the image of O in the mirror.

Reflection of Sound-Waves. — A striking analogy exists between sound and light; we can show nearly all of the phenomena of reflection, refraction, and diffraction by means of sound-waves. An electric spark is the centre of a spherical sound-wave, which expands at the rate of about three hundred and thirty-one (331) metres per second, and at the same time is the centre of spherical light waves, which expand at the rate of three hundred thousand (300,000) kilometres per second. We have no means of directly showing the form of the wave-front of the *luminous* disturbance. We can calculate its form before and after reflection and show the agreement between these forms and the rays as actually observed, but we cannot actually show the wave-front. In the case of the *sonorous* disturbance, however, the wave-front can not only be seen, but photographed. We are dealing with a spherical shell of condensed air and, by a suitable optical contrivance which will be described in the next chapter, we can study at our leisure the changes which the wave-front undergoes.

The author has prepared an extensive series of photographs of sound-waves for the purpose of illustrating optical phenomena.

The case that we have just considered, namely, the reflection of a

Fig. 32.

spherical wave from a plane surface, is shown in Fig. 32. The sound wave is started by an electric spark which has just passed between two brass balls, seen in line, one behind the other at the centre of each picture. The wave of condensed air is illuminated and photographed by the light of a second spark occurring a moment later. By properly regulating the time interval between the two sparks a progressive series of views is obtained showing the wave-front at different stages of its development.

The form of the reflected wave or echo is seen to be identical with the form of the light-wave as calculated by Huygens's principle.

Reflection by Ellipsoidal Mirror. — If a spherical wave start at one focus of an ellipsoid of revolution, the reflected wave will be spherical in form, and will collapse to a point at the other focus, or rays of light issuing from one focus come accurately together at the other focus. A surface capable of bringing rays of light accurately to a focus, either by reflection or refraction, is said to be aplanatic, consequently an ellipsoidal mirror is aplanatic for rays issuing from a

point situated at either focus. This can be shown by the following construction.

Around one focus of an ellipse describe a circle which falls just outside of the furthest extremity of the ellipse. Draw a number of radii to the circle, and around the points where the radii cut the ellipse describe circles with radii equal to the distances from the respective points to the outer circle (measured along the radii). The circles will be enveloped by another circle (the reflected wave-front), the centre of which is at the other focus of the ellipse.

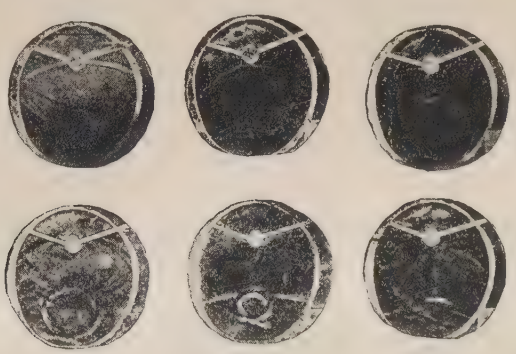


Fig. 33.

In Fig. 33 we have a series of photographs showing a sound-wave starting at the focus of an elliptical mirror. The expanding sphere is seen to have been changed by reflection into a contracting sphere, which shrinks to a point at the other focus. The same thing can be shown by making a shallow, flat-bottomed, elliptical dish of wood, filling it with mercury, and touching the surface of the fluid at one focus of the ellipse.[1]

Reflection from a Parabolic Mirror. — If we construct the reflected wave-front in the case of a wave starting at the focus of a parabolic mirror, we shall find that the reflection transforms the spherical wave into a plane-wave. The reflected rays, being normal to the wave-front, are parallel, and are consequently projected in a narrow beam out of the mirror. This is the principle on which the naval search lights are constructed.

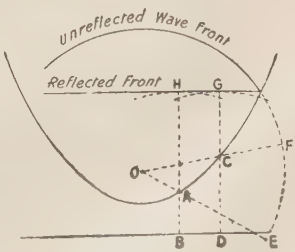


Fig. 34.

Let O be the focus of the parabolic section of the mirror (Fig. 34). Construct the imaginary spherical wave-front (unreflected) EF, and around points A, C, etc., on the parabola construct circles, or secondary wavelets, with radii equal to the distances of the points from the imaginary wave-front measured along the radii of the circle EF.

[1] By placing the dish in the sunlight, and receiving the reflected light on a screen, the experiment can be shown to a class.

The wavelets will be enveloped by a straight line, the section of a plane. It is easy to see that this line is straight, or that the reflected wave-front is accurately plane. Every point on the parabola is equidistant from the focus and the directrix; $\therefore OA = AB$ and $OC = CD$. Around A and C we have circles with radii equal to AE and CF respectively. Now, $OE = OF$, being radii of same circle, and $DG = OF$ and $BH = OE$; $\therefore DG = BH$ or the enveloping line is everywhere equidistant from the directrix, and consequently parallel to it. The reflected wave is, therefore, a true plane. The projection of a truly plane-wave from a paraboloid mirror is practically never realized, since the source of light is always of finite size — that is, the waves do not all start from the focus.

In Fig. 35 we have the reflection of a sound-wave from a parabolic mirror. The converse of this case is also true. Plane-waves entering a parabolic mirror are transformed by reflection into converging spherical waves which shrink to a point at the focus of the parab-

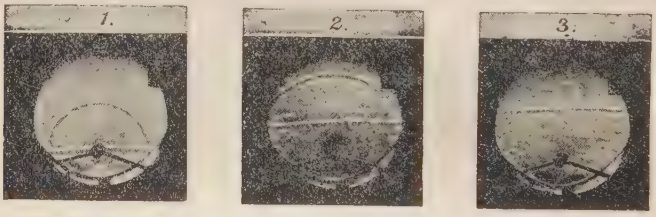


Fig. 35.

oloid. This means that parallel rays, or rays coming from an object situated at a great distance, are brought accurately to a focus by a mirror of this form, or the parabolic mirror is aplanatic for parallel rays.

The surface of a liquid in uniform rotation assumes the form of a paraboloid under the influence of centrifugal force, and the author has recently constructed a reflecting telescope of mercury twenty inches in diameter which operates on this principle. The focal length can be varied from three to twenty or more feet by altering the speed of rotation. The instrument resolves stars three seconds apart, shows the smallest craterlets on the moon, and yields wonderfully bright images of nebulae when running with a short focus. See *Astrophysical Journal*, 1909 and 1910.

Reflection by Hyperboloid. — A spherical wave originating in one focus of an hyperboloid is, by reflection, changed into a sphere whose centre is at the other focus. The reflected rays appear, therefore, to come from this point, each focus of the hyperboloid being the virtual image of the other. This follows at once from the fact that lines joining the two foci of the hyperbola with any point on the curve make equal angles with the normal at the point. Let F and F' be the two foci (Fig. 36), and consider any ray drawn from F' to the branch of the hyperbola nearest it, say to the point A. Then FA and $F'A$ make equal angles with the normal, and the reflected ray AB produced backwards coincides with AF. A spherical wave started at F' will, after reflection, be a sphere with its centre

at F. If a wave is started at F and reflected in the *convex* surface of the same branch of the hyberbola (the branch nearer F being removed), it will be transformed into a sphere expanding from F' as a centre.

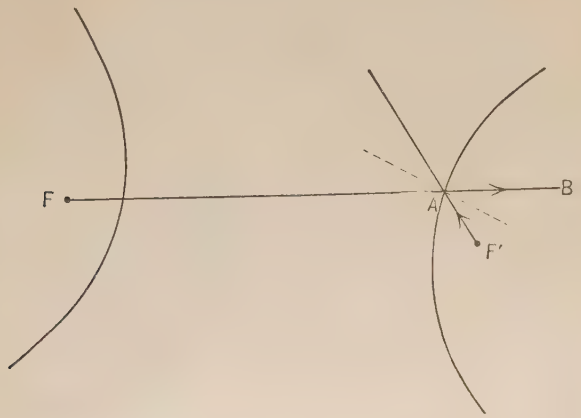


Fig. 36.

If now we consider a spherical wave front contracting towards the focus F' (*i.e.* reversing the last case) it will, after reflection, converge towards F. In other words, the convergence of the rays will be decreased without interfering with their ability to come to a focus at a point. This is important in connection with the use of hyperboloidal mirrors in modern reflecting telescopes.

Reflecting Telescopes. — A concave paraboloid of glass silvered on its curved surface forms the objective of the reflecting telescope. The image is sharpest when it falls upon the axis of the paraboloid. Newton, who constructed the first reflecting telescope, placed a small mirror on the axis between the objective and the image (near the latter), which reflected the rays out through the side of the tube, where the image was viewed with an eye-piece. Telescopes of this form are called Newtonian reflectors. The mirror, however, obstructs some of the incident light, and Herschel accordingly inclined his mirror so that the image fell at the side of the tube. This produces a little distortion, however, though it is small if the inclination is not more than two or three degrees. If we could make a paraboloid in which a portion of the true surface a little to one side of the axis was used, the mirror when properly oriented would give an image free from distortion in an oblique direction. See section on Focal lines. This could be accomplished by local corrections of the surface. In practice it is found that the definition of the Herschel reflector is best with the mirror in a certain position, which can be found by rotating it in its own plane. In this position we have the nearest approximation to the ideal condition mentioned above. The circumstance results from slight irregularities in the curvature. The largest modern reflector in the world is the monster instrument of the Mount Wilson Solar Observatory in California. The parabolic

mirror is 60 inches in diameter and has a focal length of 25 feet. It can be used first as a Newtonian, as shown in Fig. 37; second as a Cassegrainian with a focus equivalent to 100 feet, Fig. 38; and

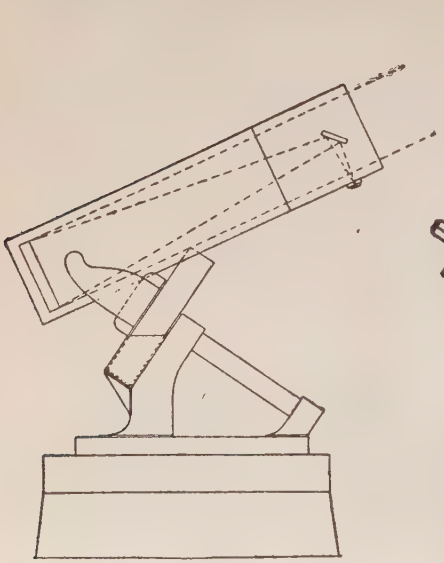


FIG. 37.

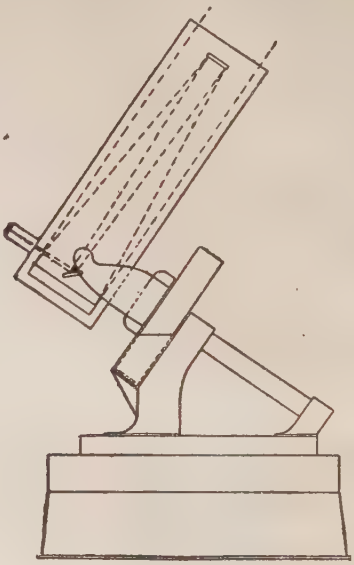


FIG. 38.

third as a Cassegrainian with a focal length of 150 feet. To transform it into the Cassegrain type a convex hyperboloid is put in place of the plane mirror used in the Newtonian type. This decreases the convergence of the rays, which are either reflected to one side at the base of the tube, as in Fig. 38, or thrown down through the hollow polar axis as in Fig. 39 (150-foot focus). In using the hyperboloidal mirror, the actual distance traversed by the rays in coming to a focus is not 150 feet, but the size of the image is the same as that produced by a mirror having an actual focal length of 150 feet. The theory of this method of amplification will be taken up when we come to lenses.

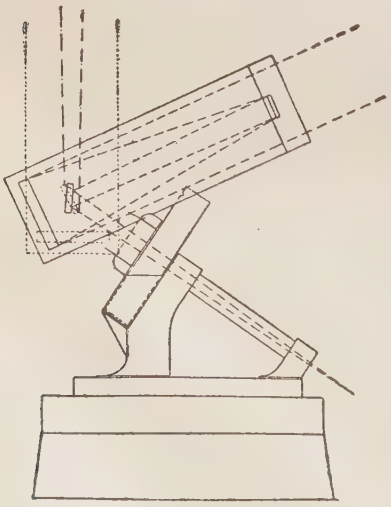


FIG. 39.

Reflection from Spherical Surfaces. — If the reflecting surface be a portion of a sphere, the effects are more complicated, except in the special case of waves starting at the centre of curvature. The rays do not all meet at a point, as in the cases which we have con-

sidered, but envelope a surface known as the *Caustic*. An example of a caustic is the cusped line of illumination seen on the tablecloth when the light of a lamp strikes the inner surface of a silver napkin ring. We have seen that a concave paraboloid brings parallel rays accurately to a focus. A concave spherical mirror does not do this. Rays near the axis come to a focus approximately at a point, but as we recede from the axis we soon find the reflected rays falling wide of the focus. This effect is known as Spherical Aberration. In constructing telescope mirrors, opticians strive to give the surface as nearly as possible the figure of a paraboloid. The nature of the wave-front in cases where caustics are formed is not at once apparent. The subject is usually treated by ray methods, and we shall accordingly begin by considering one or two examples geometrically, although the evolution of the wave-front, and the relation between the wave-front and the caustic, form a more interesting study.

Reflection from Convex Spherical Surfaces. — In studying reflection by ray methods we can regard a curved surface as made up of an infinite number of plane surfaces, for each one of which the law of equal angles holds. Let us take as the first case the reflection of light radiating from a luminous point at a convex spherical mirror.

When the light is incident on a small portion of a sphere in a nearly normal direction, we may regard the reflected rays as emanating from a point behind the mirror; the virtual focus, as will appear presently, is not at once apparent. If, however, we employ a large arc of the mirror this does not hold even approximately, and we require an expression for the position of this focus in terms of the angle of incidence.

Let the radius of curvature of the mirror be r, the distance of the radiating point from the centre of curvature be b, and the angle of incidence be i; find g, the distance of the focus D from the centre.

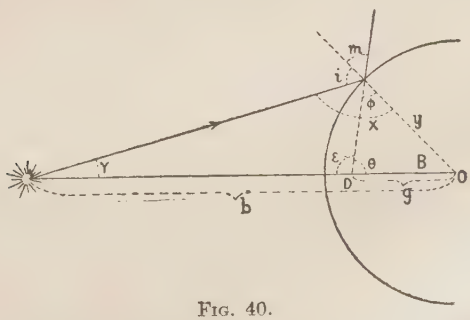


Fig. 40.

(1) $g : r = \sin \Phi : \sin \theta$ (Sides of $\triangle$ in ratio of sines of opp. $\measuredangle$s).
$b : r = \sin X : \sin Y$.

Dividing $\frac{g}{b} = \frac{\sin \phi}{\sin \theta} \frac{\sin Y}{\sin X}$ and $\sin \phi = \sin x$ (since X is the supplement of Φ).

$$\therefore \frac{g}{b} = \frac{\sin Y}{\sin\theta} = \frac{\sin(i-\beta)}{\sin(i+\beta)} \text{ since } \begin{matrix} Y+\beta+X=X+i \text{ and} \\ \sin\theta=\sin(\beta+\phi)=\sin(\beta+i). \end{matrix}$$

$$\frac{g}{b} = \frac{\sin i\cos\beta-\cos i\sin\beta}{\sin i\cos\beta+\cos i\sin\beta}.$$

Adding 1 to both sides,

$$\frac{g+b}{b} = \frac{\sin i\cos\beta-\cos i\sin\beta+\sin i\cos\beta+\cos i\sin\beta}{\sin i\cos\beta+\cos i\sin\beta},$$

$$\frac{g+b}{b} = \frac{2\sin i\cos\beta}{\sin(i+\beta)}, \text{ but } \frac{\sin i}{\sin(i+\beta)} = \frac{\sin\phi}{\sin\theta} = \frac{g}{r} \text{ (by (1))};$$

$$\therefore \frac{g+b}{b} = \frac{2g\cos\beta}{r} \text{ and } \frac{g+b}{g} = \frac{2b\cos\beta}{r} \text{ or } \frac{b}{g}+1 = \frac{2b\cos\beta}{r};$$

$$\therefore g = \frac{br}{2b\cos\beta-r}, \text{ the equation for a CONVEX MIRROR.}$$

Considering D as the luminous point, and solving for b, we get

$$b = \frac{gr}{2g\cos\beta-r}, \text{ the equation for a CONCAVE MIRROR.}$$

These equations show us that the rays reflected from different annular elements of the sphere (determined by β) if produced backwards cut the axis of the mirror nearer and nearer to its surface as B is increased.

For rays near the axis ($\beta=0$ and $\cos\beta=1$) we have

$$g = \frac{br}{2b-r}.$$

If the source is at a great distance, *i.e.* b very large in comparison to r, we have

$$g = \frac{r}{2};$$

that is, the focus is midway between the surface and the centre of curvature. If we construct a number of reflected rays we shall find that they envelop a caustic surface, which is virtual in the case of a convex spherical mirror and real in the case of a concave. We cannot therefore form a clear image with a spherical mirror of large aperture, since the rays do not focus all at the same point; in other words, the reflected wave is not spherical as in the case of the ellipsoid and paraboloid.

Reflection of Plane-Waves from Concave Spherical Mirrors. — Let us now determine what happens to the wave-front in the case just considered. We might employ the method already used for constructing reflected fronts, but this method has certain disadvantages in the present instance: it does not readily give us the complete wave-front, and it gives but a single front. In this particular case the reflected front is rather intricate, and a construction

that will enable us to follow it in its evolution is desirable. This can be accomplished by employing a second method.

ABC is the mirror, *AOC* the plane-wave (Fig. 41). Around points on *ABC* as centres describe circles tangent to the wave. These circles will be enveloped by another surface, *ADE*, below the mirror (the orthogonal surface). If we erect normals on this sur-

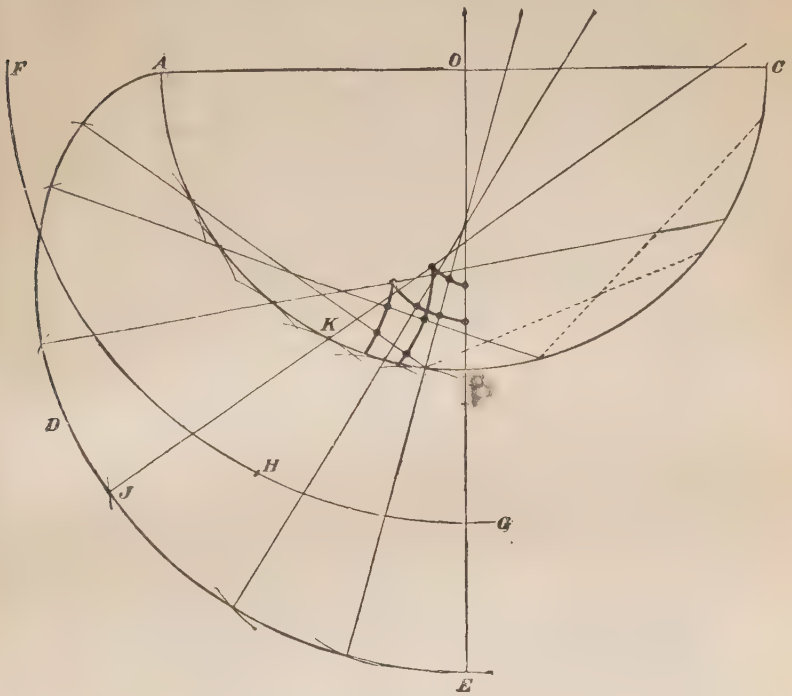


Fig. 41.

face, we have the reflected rays, and if we measure off equal distances on the normals, we have the reflected wave-front. By drawing the orthogonal surface we avoid the complication of having to measure off the distances around a corner. The orthogonal surface is an epicycloid formed by the rolling of a circle of a diameter equal to the radius of curvature of the mirror on the mirror's surface, and the normals can be erected by drawing the arc *FG* (the path of the centre of the generating circle), and describing circles of diameter *BE* around various points on it. A line joining the point of intersection of one of these circles with the epicycloid, and the point of tangency with the mirror, will, when produced, give a reflected ray; for example, *JK* produced, for circle described around *H*. This construction once prepared, the series of wave-front pictures can be very quickly made. Three or four sheets of paper are laid under the construction and holes punched through the pile by means of a pin, at equal distances along each ray (measured from orthogonal surface).

The centre of the mirror and the point where its axis meets the surface are also indicated in the same manner. The sheets are now separated, and corresponding pin-holes are united on each sheet by

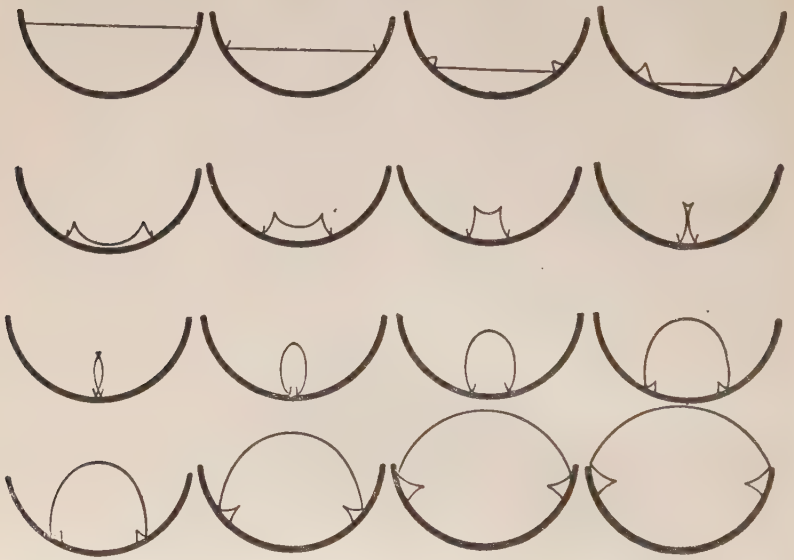


Fig. 42.

a broad black line, which represents the wave-front. After a time it becomes necessary to consider double reflections, and to do this

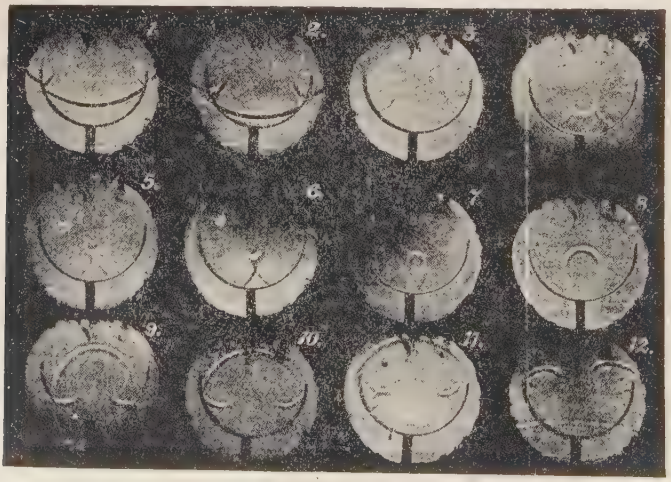


Fig. 43.

we are compelled to construct twice-reflected rays (indicated by dotted lines), and measure around a corner each time.

The geometrically constructed fronts are shown in Fig. 42.

These are diagrams taken at intervals on a kinetoscope film prepared by the author for illustrating the wave evolutions.

About one hundred constructions were made, in the manner just described, and photographed in succession on the film, which, when run through the machine, gives us the moving wave on the screen in a most graphic manner.[1]

A series of photographs of a sound-wave entering a hemicylindrical mirror [2] is shown in Fig. 43, and it will be seen that the forms are identical with the geometrical constructions. The reflected front is cusped, and in certain stages (No. 4) has a form not unlike a volcanic cone with a bowl-shaped crater.

In Fig. 44 we have a number of wave-fronts in different stages of reflection, and it is at once apparent that the cusp traces the caustic surface, indicated by a dotted line.

FIG. 44.

This gives us at once a *physical*, as distinguished from a *geometrical*, definition of a caustic, which is a surface traced by a moving cusp of the wave-front.

Let us examine a little more carefully the manner in which this cusped wave is propagated. A superficial examination of the forms might lead one to imagine that the bowl of the crater collapsed to a point at the principal focus of the mirror. This can of course only be true in the case of a concave spherical wave, which is only given by a parabolic mirror. We shall find as a matter of fact, if we examine the geometrical construction, that the cusp of the wave, or the rim of the crater, which traces the caustic as we have seen, is continuously passing through a focus. In other words, the curvature of the crater increases as we go from the bottom to the rim, at which point the radius becomes zero. The inner edge is then continually passing through a focus and appearing on the outside, building up, as it were, the sides of the cone. These wave-fronts were drawn by constructing the orthogonal surface, which was shown to be, in section, an epicycloid formed by rolling a circle, whose diameter was equal to the radius of curvature of the mirror, around the outside of the mirror. The evolute of this curve is the caustic, itself an epicycloid, and the reflected wave-fronts form a family of parallel curves, which are the involutes of the caustic.

Though the caustic and orthogonal surface (evolute and involute) are similar epicycloids, the reflected wave-fronts, or parallels to the orthogonal surface, are not epicycloids. It may be well to point out here an error that sometimes appears in text-books on Optics, namely, the assumption that the wave-front (say in the case of a spherical wave refracted at a plane surface) is an hyperboloid in the second medium, because the caustic is the evolute of an hyperboloid.

[1] "Photography of Sound-Waves, and Kinematographic Demonstration of Reflected Wave-Fronts." Wood, *Proceedings Royal Society*, Vol. 66.

[2] Cylindrical surfaces have been used instead of spherical for obvious reasons. The sectional view is of course the same in each case.

An hyperboloid wave will not propagate itself as an hyperboloid, nor an ellipsoidal wave as an ellipsoid (except in an anisotropic medium), the parallels to a conic being in general curves of the eighth degree. In the case above cited, we should speak of the wave-fronts after refraction as the parallels to an hyperboloid.

Let us suppose the wave to be just entering the mirror. The form of the portion which has already suffered reflection is a cusp extending around the upper edge of the hemisphere (Fig. 45). The upper branch of the cusp is concave upward, and is the portion of the wave which left the reflecting surface and has passed through a focus. The lower branch is concave downward, or in the direction of propagation, and represents the portion of the wave which has just left the surface and is on the way to its focus. The radius of curvature increases from zero as we go away from the cusp-point along either branch, as has been said before. This cusped wave moves down the mirror, the lower branch being continually replenished by consecutive portions of the incident wave as it encounters the mirror, the upper branch being continually added to by elements of the lower branch as they pass through their foci at the cusp.

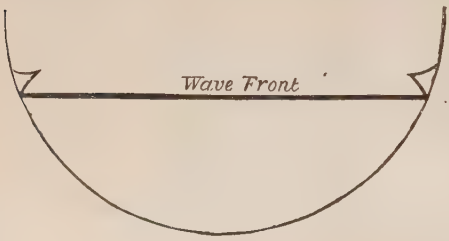


Fig. 45.

As has been said, the cusp traces the caustic surface, and since the wave is always coming to a focus on the cusp, the increased illumination along the caustic is accounted for.

The difference between a parabolic reflector and a spherical one is now clear. The former gives us a spherical wave which will collapse to a point, the latter an approximately spherical wave near the axis only, the rest of the wave being incapable of shrinking to a point.

We will next consider the opposite case, starting a spherical wave at the principal focus of the concave, spherical mirror, and determine the form of the reflected wave, which we found to be plane in the case of the paraboloid. We will use the first method, constructing a single reflected front only. Let ABC be the mirror, with its focus at D, where the wave originates. Draw dotted lines representing the wave in a subsequent position (Fig. 46), and around the points B, F, G, H on the mirror describe arcs with radii equal to BE, FH, and GI respectively. These arcs will be enveloped by the reflected wave-front, which is approxi-

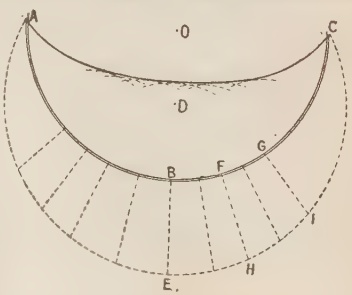


Fig. 46.

mately plane near the axis of the mirror, curling up at the edges, however, the whole resembling a shallow, flat-bottomed saucer.

Roughly sketch in a few normals to this wave, and determine its form in subsequent positions, and it will be found that the curved

Fig. 47.

sides of the saucer run in to a focus around the edge of the flat bottom (a ring focus), disappearing for a moment and then reappearing on the under side, but turned over the other way. As the wave advances, the flat bottom contracts, and the cusps formed by the union of the turned-over sides with the bottom trace a caustic which has the form of a long, tapering funnel. Photographs of a sound-wave taken under these conditions, and diagrams from the kinematograph films, are shown in Figs. 47 and 48. The reflected wave-fronts, and the caustic traced by the cusps in this case, are shown in Fig. 49.

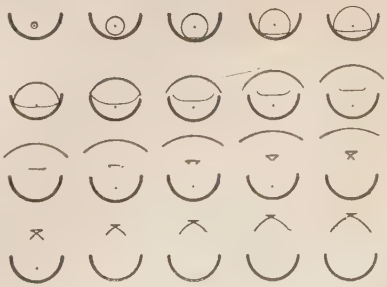


Fig. 48.

A useful piece of apparatus can be made by silvering the outside of a hemispherical glass evaporating-dish or half of a large, round-bottomed flask. The concave mirror thus formed should be mounted on a stand, and a two-candle" pea " electric lamp arranged so that it can be moved along the axis of the mirror.

If we place the lamp in the focus of the mirror, and hold a sheet of ground-glass in front of it at the proper distance, we can show the luminous ring formed by the passage of the sides of the saucer-shaped wave through a focus.

The illumination within the ring is due in part to unreflected light, and in part to the flat portion of the reflected wave.

Formation of Images. — In the formation of a real image by a mirror, the rays radiating from a point are brought together again approximately at a point, or the reflected wave-fronts are converging spheres. A luminous object being made up of a collection of adjacent, radiating points, we have vast numbers of spherical waves entering the mirror from these points, and converging to points similarly situated with reference to one another. The formation of these images, and the study of their position and distance from the mirrors, belongs to geometrical optics. The influence of the form of the wave-front on the distinctness of the image may, however, be considered in connection with what has gone before. We have seen that the paraboloid and ellipsoid are the only surfaces that reflect spherical fronts; other curved surfaces give, in general, caustics. If we use only a very small portion of the sphere, and receive the light normally, we have only the cusp of the caustic, which is very nearly a point. There is a certain amount of what is called longitudinal aberration, owing to the fact that the rays reflected from the periphery come to a point situated nearer the mirror than those reflected from the points near the axis. If the image be thrown on a screen, the bright spot formed by the central rays will be surrounded by a circular ring of light formed by the peripheral rays which have already come to a focus and are diverging once more. The position of sharpest definition lies between the focal points of the marginal and central rays. The circle of light is smallest at this point, and is known as the circle of least confusion. The line AB in Fig. 50 shows the position of this point.

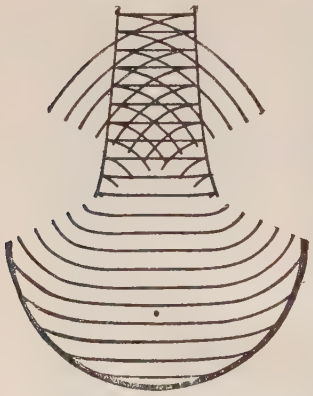


Fig. 49.

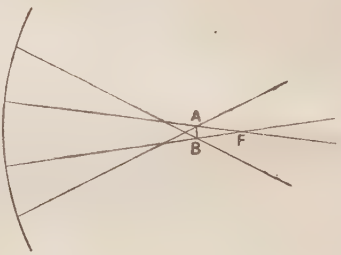


Fig. 50.

Focal Lines. Suppose now that the incident light falls on the mirror in an oblique direction. This is the condition if we consider a small portion of the hemispherical, concave mirror far removed from the axis. The reflected wave will come to a focus in a line instead of a point; as we increase the distance of the screen from the mirror, the line will decrease in length, increasing in width until it is transformed into a line at right angles to the first.

These lines are known as the primary and secondary focal lines respectively, and can be shown by holding a concave mirror in an

oblique position and reflecting the light coming from a small, brilliant source on a screen placed at various distances from the mirror. We can best form an idea of how these lines are formed by considering the question first by a ray method and then by a wave-front method.

Let AB be the axis of the mirror, near the edge of which a bundle of rays parallel to the axis falls. Construct the reflected rays from a linear strip BC as shown in the sectional view (Fig. 51): we shall have a flat, converging fan coming to a focus at F', then diverging and cutting the axis at F''. Now rotate the whole figure through several degrees around AB as an axis; the parallel sheet of incident rays will trace the rectangular incident bundle, the line BC will trace an approximately rectangular area of the mirror, F' will move through a short circular arc, approximately a straight line (the primary focal line), while the diverging fan will trace out wedge-shaped portions of space on each side of the axis, which have a common linear boundary at F'' (the secondary focal line). The reflected rays between the two focal lines fill a space similar in shape to the sphenoid of crystallography. If any difficulty is found in forming a picture of this rotation figure in the mind, it can be removed by cutting out of cardboard a diagram representing a section of the mirror, incident and reflected rays as figured above, and mounting it on a knitting needle placed in coincidence with the axis AB. By rotating the needle through a small angle, the formation of the focal lines and the sphenoidal bundles of rays can be readily seen.

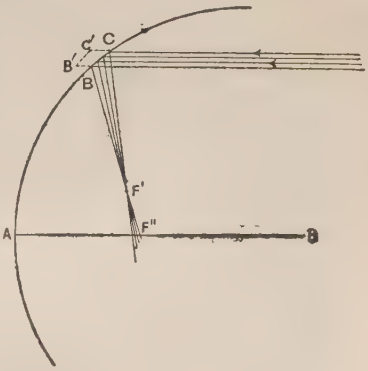


FIG. 51.

If we require a mirror which will form a focal point in an oblique direction, the portion BC must be cut from a paraboloid surface the axis of which is AB. This is for parallel rays, and the subject has already been discussed under reflecting telescopes.

If we require a mirror which will form a focal point in an oblique direction for rays coming from a source at a small distance, the mirror must be a portion of an ellipsoid, say the portion at D in Fig. 54, where A and B are the conjugate foci. Such mirrors would be very difficult to make, as the curvatures are different along different meridians.

Let us next endeavor to explain the formation of focal lines by considering the form of the wave-front.

The curvature of the wave-front as it leaves the mirror under these conditions is different along different meridians. If we cut a piece out of the side of a hen's egg we shall have something of analogous form.

Let AB be the direction of greatest curvature and CD that of

least curvature (Fig. 52). To start with, suppose the curvature be equal along all lines parallel to AB, and suppose all lines parallel to CD to be straight. This will give us a cylindrical wave which will come to a linear focus at F_1, the length of the line being equal to the length of the cylindrical wave.

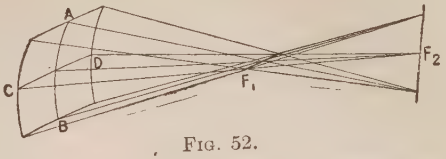


Fig. 52.

If we start with a square wave-front we shall find it contracting to a line as we approach F_1, and expanding beyond F_1, first as a horizontal rectangle, then a square, and finally a vertical rectangle. Now let us impress a slight curvature parallel to CD. The result of this will be that our square will now contract in both directions, only in one less rapidly than in the other, and the line at F_1 into which it shrinks will be shorter than before, and instead of being straight will be slightly concave towards F_2. From here it can be regarded as an expanding wave in a vertical plane, and a contracting wave in a horizontal plane. It is easily seen that the line at F_1 will now open out, first into a horizontal rectangle, as before, then a square (as the two sides *closing in* become equal to the top and bottom *moving out*), then a vertical rectangle, and finally a vertical line at F_2, as the sides come together.

It is interesting to inquire as to the nature of the rectangular wave surface between F_1 and F_2. From its nature we see that it must be concave towards F_2 in the horizontal plane, and convex in the vertical, the surface resembling a small portion cut out from the inside of a thick cylindrical ring. We can, indeed, find surfaces of this form on our geometrically constructed wave-fronts.

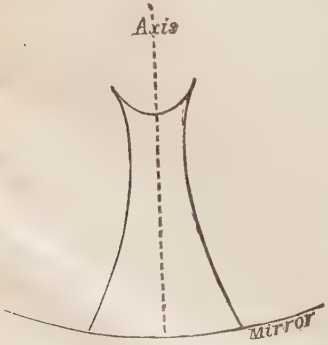


Fig. 53.

Consider the diagram shown in Fig. 53 (which will be recognized as the "volcanic cone" form), remembering that the complete wave-front is formed by the rotation of this figure around the axis of the mirror.

The bowl of the crater is concave along every meridian, but it is at once apparent that any portion of the outer slope has the required saddle-shape, being concave in horizontal planes and convex in vertical planes. From this it is evident that the outer wall of the volcanic cone, before it crosses the axis of the mirror, always represents the portions of the wave-front between the primary and secondary focal lines.

That this is true is evident, when we recollect that the first focal line is formed by the intersection of rays on the caustic surface, or,

regarded from the wave point of view, by the passage through their foci on the cusp of the wave, of adjacent elements of the wave-front. The second focal line lies on the axis of the mirror; consequently the wave-front between the lines is that portion of the surface which has passed through a focus on the cusp, but which has not crossed the axis.

It will be found that a small glass model of the wave-front, shown in cross section in Fig. 53, is extremely useful in making the whole matter clear. It can be made by drawing down a large thin tube, melting the end down flat, and then sucking it in a little.

Fermat's Principle. — We sometimes find it stated that a ray of light in passing from one point to another by way of either a reflecting or refracting surface, chooses a path such that the time of transit is a minimum. This principle was stated by Fermat more than two centuries ago. It is true, however, only for plane surfaces. In the case of reflection from a plane surface the incident and reflected rays make equal angles with the normal, and we know from elementary geometry that this path is the shortest that can be traced from one point to the other by way of the surface. The same is true for convex surfaces, but for concave surfaces we find that in certain cases the path is a maximum instead of a minimum.

That the path is sometimes a maximum can be seen by the construction shown in Fig. 54. We will consider the passage of a ray from the point A to the point B by way of the reflecting spherical surfaces CDE and FGH. Around the points A and B as foci we construct an ellipse which we will suppose to be tangent to the two reflecting surfaces at D and G. This ellipse is an aplanatic surface for rays issuing from either focus, consequently the time of transit of a ray from one focus to the other by way of the elliptical surface is the same for every point on the surface. Now the ellipse and the two spherical surfaces have common tangent planes at D and G, consequently D and G will be the points on the spherical mirrors so oriented that they can reflect rays from A to B. It is easy to see that the path AGB is shorter than any other path between A and B by way of the sphere which is exterior to the ellipse, while in the case of the other sphere the path actually pursued (ADB) is longer than any other path which we can draw from A to the surface and from thence to B. In this case we see that the path chosen by the ray is such as to make the time of transit a maximum. The conditions for a maximum or minimum may be expressed by saying that the variation of the time of transit with the change of path, ceases at the points for which the path is either a maximum or minimum, or $\delta(AD+DB)=0$. This matter will be further discussed under refraction.

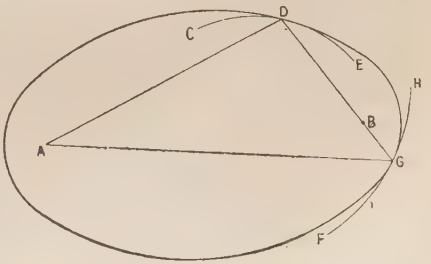


FIG. 54.

CHAPTER IV

REFRACTION OF LIGHT

In the preceding chapter we have discussed the forms and behavior of the wave-fronts reflected back into the first medium, when light falls upon the boundary between two media of different optical density. A portion of the energy, however, always passes into the second medium, except, perhaps, in the special case of total reflection, and even in this case mathematical analysis shows us that there is a disturbance beyond the boundary, though only penetrating to a distance of a few wave-lengths. The energy crossing the boundary may either be absorbed by the second medium, or propagated according to the laws governing luminous disturbances in it.

In the present chapter we shall consider only the case of wave propagation in an isotropic medium, or one in which the velocity of propagation is independent of direction. Later on we shall investigate the refraction of light in bodies in which the velocity is different in different directions.

We will begin by considering the refraction of a plane-wave at a plane surface.

Refraction of Plane-wave at Plane Surface. — Suppose a plane-wave incident at an angle of 45° on a flat surface of glass, and assume the velocity in the glass to be less than the velocity in air, as we shall subsequently show it to be.

The various points on the glass surface become in succession centres of secondary disturbances as they are struck by the incident wave. These secondary wavelets spread out in both media, and it has been shown by Huygens's construction that the reflected wave is the envelope of those spreading out in the first medium. If we apply the same construction to the second medium, supposing for the sake of simplicity that the velocity of the wave propagation in it is only one half as great as in the first, the wavelets in the glass will have radii half as large as the corresponding wavelets in air, and the enveloping surface or refracted wave-front is turned through an angle (Fig. 55). The rays, or normals of the wave, are therefore bent an equal amount.

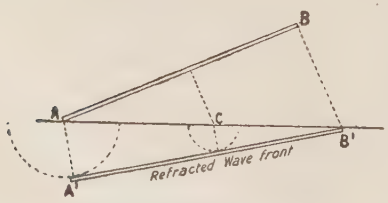


Fig. 55.

It was determined in 1621 by Snell that in every case of refraction the incident and refracted rays make such angles with the normal to the surface, that the ratio of their sines is constant for

any two given media. Snell's law of refraction we now know holds only for isotropic media. It can be easily deduced from Huygens's construction in the following way.

Construction. — The angle of incidence i is the angle between the incident ray and the normal to the surface. It is also the angle between the wave-front and the surface. The same is true for the angle of refraction. Let the velocity in air be v, and the velocity in glass be v' $\left(\text{equal to say } \frac{v}{2}\right)$, and let t equal the time required for the wave in air to traverse the distance BB' (Fig. 51). Then $BB' = vt$, and the radius of the secondary disturbance around A in the glass will be AA' or $\frac{BB'}{2}$. We have then $\frac{BB'}{AB'} = \sin i$ and $\frac{AA'}{AB'} = \sin r$,

or
$$\frac{\sin i}{\sin r} = \frac{BB'}{AA'} = \frac{v}{v'} = \text{constant, in this case 2.}$$

This constant is the relative refractive index between the media, and the above relation holds for every value of i, if the second medium is the one in which the disturbance travels at a lesser velocity. The refractive index is usually designated by μ, and in the above case is of course 2, which is higher than is usually the case.

Total Reflection. — We have seen that in passing from a rare to a denser medium a refracted ray always exists, no matter how great the angle of incidence. This is not true if we reverse the conditions, for now the relative refractive index will be less than one, and we shall find that, if i exceeds a certain value, sin r is greater than one. But no angle has a sine greater than one, therefore there can be no refracted ray. Let us apply Huygens's construction to the case. The secondary waves in the second medium will have radii greater than the corresponding ones in the first medium, since their velocity of propagation is greater. By dividing the radii of the reflected wavelets by μ, we obtain the dimensions of the refracted wavelets. We shall find that, up to a certain value of i, these secondary disturbances will intersect the surface *within* the projection of the incident wave upon it; in other words, a tangent plane can be drawn from the point B' (Fig. 52). At a certain value of i, however, the secondary wavelet around A will intersect the surface at B', and the same will be true for all of the other wavelets: (since, if the short leg of

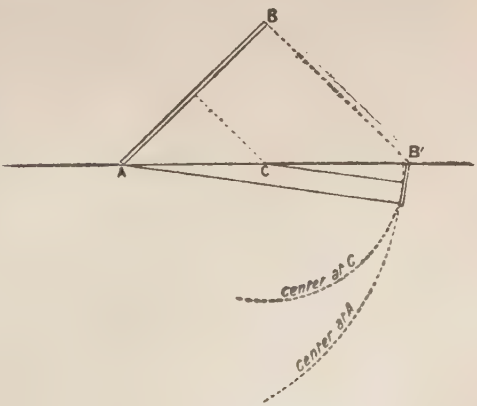


Fig. 56.

one of the right triangles divided by μ gives us the hypotenuse, the same will be true of all the other similar right triangles). The tangent plane drawn from B' to these wavelets will be normal to the surface, and will touch the wavelets in a single point only (in the sectional diagram). The refracted ray therefore will travel along the surface.

The value of i for which this condition exists can be found by combining $\sin i = \frac{AC}{AB}$ with $\frac{AC}{\mu} = AB$, which gives us $\sin i = \mu$, μ of course being the refractive index of the rarer medium with respect to the denser. If, as is customary, we consider μ as the refractive index of the denser with respect to the rarer the last equation becomes

$$\sin i = \frac{1}{\mu}.$$

The angle determined by the above expression is known as the *Critical Angle*. If it be exceeded, the secondary wavelets cut the surface beyond the point B' and no tangent plane can be drawn, therefore no refracted ray exists. The energy in this case is totally reflected.

If in the formula $\frac{\sin i}{\sin r} = \mu$ we assign various values to i, and solve for r, we shall find that every possible value of i between 0° and 90° gives a corresponding value of r if μ is greater than one, which is always the case when the ray passes from a rare into a denser medium.

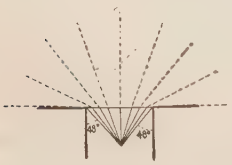


FIG. 57.

Take the case of rays of light entering the level surface of a pond, at all possible incidences from 0° to 90°. The zenith light passes straight down, the horizon light is refracted in a direction given by $\frac{\sin 90°}{\sin r} = 1.33$, or $\sin r = \frac{1}{1.33}$, which gives for r a value slightly greater than 48°. In other words, no ray in the water makes an angle with the normal greater than about 48°. The light therefore which enters an eye under water consists of rays embraced by a cone of 96° angular aperture (Fig. 57) instead of 180°, as is the case when the eye in air is directed towards the zenith. If therefore, when submerged in water, the eye be directed towards the surface, the sky appears compressed into a circle of light subtending an angle of 96°, the appearance being precisely as if the water were covered with an opaque roof with a round hole directly overhead. If, however, we are in diving armor, and look upward through the plate glass window of the helmet, the illusion of the hole vanishes, for now the ho-

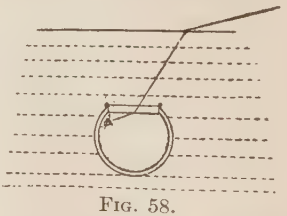


FIG. 58.

rizon rays are refracted back into their original direction on passing into air once more, as is shown in Fig. 58, the 96° cone widening out to 180°.

In this connection it is of interest to ascertain how the external world appears to a fish below the surface of smooth water. The objects surrounding or overhanging the pond must all appear within the circle of light previously alluded to. There must be a great deal of distortion of objects which are not very nearly overhead, but we can gain absolutely no idea of their appearance by opening the eyes under water, since the lens of the human eye is only adapted to vision in air, and when submerged is quite unable to distinguish the shape of objects. There is, however, no difficulty in photographing the circular window of light and the external world as seen through it. It was found after a little experimenting that better results were obtained with a pin-hole than with a lens, and a small camera was constructed which could be filled with water and pointed in any direction. If pointed vertically it recorded the view seen by a fish in a pond; if horizontally, the view as seen by a fish looking out through the side of an aquarium. It is obvious that the plate must be immersed in water, as otherwise refraction occurs as in the helmet of diving armor.

The fish-eye camera can be made of a wooden or metal box measuring about $12 \times 12 \times 5$ cms. (inside measure). A hole 3 cms. in diameter is bored through the centre of one of the sides, over which is cemented a piece of mirror glass with the silvered and varnished side facing the interior. The glass must be quite opaque, *i.e.* free from pin-holes in the silvered film. A very small hole should be made through the film by scratching it carefully with a needle, before the plate is cemented to the box. This small aperture passes the rays of light which form the image to the photographic plate which lies against the opposite side of the box. The box must be light-tight, and filled with clean water. A little consideration will show that the part played by the water in the pond is, in this case, played by the glass plate. A number of views secured with the apparatus are reproduced below, Fig. 59. The camera obviously has an aperture of 180°.

One of the views is of a railroad bridge passing overhead, the other represents the appearance of a crowd of men standing around a pond, to a fish below the surface. The two lower views were taken with the camera pointing in the horizontal direction, *i.e.* the views correspond to what a fish sees when looking out through the side of an aquarium. One of them shows a view looking both up and down a street, the other a row of men standing in a straight line taken from a point only 50 cms. in front of the central figure. These last two show in a very effective manner that the angle of view embraces 180°.

Effect of Refraction on the Width of the Beam. — It is at once apparent, by reference to the diagrams for the construction of the refracted wave-fronts, that the width of the beam is increased in a direction parallel to the plane of incidence, when the rays pass

from a rare to a dense medium, or a given portion of the wave-front is spread out over a larger surface. In passing from a dense to a rare medium the reverse is the case, the wave-front being compressed into a smaller area. This change in the width of the beam diminishes or enlarges (in one direction) the apparent size of objects seen under these conditions, a matter which will be more completely investigated later on.

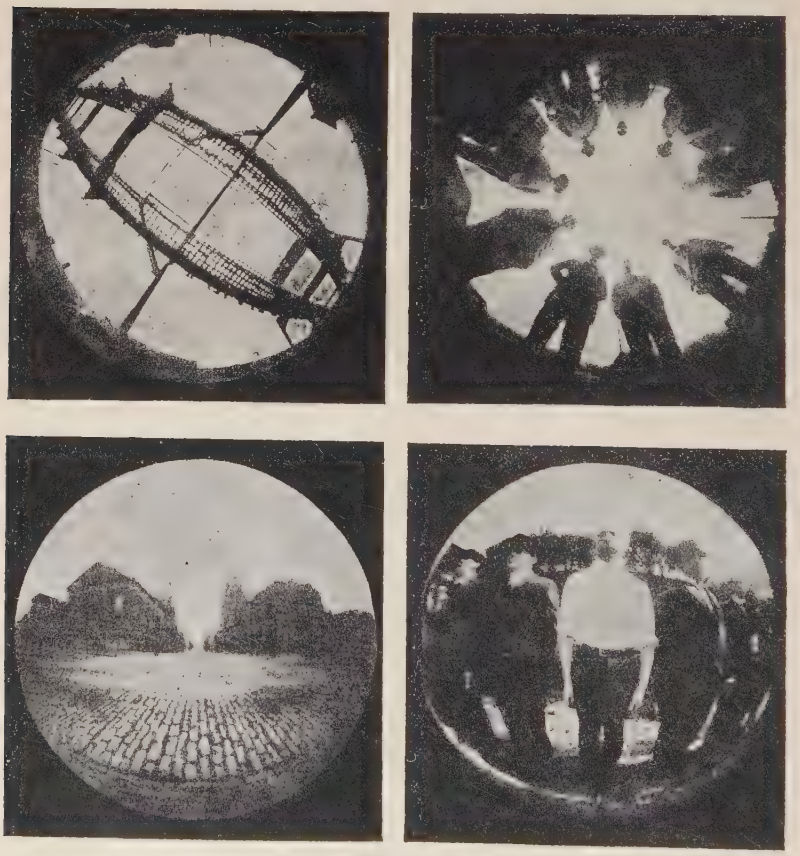


Fig. 59.

Refraction of a Wave by a Plane Parallel Plate. — The application of Huygens's construction to the passage of a plane-wave through a glass plate bounded by parallel planes, shows at once that the emergent wave-front is parallel to the incident, no matter how great the angle of incidence. The direction of the ray is therefore unchanged, though each individual ray is shifted to one side by its passage through the plate. Inasmuch as the position of an object at a great distance depends solely on the direction of the parallel

rays reaching the eye, it will not be changed by the interposition of a thick plate, at any angle. We can test this by viewing a very distant object through a thick piece of plate glass and turning the plate rapidly around a vertical axis to the right and left. Objects near the plate, however, will be found to shift their apparent position considerably as the plate is turned. If the two objects and the eye be in the same straight line it may seem at first sight as if the intervention of the oblique plate would in no way affect their apparent positions, for parallel rays from the distant object are unchanged in direction by passage through the plate, and the same is true of the rays from the near object. If, however, we remember that the ray is shifted laterally, the difficulty disappears, for the lateral shift, while it does not alter the apparent position of an object at infinity, displaces an object situated at a finite distance. This will be made clear by reference to Fig. 60.

Let A be a point not far from the plate. It is seen by an eye at E by means of the rays pursuing the path $ABCE$, and its apparent position is A^1. If the oblique plate be removed, the point A will be seen by the direct pencil of rays AE, and will appear in its true position A.

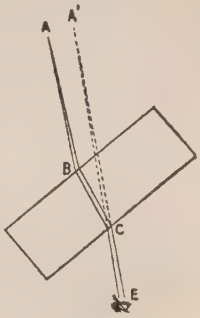


FIG. 60.

If we make the same construction for parallel rays coming from a distant point we shall find that the apparent position, or the *direction* from which the rays by which it is seen come, is unchanged.

The Opthalmometer. — This principle is made use of in the opthalmometer, an instrument devised by Helmholtz for determining the curvature of the lens of the eye, by measuring the diameter of the image of a source of light seen reflected from the curved surface.

The instrument enables us to measure the distance between two points, or the diameter of an object, without taking into account its distance, by an optical method.

It consists of a small telescope with two plane parallel thick glass plates in front of the objective, arranged so as to rotate about a common axis, the angle of inclination between the plates being measured by a graduated circle. If the object is at a great distance rotation of the plates produces no effect of course. If at a small distance we see it doubled as soon as the plates are inclined, and by setting the plates in such a position that the two images touch each other end to end, and reading off the inclination we can determine the length l of the object by the formula

$$l = 2a \sin \Phi \frac{\sqrt{n^2 - \sin^2 \Phi} - \cos \Phi}{\sqrt{n^2 - \sin^2 \Phi}},$$

in which a is the thickness of the plates, n their refractive index, and Φ the angle through which each plate is turned from the zero position (plates parallel).

As will be seen, the distance of the object does not come in at all. As we bring the object nearer, it appears larger, but the angular shift necessary to produce a given displacement increases at the same rate. In other words there is compensation. The instrument can be used only for measuring objects at a moderate distance.

Refraction of a Spherical Wave at a Plane Surface. — Suppose a spherical wave originating at O (Fig. 61) to be refracted at the plane surface AB. If we construct the refracted wave-front by the method of Huygens, making the points on the refracting surface the centres of secondary wavelets whose radii are found by dividing their distances (measured along rays) from the wave in its unrefracted position, by the refractive index of the medium, we shall find that the incident wave is flattened down into what at first sight appears to be a sphere of less curvature. Let us investigate the form of the refracted wave, which is sometimes erroneously stated to be an hyperboloid.

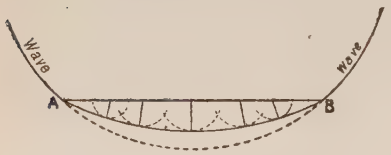


Fig. 61.

Suppose light diverging from O to be refracted at the surface AP (Fig. 62). Draw an incident ray OP, which is refracted in the direction PQ. Draw $OD \perp$ to the surface, and produce it to O', making $OD = DO'$. Draw a circle passing through the points O, P, and O', and produce PQ backwards until it intersects the circle at M, and the prolongation of OO' at J. $\angle DOP$ = incidence $\angle$, also $O'MP$, both being measured by arc $O'P$. Moreover $\angle OMJ$ = incidence $\angle$, being equal to $O'MP$, because $OMO' = OPO'$ (measured by $\frac{1}{2}$ arc OAO') and $OPO' = \pi - 2i$. We can now write

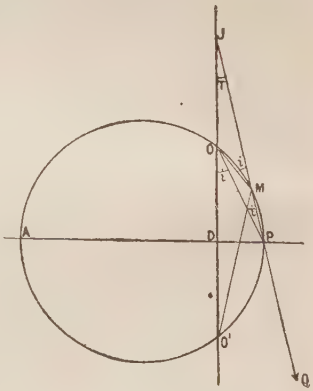


Fig. 62.

$$\frac{\sin i}{\sin r} = \frac{O'J}{O'M},$$

since $\sin O'MJ = \sin i$ (supplementary angles) and the sides of $\triangle$ in ratio of sines of opposite angles.

$$\therefore \mu = \frac{O'J}{O'M} = \frac{OJ}{OM};$$

$$\therefore \mu = \frac{O'J - OJ}{O'M - OM} = \frac{OO'}{O'M - OM} \text{ and } O'M - OM = \frac{OO'}{\mu} = \text{Constant.}$$

This same relation holds no matter where P be taken, conse-

quently the locus of M is an hyperbola having O and O' for foci, and the refracted ray PQ is normal to the hyperbola at M. The hyperbola is therefore the orthogonal surface of the refracted wave, since it is everywhere normal to the refracted rays, and the refracted wave-fronts are parallel curves, located by measuring off equal distances on the rays from the hyperbola. They will *not* be themselves hyperbolae, for the parallels to a conic are in general curves of the eighth degree. The evolute of the hyperbola is the caustic of the refracted wave, in this case virtual of course. After refraction, then, the different elements of the wave-front appear to come from points distributed along the caustic. If then we transfer our eyes from one position of the wave-front to another, the position of the radiant point in space will apparently alter. The same thing is true when the waves are refracted from a dense to a rare medium, the caustic in this case being the evolute of an ellipse, and the refracted waves parallels of an ellipse. The formation of a caustic under these conditions is shown in Fig. 63.

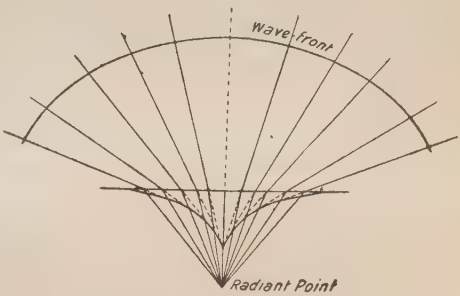


FIG. 63.

A small portion of the wave around the ray leaving the surface normally comes from the cusp of the caustic, which we may regard as a point, consequently this portion of the wave is approximately spherical.

The cusp of the caustic from which this portion comes is elevated above the true radiant point, consequently the refraction appears to bring the point nearer the eye. The bottom of a vessel of water consequently appears to be nearer than it really is.

The apparent elevation of the bottom of a body of still water is a matter of common observation. It is most marked when the eye is only a little above the plane of the surface and the bottom at a considerable distance is under observation. The rays which leave the surface at nearly grazing emergence come from that portion of the caustic which is very near the surface, as is apparent from Fig. 63. The bottom at a distance may thus appear elevated almost to the surface.

In the case just considered the caustic is virtual and not real, and since it is in reality non-existent, there are no moving cusps on the wave-front as in the cases considered under reflection. If we could reverse the emergent wave, and at the same time remove the water, the caustic would become real, and cusps would develop upon the wave-front. This cannot of course be done experimentally, but attention is drawn to it in order to clearly define the difference between a real and virtual caustic.

Refractive Index of a Plate measured by the Microscope. — This apparent elevation of a point due to the decrease in the radius

of curvature of the wave-front when it emerges into the air can be used for measuring the refractive index of a glass plate, of which we know the thickness d. An object seen through the plate appears nearer by the amount $a=\frac{d(n-1)}{n}$.

If we focus a microscope upon an object, and then place the glass plate over it, we shall have to raise the microscope through a distance a to bring the object into focus. The refractive index is then given by $n=\frac{d}{d-a}$. We should use an objective of as short focus as possible, consistent with its use with the plate. It is best to provide the eye-piece with cross hairs and focus by absence of parallax, *i.e.* so that there is no relative motion between the object and the cross hair as the eye is moved from side to side.

Two modifications of the method may be cited. Make a mark on the upper and lower surface. To change from one to the other the microscope must be raised a distance a: then $n=\frac{d}{a}$. Make a small dot with white paint on the upper surface and illuminate it from above, on a dark background. Focus first on the object and then on its image seen reflected from the lower surface, moving the microscope a distance a: then $n=\frac{2d}{a}$. The values are correct to the third place of the decimal if the observations are made with great care.

Fermat's Law. — In the case of reflection we have seen that the path of a ray from one point to another by way of a reflecting surface is either a maximum or a minimum. The same is true in the case of refraction, as we shall now show. If the refracting surface is plane, the time of transit is a minimum, and we have what is known as the principal of least time. If the refracting surface is curved, the time may be either a maximum or a minimum, according to whether the refracting surface lies within or without the aplanatic surface, the same as in the case of reflection. Fermat's law may be deduced from Snell's law by the maximum and minimum method of the calculus.

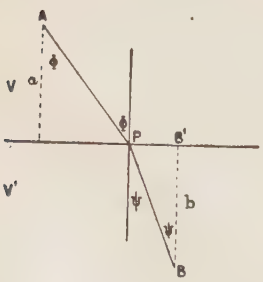


FIG. 64.

Let A be a luminous point at height a above the refracting surface $A'B'$, and B a point illuminated by a disturbance reaching it by way of any point on the refracting surface, for example, over the path APB (Fig. 64). The time of transit obviously changes with the position of P. We will start by assuming it to be a maximum or minimum, and see if the ordinary law of refraction follows.

From A and B drop perpendiculars on the refracting surface, of length a and b respectively: let the distance $A'B'=p$, then $A'P=x$ and $PB'=p-x$. Call the velocities in the two media v and v',

then the time along AP is $\frac{AP}{v}$ and the time along PB is $\frac{PB}{v'}$. The whole time, which we require to be either a maximum or a minimum,

$$t = \frac{AP}{v} + \frac{PB}{v'}, \text{ or } t = \frac{\sqrt{a^2+x^2}}{v} + \frac{\sqrt{b^2+(p-x)^2}}{v},$$

$$\frac{dt}{dx} = \frac{x}{v\sqrt{a^2+x^2}} - \frac{p-x}{v'\sqrt{b^2+(p-x)^2}} = 0.$$

Now $\frac{x}{\sqrt{a^2+x^2}} = \sin\Phi$, and $\frac{p-x}{\sqrt{b^2+(p-x)^2}} = \sin\Psi$;

$\therefore \frac{\sin\Phi}{v} = \frac{\sin\Psi}{v'}$, or $\frac{\sin\Phi}{\sin\Psi} = \frac{v}{v'} = \text{Const.}$, which is Snell's law.

A second differentiation, or in this case a mere inspection of the figure, shows us that the time is a minimum.

We will now examine the case of refraction by a curved surface, following a demonstration by Czapski.

In Fig. 65 let AB be a portion of an aplanatic refracting surface. Though we have not yet discussed this surface, we can make use of it in the present case. It is the surface which will bring all rays emanating from O accurately together at O', or change the expanding spherical waves into contracting spheres with centres at O'. The times of transit over all paths from O to O' by way of the aplanatic surface (a surface capable of bringing rays together at a point) are equal, or if n and n' be the refractive indexes of the media the reduced path is $n(OP)+n'(PO') = \text{Constant}$. (By reduced path we mean the length of path in vacuo, which will contain the same number of waves as the number contained in the actual path.)

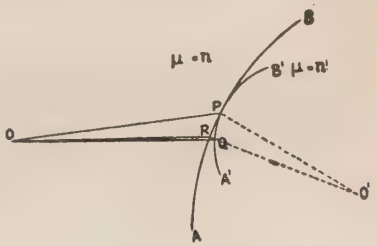


FIG. 65.

Now suppose that we have a refracting surface $A'B'$ of greater curvature than the aplanatic, which it touches at P. The ray incident at P is obviously the one which passes through O', and we are to ascertain whether the reduced path is greater or less than any hypothetical path through some other point on AB. Let this point be at Q, for which the reduced path will be $n(OQ)+n'(QO')$. The ray which reaches O' by way of the point R on the aplanatic surface has a reduced path $n(OR)+n'RQ)+n'(QO')$ and the difference between them is

$$[n(OQ)+n'(QO')]-[n(OR)+n'(RQ)+n'(QO')],$$

or

$$n(OQ-OR)-n'(RQ).$$

Now $(OQ-OR)<RQ$ (sides of a triangle).

Therefore, since $n<n'$, $n(OQ-OR)<n'RQ$, and the path by way

of Q is less than the path by way of R; but the latter is the same as the *actual* path by way of P, therefore that path is a maximum. In the same way we may show that if the refracting surface has a curvature less than the aplanatic, *i.e.* lies without it, the path will be a minimum.

Refraction by a Prism. — In the case of refraction by a prism we have to determine the deviation of a ray or wave-front, by passage through a medium bounded by two planes which make an angle with each other; this angle is called the angle of the prism. If the refractive index of the prism be greater than that of the medium in which it is immersed, as is usually the case, the deviation of the ray is always away from the vertex, that is, towards the base of the prism. This is obvious at first sight, except perhaps in the case in which the incident ray falls on the prism in the direction shown in Fig. 66, for here the deviation at the first surface is *towards* the vertex, while that at the second is towards the base, the final direction depending on the relative magnitude of these two deviations. The angle of refraction at the second surface is greater than that at the first, and since the deviation increases as this angle increases, the deviation towards the base at the second surface is greater than the deviation towards the apex at the first.

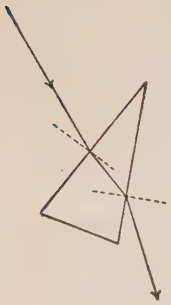


Fig. 66.

We will now derive an expression for the deviation. Let the angle of the prism be α (Fig. 67) and let i and r be the angles of incidence and refraction at the first surface, r' and i' at the second. The deviation is obviously D, the angle between the emergent ray and the incident ray. The deviation at the first surface is $i-r$, at the second $i'-r'$, while

$$D=(i-r)+(i'-r')$$
$$=i+i'-(r+r').$$

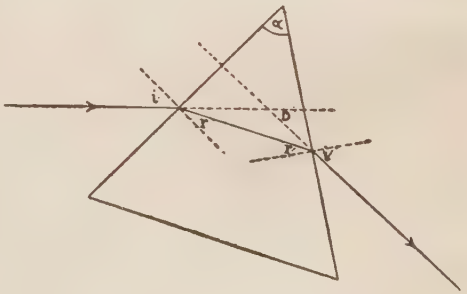


Fig. 67.

But $r+r'=\alpha$, since α + the two base angles of the prism = 2 rt. $\angle$s and $(r+r')$ + the base angles = 2 right angles.

$$\therefore D=i+i'-\alpha.$$

This formula holds for the condition shown in the previous figure, except that in this case the negative sign must be prefixed to the angles i and r. Suppose the angle α becomes zero, the prism then becoming a plane parallel plate. The deviation then becomes $i(-i')-0$, which is equal to zero (since in this case $i=i'$).

Refractive Index of a Prism. — Let us now suppose the angle of incidence to be such that the ray passes through the prism parallel to the base. In this case $i=i'$ and $r=r'$, and if we can measure D and know the angle α we can easily determine the refractive index of the prism.

We have $D=2i-\alpha$, or $i=\dfrac{\alpha+D}{2}$, also $r=\dfrac{\alpha}{2}$.

Substituting these values in $\mu=\dfrac{\sin i}{\sin r}$ we have $\mu=\dfrac{\sin \frac{1}{2}(\alpha+D)}{\sin \frac{1}{2}\alpha}$.

We must now find some method of arranging the angle of incidence so that the path of the ray through the prism will be parallel to the base, *since it is only for this condition that the above formula holds.*

This adjustment is very easily made, for the deviation of the ray can be shown to be a minimum when the passage through the prism is symmetrical. There are several methods of proving this, the most direct and rational being the method of maxima and minima of the calculus. We must obtain an expression showing the change of deviation with the change of the angle r, and by equating this to zero derive the condition for a maximum or minimum. In other words

$$\frac{dD}{dr}=0 \text{ and } \frac{d^2D}{dr^2}>0 \text{ (condition for minimum)},$$

$$\frac{dD}{dr}=\frac{d(i+i'-\alpha)}{dr}=0.$$

We have $\sin i=\mu \sin r$, and $\sin i'=\mu \sin r'=\mu \sin(\alpha-r)$.

Then $i=\sin^{-1}(\mu \sin r)$ and $i'=\sin^{-1}[\mu \sin(\alpha-r)]$;

$$\therefore D=\sin^{-1}(\mu \sin r)+\sin^{-1}[\mu \sin(\alpha-r)]-\alpha,$$

$$\frac{dD}{dr}=\frac{\mu \cos r}{[1-\mu^2 \sin^2 r]^{\frac{1}{2}}}-\frac{\mu \cos(\alpha-r)}{1-\mu^2 \sin^2(\alpha-r]^{\frac{1}{2}}}=0.$$

Expressing the cos by the sin we have

$$\frac{\mu(1-\sin^2 r)^{\frac{1}{2}}}{[1-\mu^2 \sin^2 r]^{\frac{1}{2}}}-\frac{\mu[1-\sin^2(\alpha-r)]^{\frac{1}{2}}}{[1-\mu^2 \sin^2(\alpha-r)]^{\frac{1}{2}}}=0.$$

Equating the above terms, multiplying the numerator by the denominator and cancelling, gives us,

$$(\mu^2-1)\sin^2 r=(\mu^2-1)\sin^2(\alpha-r),$$

$$\text{or } r=\alpha-r \text{ and } r=\frac{\alpha}{2}=r'.$$

By symmetrical passage $r=r'$; $\therefore$ D is either a maximum or minimum. A second differentiation gives

$$\frac{d^2D}{dr^2}=\frac{\mu^2-1}{\left(1-\mu^2 \sin^2\frac{\alpha}{2}\right)\left(1-\mu^2 \sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}}$$

If $\mu > 1$ all the factors are positive and the whole expression is positive, therefore D is a minimum.

There is another condition for which we can get a simple expression for μ, namely: when either the incident or emergent ray is normal to the surface of the prism.

If $i = 0$, $r = 0$, and $r' = \alpha$, $D = i' - \alpha$.

Therefore $$\mu = \frac{\sin (D + \alpha)}{\sin \alpha}.$$

Magnifying Power of Prisms. — When the prism is set at minimum deviation the widths of the incident and emergent beams are the same, otherwise not. For example: in Fig. 68 when the incident beam falls normally on the first surface, and leaves the second surface at a large angle with the normal, the width of the beam has been contracted. If we view an object under these conditions, the eye being placed in the contracted beam, we shall find that it is magnified in the direction in which the beam has suffered contraction. A circular opening in a card backed by a sodium flame is a suitable object, and will be found to appear as an ellipse. If an achromatic prism is available a circular white object can be used, when the effect is very striking. If on the other hand the incident light makes a large angle with the normal, the emergent wave-front is expanded in width, and, if the eye be placed in it, the object will appear decreased in size in this dimension, a circular card appearing as if turned edgewise. Brewster suggested that by using two achromatic prisms at right angles to each other, magnification might be shown in both directions, and the action of a telescope imitated.

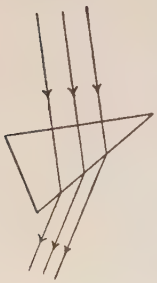


FIG. 68.

Lord Rayleigh has given a very neat demonstration of magnifying power, based on Fermat's law, which is applicable to telescopes as well as to the case just cited. It proves by a wave-front method that the contraction of a beam of light, or the compression of a wave-front, causes magnification. Consider a wave-front of width AB (Fig. 69) refracted at the surface CP, and compressed thereby to width $A'B'$. By Fermat's law the time of transit over the path ACA' is equal to the time of transit over the path BPB', being a minimum in each case. This we may express by saying that $\int \mu ds$ (the reduced path) is the same along each ray. If from any cause B is retarded relatively to A, say an amount BE, B' will be retarded an equal amount relatively to A', namely $B'E' = BE$.[1] If this retardation be considered as

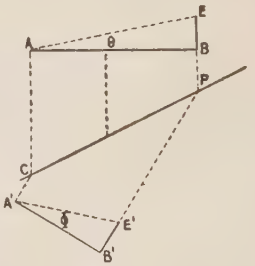


FIG. 69.

[1] This is of course only true if AB and $A'B'$ are in the same medium: the first surface of the prism, parallel to AB, is not represented.

represented by a rotation of the wave-front AB through angle θ it will be measured by $(AB)\theta$. The wave-fronts AB in the two positions can be thought of as two separate fronts coming from two distant stars subtending an angle θ at the point of observation. The retardation of B' must be of the same amount, consequently the rotation of the wave-front $A'B'$ will be much greater than θ, being measured by $\Phi(A'B')$. Since the retardations are equal we can write

$$\theta(AB) = \Phi(A'B') \text{ or } \frac{\theta}{\Phi} = \frac{A'B'}{AB}.$$

Now Φ is the angle formed by the rotation of $A'B'$ the compressed wave-front, consequently we may regard it in its two positions as two fronts coming from stars which subtend an angle Φ, as much greater than θ as AB is greater than $A'B'$.

The same reasoning can be applied to telescopes, the compression here being symmetrical, a plane-wave of large area emerging from the eye-piece as a plane-wave of small area, the magnifying power being equal to the *ratio of the widths of the stream of light before and after entering the telescope.*

Refraction by a Lens. — In the chapter on reflection it has been shown that a parabolic mirror transforms a plane-wave into a contracting spherical wave, while an ellipsoidal mirror exerts the same action on spherical waves originating at one of the foci. It is possible to construct refracting surfaces having the same property. We will begin by computing the refracting surface, which shall be aplanatic for spherical waves.

Let O (Fig. 70) be the luminous point, and O' the conjugate focus where the converging waves are to shrink to a point. By Fermat's principle the reduced paths along the different rays will be equal, and the disturbances will all reach O' in the same phase, resulting in intense illumination. Suppose O to lie in a medium of ref. index 1, practically in air, while O' is in a medium of ref. index 2 between which we require an aplanatic surface of separation.

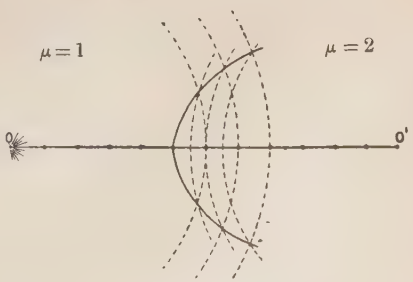


Fig. 70.

Let r and r' be the distances of any point on the surface from O and O', then $r + \mu r' = \text{Constant}$, the equation of a Cartesian oval. Choose a point on the line joining O and O' such that $r = 5$ and $r' = 7$. The constant for this particular case will be 19. Now describe around O a circle of radius 6 and around O' a circle of radius of $\frac{19-6}{2}$. The intersection of these circles will give two more points on the aplanatic surface, which can be gradually built up by giving to r constantly increasing values.

The general form of the equation of a Cartesian oval is

$$\mu r + \mu' r = \text{Constant, from which we get } \mu \frac{dr}{ds} + \mu' \frac{dr}{ds} = 0.$$

In the case just considered the conjugate foci lie in different media. If they are to be in the same medium we require an intervening medium capable of effecting the required change in the form of the wave-front. We thus come to the Aplanatic Lens.

Spherical light waves, originating at a point in air, are to be transformed by a lens into converging spheres which come to a focus at another point also situated in air. Suppose the lens to be midway between the two points and the curvature of its two surfaces the same. The spherical wave will be changed into a plane-wave by the first surface, consequently the simplest way to construct the form of the lens will be to trace the surface aplanatic for a plane-wave by the method given above, modifying it, however, in such a way as to make the sum of any path measured along a ray from the plane-wave to the aplanatic surface, and the reduced path from this point to the focus, a constant. These aplanatic surfaces are, however, of very little practical importance, for they can only be reproduced approximately, and then only when the departure from a spherical surface is very slight. In the process of lens making the surfaces which are being ground together assume of their own accord a spherical form, since two surfaces, to fit together in all positions, must be of constant curvature. Lenses with spherical surfaces do not bring rays accurately to a point, or in other words do not give converging waves which are truly spherical. This results in what is known as spherical aberration, which has been treated sufficiently for the purposes of this book under *reflection*. A spherical surface may be made approximately aplanatic by local grinding, if the amount of material to be removed be not too great. This process is known as correcting the lens for spherical aberration, or figuring, and is largely a cut and try operation.

Refraction by Sphere. — There is one special case in which the spherical lens is aplanatic, which is made use of in the construction of the microscope. Weierstrasse gives this simple method of constructing the refracted rays when incident upon a sphere.

Suppose a sphere of refractive index n' and radius r immersed in a medium of refractive index n. If the sphere is in air, $n=1$.

Describe a circle of radius r representing the sphere, and around its centre two other circles of radii $\frac{n'}{n}r$ and $\frac{n}{n'}r$, respectively, as in Fig. 71. Draw the ray LE incident upon the sphere at E, at incidence angle α, and continue it until it cuts the outer circle at A: then join this point with the centre, the line cutting the inner circle at A'. The line EA' is then the refracted ray. If now we have a convergent system of rays falling upon the sphere, which would unite at A if the sphere were absent, it is clear from the construction that the sphere will bring them all together in a point focus at A'. Conversely if rays emanate from A' within the sphere, they will, after refraction out through the lower half of the

sphere, traverse paths which, if produced backwards, meet at A. A and A' are called the aplanatic points of the sphere, and the sphere is an aplanatic surface for these two points, one of which is real, the other virtual.

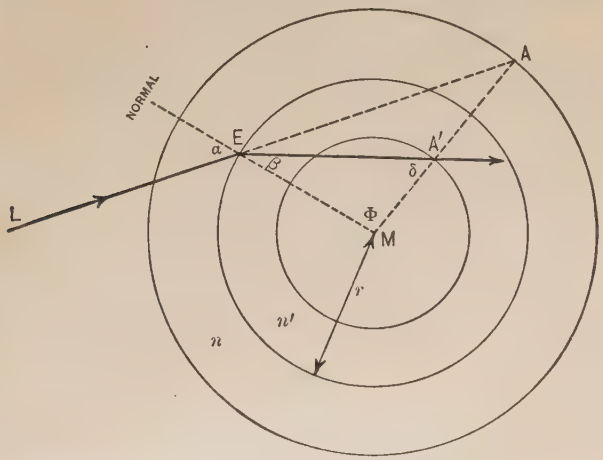


FIG. 71.

The proof of the above construction is as follows.

Triangles EMA and EMA' are similar having $\angle\Phi$ in common, and its enclosing sides in equal ratio

$$\frac{EM}{MA'}=\frac{AM}{EM}=\frac{n'}{n}.$$

$$\therefore \angle EA'M=\angle AEM,$$

and since $$AEM=\alpha,\ \delta=\alpha.$$

In triangle EMA'

$$\frac{\sin\delta}{\sin\beta}=\frac{EM}{A'M}=\frac{n'}{n}.$$

$$\therefore \frac{\sin\alpha}{\sin\beta}=\frac{n'}{n}.$$

Microscope Objective. — The existence of the aplanatic points just proved was utilized by Amici in the construction of microscope objectives of wide aperture. A section of such an objective is shown in Fig. 72.

A hemispherical lens I receives the wide cone of rays from a point at L. After refraction by the plane surface, they pursue directions as if coming originally from L^1. If L^1 is the aplanatic point refraction by the spherical surface will render them still less divergent without introducing any spherical aberration. Since refraction at a plane surface introduces aberration, better conditions obtain if a drop of some oil having the same refractive index as the glass is introduced between the plane surface and the object,

which must now be at L^1, the aplanatic point. This virtually imbeds the object within the medium of the sphere, and the wave leaving it has its centre at L^2. The next lens II is a meniscus, its

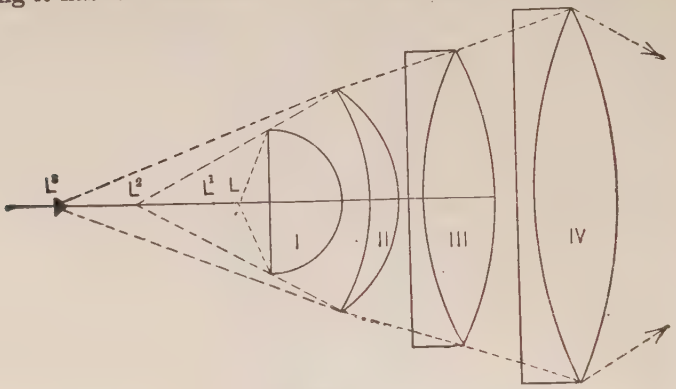


Fig. 72.

first or concave surface having the same radius of curvature as the spherical wave leaving lens I, *i.e.* its centre of curvature is at L^2. The wave therefore enters lens II without change of form. The second surface of the meniscus is spherical and of such a curvature that L^2 is an aplanatic point for it. The wave on leaving the meniscus is still spherical, the centre being now pushed back to L^3. The rays are rendered still less divergent and finally convergent by the two achromatic lenses III and IV.

The Amici principle has a disadvantage which makes it impossible to use it more than twice. It introduces chromatic aberration, which can be compensated by the overcorrected (for color) achromatic lenses III and IV provided the divergence is not too small. If it is too small, as it would be if the principle was made use of again, it would be impossible to compensate it, and at the same time render the rays convergent.

Focal Length of a Lens. — The distance from the middle of the lens to the focus varies with the distance of the source of light: if it be at infinity the waves are plane and the distance between the lens and the focus is called the focal length of the lens. This may be expressed in terms of the radii of curvature of the two surfaces and the refractive index of the glass, and the formula which we will now deduce can be used for determining the refractive index of a lens.

Let μ = ref. index of lens, r = radius of surface AMB, s = radius of surface ANB and $y = AD$ (Fig. 73).

All disturbances from O reach O' at the same time;

$$\therefore OA + O'A = OM + \mu(MN) + NO',$$
$$y^2 = 2rMD - MD^2 = 2sND - ND^2.$$

(MD^2 and ND^2 can be rejected since MD and ND are small in comparison to r and s.)

$$\therefore MD = \frac{y^2}{2r},\ ND = \frac{y^2}{2s};$$

$$\therefore MN = \frac{y^2}{2r} + \frac{y^2}{2s} = \frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right). \quad . \ . \ . \ . \quad (1)$$

Denote OD by u and $O'D$ by v, and we have

$$OA = u + \frac{y^2}{2u} \text{ and } O'A = v + \frac{y^2}{2v} \text{ (approximately);}$$

$$\therefore OA + O'A = OO' + \frac{y^2}{2}\left(\frac{1}{u} + \frac{1}{v}\right). \quad . \ . \ . \ . \quad (2)$$

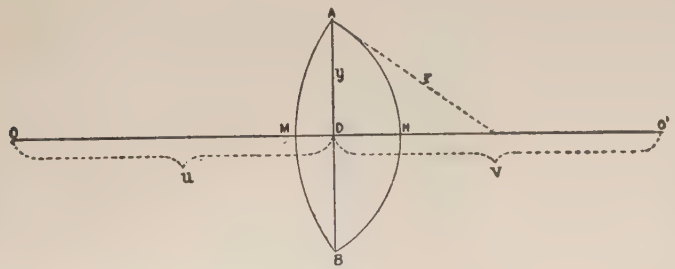


Fig. 73.

But $OA + O'A = OM + O'N + \mu(MN)$, and substituting from (2) we get

$$(\mu - 1)MN = \frac{y^2}{2}\left(\frac{1}{u} + \frac{1}{v}\right).$$

Substituting for MN the value given in (1),

$$\frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right)(\mu - 1) = \frac{y^2}{2}\left(\frac{1}{u} + \frac{1}{v}\right),$$

$$\frac{1}{u} + \frac{1}{v} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) = \frac{1}{f},$$

where f is the value of v when $u = \infty$, that is when the incident rays are parallel. The focal length of the lens is therefore f. Measure the radii of curvature of the two surfaces of a lens and its focal length, and determine μ.

Refraction of Light in Non-Homogeneous Media. — The consideration of the laws of refraction in media in which the refractive index varies continuously from point to point leads us to a most interesting class of phenomena, the most common examples of which are the illusions known as Mirages.

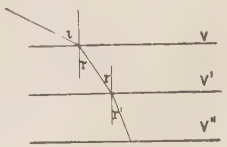


Fig. 74.

As an introduction to the subject it will be well to investigate the refraction of a ray of light by a number of media of different refractive indices arranged in horizontal strata of equal thickness. Let the velocities of light in the different strata be represented by v, v', and v'', and let the angle of incidence on the first layer be i (Fig. 74) — we require the deviation of the ray by the two boundaries.

We have $\frac{\sin i}{\sin r} = \frac{v}{v'}$ for first boundary. $\therefore \sin r = \sin i \frac{v'}{v}$.

The incidence angle at the second boundary is obviously r, therefore we have

$$\frac{\sin r}{\sin r'} = \frac{v'}{v''} \text{ or } \frac{\sin i \frac{v'}{v}}{\sin r'} = \frac{v'}{v''}, \text{ which gives } \frac{\sin i}{\sin r'} = \frac{v}{v''},$$

showing that the direction of the ray in the third medium is the same as if the intervening medium were not present. Now suppose the number of layers to be increased indefinitely, and the thickness of each to be reduced indefinitely. This gives us a medium of continuously varying refractive index, and we see that the direction of the ray at any point is the same as if the upper layers were removed, and the ray entered the flat surface of a medium of refractive index equal to that which the non-homogeneous medium has at the point in question. Suppose a ray to be travelling in a horizontal direction in a medium of this nature. As the ray is moving in a direction in which the refractive index does not change, it may seem at first sight as if there would be no change of direction. The discussion of the case by ray methods would lead to this conclusion, a result which plainly shows the danger of handling optical problems in this way. No matter how limited the width of the ray, the wave-front, — the motion of which constitutes the rays, — must have a finite size, and the upper and lower edges of the front are moving in regions of different optical density. The upper edge will consequently move faster than the lower, and the front will gradually wheel around, which means that the direction of propagation, or the direction of the ray, is constantly changing. We can treat the case by Huygens's construction by describing secondary wavelets of constantly decreasing radii around points on the wave-front, the enveloping plane representing the front in its next position (Fig. 75). By repeating the process we can show the gradual change of direction. The resulting curved rays are concave towards the direction of higher refractive index.

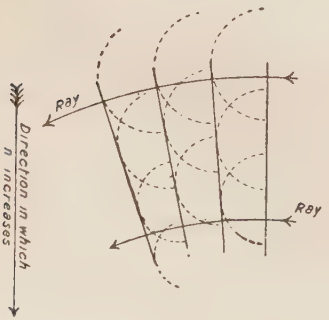


Fig. 75.

Astronomical Refraction. — The optical density or refractive index of the earth's atmosphere decreases as we ascend from the surface, consequently the rays of light, which reach our eyes from the stars, move in curved paths, except when the star is in the zenith. Since the direction in which the star appears to be is the direction from which the ray comes when it enters the eye, the true position of the star can only be determined by taking the refraction

of the atmosphere into account. The effect of refraction is to make the star appear higher up above the horizon, or nearer the zenith than it really is. For stars at the horizon the elevation amounts to 36 minutes of arc.

Now we have seen that the final direction of the ray is independent of the layers intervening between the medium in which the observation is made and the region from which the light comes; it is therefore apparent that the change in direction can be determined by determining the refractive index of the air at the point where the instrument is situated, which can be done by observing its temperature, pressure, etc.

The curvature of light rays in the atmosphere also influences the apparent positions of objects on the earth's surface, the usual effect being an elevation of an object above its true position, a circumstance which must be taken into account in all geodetic observations. As a result of this refraction it is possible to see the sun and the eclipsed moon above the horizon at the same time.

While the radius of curvature of a ray of light travelling parallel to the earth's surface is much greater (about 7 times) than the radius of the earth, it is possible to conceive of an atmosphere with a density gradient sufficient to lessen the radius of the ray to that of the planet. If such a condition prevailed a ray would travel completely around the planet, if the atmosphere were perfectly transparent. In some cases we may even have an atmosphere with a density gradient sufficient to give us an even smaller radius of curvature.

Schmidt's Theory of the Sun. — Schmidt has made the suggestion that the sun may be such a body, and that the disk as we see it may be an optical illusion. He considers the sun to be a mass of gas the density of which increases from the surface towards the centre. At a sufficient depth the radiation of the gas will be "white" light, *i.e.* it will give a continuous spectrum. What we shall see, however, will be a white-hot ball with a sharply defined rim. The radius of the ball will be the radius of the sphere taken within the gas mass, upon the surface of which light rays will have a similar radius of curvature. Schmidt calls this the critical sphere.

In Fig. 76 consider the circle to be the critical sphere, and assume light radiated in all directions from a point A deep down within the gas mass. The ray B, with a small radius, will turn back into the mass; the ray D, leaving A at a smaller angle with the normal will travel around the critical sphere; while the ray E, leaving A at a slightly less angle, will pass off into space. Other rays, such as C, will also pass off into space, but will not reach us. If the gas without the critical sphere does not emit light the ray E will come apparently from the edge of the critical sphere, notwithstanding the fact that it originated much deeper down in the mass, where the gas is radiating light as a result of its high temperature and density. On this theory what we call the diameter of the sun is merely the diameter of the critical sphere, plus a slight increase due to the refraction of the gas outside of it. An atmosphere can

in the same way cause an increase in the apparent diameter of the body which it surrounds. We can show this very nicely in the following way. Make a small rectangular glass tank by cementing five squares of glass together with sealing wax. Fill it with melted gelatine and support an empty test tube in the fluid with a clamp stand. The bottom of the test tube should be within half a centimeter of the bottom. After the jelly has solidified, pour hot water into the test tube, and immediately withdraw it. It will leave a cylindrical hole in the jelly, with a hemispherical bottom. Now pour a mixture of glycerine and powdered chalk into

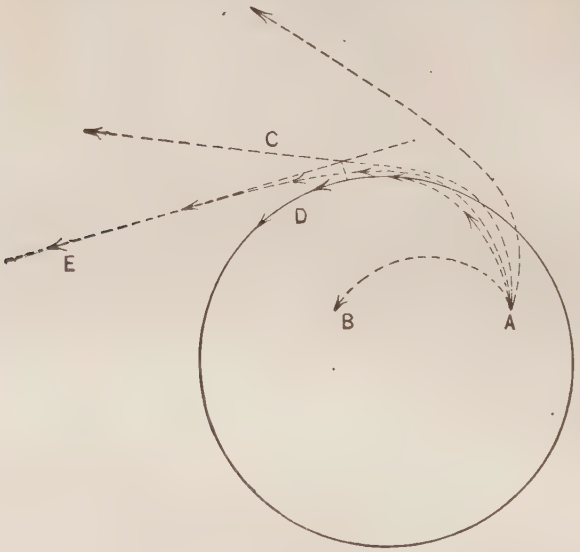


FIG. 76.

the cavity until it is half full. Fill the remainder with water to which a few drops of milk have been added. The glycerine will gradually diffuse into the gelatine, increasing its refractive index. The condition at the end of a few minutes will be not unlike that of a white body surrounded by a dense atmosphere, for the refractive index will be high at the boundary between the jelly and glycerine, gradually decreasing as we pass out into the jelly. The magnification resulting can be seen by looking through the side of the trough, the lower portion of the cavity appearing swollen out like a mushroom. If we perform the experiment with pure glycerine and clean water the same thing happens. By placing an arc light behind the tank and throwing an image of the cavity upon a piece of ground glass with a camera objective, placed at the centre of the shadow of the tank, we can see the bright ring of light which appears to surround the bottom of the cavity. This is analogous to the ring of light which would be seen surrounding the earth by an observer on the moon during a lunar eclipse, or rather a solar eclipse. As the glycerine penetrates into the jelly this ring of

light eventually separates from the line of the cavity. Photographs of the gelatine cavity at two different stages of the diffusion are reproduced in Fig. 77.

Fig. 77.

Mirage. — The normal variation of the refractive index in the atmosphere is often disturbed by temperature variations, as when the air near the surface is warmed by the heated ground, or when a layer of cold air flows over a layer of warm air, as may occasionally happen. These abnormal conditions in the atmosphere give rise to the phenomenon of mirage, the commonest type being that seen on the desert where the air is heated by the hot sand. In this case the refractive index is abnormally low along the ground, rises to a maximum as we ascend, and then decreases more slowly according to the usual law. Rays of light near the surface are therefore concave *upwards*, while those travelling at greater elevations are concave *downwards*. The result of this is, that rays which would ordinarily strike the ground are turned upwards and reach the eye, appearing as if reflected in the ground, while other rays starting originally from the same point may reach the eye by the usual path. The point thus appears double. The sky at the horizon may thus appear as if mirrored in the sand, and since the only reflecting body in nature capable of acting in this manner with which we are familiar is a smooth sheet of water, the natural inference is that a

Fig. 78.

lake exists between us and the horizon. Where the sky is broken by mountains, we see their inverted images mirrored. The paths of the rays in mirage of this type are shown in Fig. 78.

It can be very beautifully reproduced by a method which was described by the author in the *Philosophical Magazine* in 1899, and which has since been somewhat improved. Three slabs of flat sheet steel each a metre long, 20 cms. wide and 3 or 4 mms. thick, are mounted on iron tripods and carefully brought into the same

plane, so that the upper surface is continuous and flat, which can be ascertained by "sighting" it from one end (Fig. 79). The surface is sprinkled with sand, to prevent reflection, which may occur at grazing incidence. A sheet of ground glass with an arc-light behind it represents the sky, or a mirror mounted so as to reflect the sky when viewed from the opposite end of the desert. The artificial sky must come down to the level of the sanded surface, and in front of it a chain of mountains cut out of pasteboard is mounted, with peaks varying from 1 to 2 cms. in height, and valleys which come quite down to the sand. The desert is heated by a long gas burner made by drilling numerous small holes in a long piece of gas pipe. The gas should be introduced at each end of the long tube, and the flames should be about 5 cms. in height. If we look along the sand, holding the eye an inch or two above the plane of the surface, we shall see, as the desert warms up, what appears to be a brilliant pool of water on the sand, in which the inverted images of the mountains and sky appear reflected. Photographs of this artificial mirage are shown in Fig. 80, Plate 3.

Another type of mirage, sometimes seen at sea, can be referred to a stratum of hot air at a considerable distance above the earth's surface, which behaves in a manner precisely similar to the hot air on the ground, giving rise to inverted images of distant ships high above the horizon. Objects ordinarily below the horizon are frequently brought into view, by the curvature of the rays resulting from an abnormally rapid change in the refractive index of the air, a case being on record where ships moored off the French coast across the English Channel 20 miles distant were seen from Dover.

Mirages are frequently seen on cold autumn mornings over large bodies of water, the air in the vicinity of the surface of the water being warmed. A frequent illusion, known as Fata Morgana, is the apparent elevation of objects on a distant shore into pinnacles and columns. It results from a distribution of density similar to that causing the desert mirage, the transition being less abrupt, however. A medium stratified in horizontal layers, with a maximum refractive index along the central plane, will render divergent rays parallel and then convergent, the medium acting as a sort of continuous lens.

Fig. 81 is a somewhat exaggerated diagram of this effect. An eye at O receives rays from O' which have come over a number of different paths, and can be considered as situated at a focus towards which these rays converge. O' will therefore appear magnified in the vertical direction into a column AB. As the curvature of the rays is only in vertical planes there will be no corresponding horizontal magnification. Rocks and other objects lying along the shore are thus seen raised to the dignity of lofty cliffs, and blocks of ice floating in the water appear as white pinnacles.

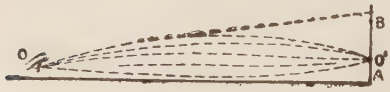


FIG. 81.

Non-Homogeneous Cylinders as Pseudo-Lenses. — If there was a similar variation in the refractive index in horizontal directions,

magnification in all directions would occur. In a medium capable of acting in this way the equi-indical surfaces, or layers, of equal refractive index will be coaxial cylinders, the highest refractive index being along the axis. Exner has shown that the eyes of some insects are arranged in this way, the convergence of the rays to a focus resulting from the action of a non-homogeneous medium.

Cylinders of gelatine soaked in water were found by Exner and Matthiessen to behave in the same way.

It is possible to prepare cylinders which have the maximum refractive index on the surface or along the axis, and act accordingly as concave or convex lenses.

The original method has been improved by the author by the use of glycerine. These pseudo-lenses are not at all difficult to prepare and are extremely interesting. A handful of photographic gelatine is soaked in clean water until thoroughly softened. The excess of water is poured off and the mass is then heated until quite fluid, and filtered through a funnel with a small piece of absorbent cotton placed at the bottom of the cone. If the gelatine refuses to run through, add a little more boiling water. Pour a small quantity into a test tube, and let it stand until solid. Evaporate the remainder over a small flame, stirring constantly until it is of the consistency of syrup. This means boiling it down to one-third or less of its original volume. Now add an equal volume of glycerine, and pour the mixture into a second test tube. After the jellies have set, crack the bottom of the tubes by a sharp blow, warm them by the momentary application of a Bunsen flame and push out the cylinders.

Cut the cylinders into disks of different thicknesses, with a warm pen-knife. The best thickness is about two-thirds of the diameter. Mount the disks between small squares of thin plate-glass (window glass will do), warming the plates slightly, to insure getting the jelly into optical contact (Fig. 82). It may be found necessary to prop the upper plate in position until the surface in contact with the glass has "set." The cylinders which are made of gelatine and water are now to be immersed in glycerine, the glycerine jelly cylinders in cold water. The glycerine should be stirred occasionally, as the layers in contact with the jelly take up the displaced water. The action will be found to be well under way in a quarter of an hour, the glycerine gradually diffusing into the jelly, driving out the water, and the water gradually replacing the glycerine. A jelly containing glycerine has a higher refractive index than one containing water, consequently the cylinders soaked in glycerine act as concave, while those soaked in water act as convex lenses.

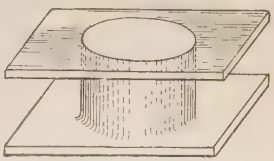


Fig. 82.

The focal length will be found to be only 8 or 10 cms., and very sharp images of the filament of an incandescent lamp or a gas flame can be obtained with them.

Schott has prepared similar cylinders of glass, by pouring the molten glass into iron tubes. The sudden chilling of the outer

layer produced tension in the glass cylinder, and a corresponding variation in the refractive index, plane parallel plates cut from the cylinder acting as concave lenses.

Following the mathematical treatment which Exner gave for cylinders of this nature, we arrive at two interesting conclusions, namely that the ordinary lens formula $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ holds for them, and that to be *aplanatic*, the equation which shows the relation between the refractive index and the distance from the axis of the cylinder is that of a parabola.

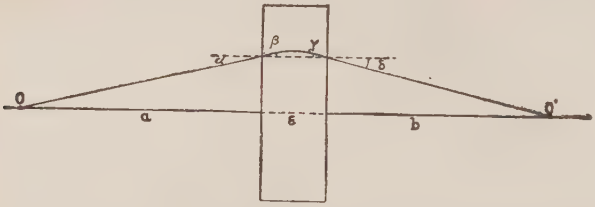


FIG. 83.

Let O (Fig. 83) be the source and O' the conjugate focus, at distances a and b from the surfaces of a cylinder of thickness ϵ.

Consider the ray incident at angle α, at a point at distance x from the axis, where the refractive index is n. The angle of refraction $\beta=\frac{\alpha}{n}$. From now on the ray moves in a path which is approximately circular for rays near the axis, turning through an angle $\frac{\epsilon}{r}$ before reaching the second surface (r = radius of curvature of ray).

The ray meets the second surface at an incidence angle $\gamma=\frac{\epsilon}{r}-\beta$, for β and γ can be regarded as angles of a triangle whose exterior angle $\frac{\epsilon}{r}$ is equal to their sum. The angle of refraction into air is

$$\delta=n\gamma=\frac{n\epsilon}{r}-n\beta=\frac{n\epsilon}{r}-\alpha.$$

Since we are considering rays near the axis $x_1=x_2=x$, where x is the distance of any point on the ray from the axis

$$\frac{x}{b}=\delta=\frac{n\epsilon}{r}-\alpha=\frac{n\epsilon}{r}-\frac{x}{a},\text{ or }\frac{x}{a}+\frac{x}{b}=\frac{n\epsilon}{r}. \quad . \quad . \quad . \quad (1)$$

FIG. 84.

We now require an expression for r, the radius of curvature of the ray.

Let GH be the element of wave-front (Fig. 84); the velocities and refractive indices at G and H are respectively v', n' and v, n, which are so related that $v'>v$ and $n>n'$.

If the radius of the secondary wavelet around G is GL and the radius of the one around H is HK, the radius of curvature r will be found by producing LK until it meets GH produced, say at J. If GH equals dx,

$$\frac{r}{r+dx}=\frac{HK}{GL}\ (\text{Homol. sides of }=\triangle)=\frac{v}{v'}=\frac{n'}{n}=\frac{n+dn}{n},$$

where dn is the increment of n corresponding to the increment dx.

$$rn=(r+dx)(n+dn),\ rdn=-ndx,\ \frac{1}{r}=-\frac{1}{n}\frac{dn}{dx}.$$

By equation (1) $$\frac{x}{a}+\frac{x}{b}=-\epsilon\frac{dn}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now $n=f(x)=n_1+c_2x+c_3x^2+c_4x^3+\cdots$ in which n_1 equals the index at the axis, and on account of the symmetrical distribution of n around the axis, n has the same value for $-x$ as for $+x$, and $c_2=c_4=c_6=0$;

$$\therefore \text{ for nearly central rays } n=n_1+c_3x^2, \quad . \quad . \quad . \quad (3)$$

$$\frac{dn}{dx}=2\,c^3x,\ \frac{1}{a}+\frac{1}{b}=-2\,c_3\epsilon=\text{Constant},$$

in which c_3 is a constant depending on the law of change of the refractive index.

If we let $-2\,c_3\epsilon=\frac{1}{p}$, we have (4) $\frac{1}{a}+\frac{1}{b}=\frac{1}{p}$, the classical formula for lenses. This formula shows us that the focus is inversely proportional to the length of the cylinder.

By combining (3) and (4), we get $n=n_1-\frac{1}{2\,p\epsilon}x^2$,

$$n_1-n=\frac{x^2}{2\,p\epsilon}\ \text{and if } n_1-n=\Delta n,\ x^2=2\,p\,\Delta\,n.$$

If we plot x and Δn on a system of rectangular coordinates, we shall get a parabola.

We have thus far confined our attention to nearly axial rays for which x is small, and obtained our final expression by neglecting powers higher than the square in the series. Let us now consider rays farther removed from the axis, and determine the law governing the change of refractive index which must hold if the cylinder bring all rays to the same focus, or be aplanatic.

By (2), we have $$-\frac{\epsilon}{x}\frac{dn}{dx}=\frac{1}{p},$$

$$-\epsilon p dn=x dx.$$

(Integrating) $$-\epsilon pn=\frac{x^2}{2}+\text{Const.}$$

For $x=0$, $n=n$, and since the above equation holds for all values of x it holds when $x=0$. From this relation we determine the constant to be $-\epsilon pn$,

$$-\epsilon pn=\frac{x^2}{2}-\epsilon pn,\ n=n_1-\frac{x^2}{2\,p\epsilon},\ x^2=2\,p\epsilon\Delta n,$$

an expression identical with the one which we obtained for the axial rays.

The nature of the function is therefore parabolic, and if we plot ordinates equal to the refractive indices, and abscissae equal to the plus and minus values of x, we shall get a parabola, which is concave down or up according to whether the highest or lowest value of n is found along the axis, or according as we consider a cylinder (Fig. 85) which acts as a concave or convex lens.

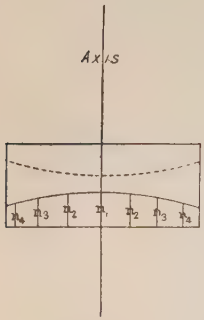


Fig. 85.

If we consider the cylinder to be of infinite length the rays will come to a focus on the axis, then diverge again, and moving in curved paths come to a focus again at a point further along on the axis.

The same thing will take place in vertical planes or in two dimensions if the equi-indical surfaces are parallel planes instead of coaxial cylinders. Such conditions sometimes exist in the atmosphere, as Professor Everett has shown, and if the eye be situated at the focus where the rays come together, vertical magnification of the object will result as we have seen.

Curved Light Rays. — A ray of light entering a medium of this description will be bent towards, and cross the line of maximum optical density, where it changes its curvature and is again bent towards the line, which it may thus cross again and again, traversing a path which is approximately a sine curve. If we are dealing with a diverging pencil of rays, the rays will alternately converge to and diverge from a focus, passing in this way through a number of successive foci. These effects can be well shown by the following device, which was described by the author in the *Philosophical Magazine* for April, 1899.

A glass trough 50 cms. long by 10 cms. high and 2 cms. wide, with plate glass ends, is filled to the depth of 3 cms. with a strong solution of alum. On this is floated a layer of water containing 10% of alcohol, which is very much lighter than the alum solution, though having about the same refractive index. A mixture of glycerine and 85% alcohol has a much higher refractive index, but a specific gravity intermediate between these two liquids, consequently it is possible by means of a glass siphon, drawn down to a small aperture which is bent in a horizontal direction, to introduce a layer of it between the alum solution and the supernatant water. The necessary precautions and fuller directions will be found in the original paper. The three solutions were previously acidified with sulphuric acid, and rendered fluorescent with sulphate of quinine in order that the paths of the rays could be followed. By cautious stirring the diffusion of the layers into each other can be assisted, and we shall have as a result a medium in which the refractive index increases from the surface towards the median plane, and then decreases from this plane towards the bottom, the condition being similar to the atmospheric condition producing the Fata Morgana.

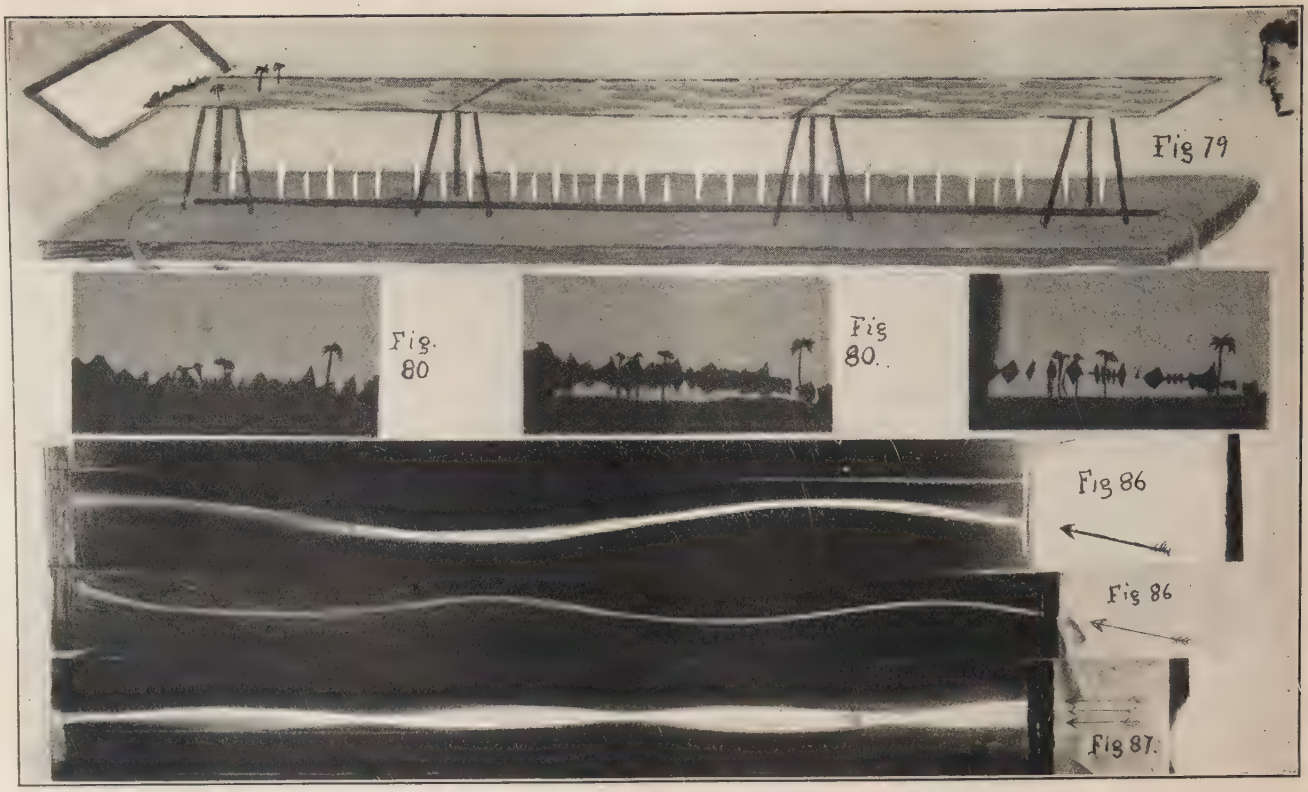


PLATE 3.

If a very narrow beam of light from an arc-lamp, made parallel by means of a condensing lens, be thrown obliquely into one end of the trough, it will be seen to traverse the liquid in the form of a most beautiful blue wave, the curvature of which varies with the angle at which the ray enters. A ray of light travelling in a sine curve is shown in Fig. 86, Plate 3, which was photographed directly from the trough.

The alternate convergence and divergence of rays, and the successive foci can be shown by allowing a parallel bundle of rays to enter one end of the trough in a horizontal direction. A photograph of this phenomenon is also shown in Fig. 87, Plate 3.

Scintillation. — In addition to the more or less regular gradations in the refractive index of the atmosphere there exist striae, or small regions of sudden change due to the mixing of hot and cold currents, somewhat similar to the conditions existing in a mixture of glycerine and water. When a wave-front of light passes through a region where the refractive index is low it gains, and while travelling in a region of high refractive index it loses. The result of this is, that the striae deform the plane-waves of light coming from the stars into corrugated waves, portions of which are convex in the direction of propagation, while other portions are concave. The concave portions naturally converge, while the convex portions diverge; the result being that the energy concentrates itself in certain areas at the expense of the adjacent areas, as shown in Fig. 88.

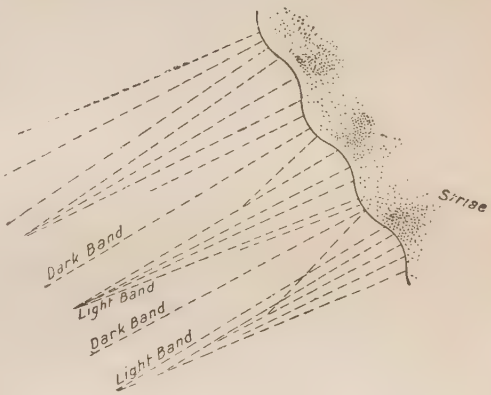


Fig. 88.

This uneven distribution of light produces the familiar phenomenon of scintillation or twinkling of the stars. If the intensity of the light from a star were sufficient, we should find that instead of illuminating a white surface uniformly, as does the sun or moon, it would illuminate it unevenly, dark and light patches alternating over the surface. This uneven illumination is actually observed during the few moments immediately preceding the total stage of a solar eclipse, the patches of light and shadow being arranged in more or less parallel bands. The bands are sometimes erroneously referred to as "diffraction fringes" bordering the moon's shadow. They move along over the ground with a velocity depending on the velocity of the upper currents of the atmosphere, usually from ten to twenty feet per second. The same phenomenon occurs also in the case of star light, except that the light is too feeble to produce

shadow-bands which are visible. Their presence, however, can be inferred from the well-known fact that the brilliancy of the star observed by the eye appears to suffer rapid periodic changes, the star appearing bright or feeble according to whether the eye is in a light or dark area of the moving system of shadows. The width of the bands is frequently not over 3 or 4 cms. This means that it may easily happen that one eye is in a dark, while the other is in a bright area at the same moment. If we look at a star with the eyes slightly converged, which we can easily do by focussing them on some object at a distance of five or six feet, and in a line with the star, the star will appear doubled and the two images will fluctuate in intensity, but the fluctuations will not be " in step," one eye seeing the star dark at the moment when the other eye sees it bright.

If a star is viewed through a telescope of large aperture, the resultant illumination at the focus is the integral of the bright and dark bands covering the object-glass at the moment, and this average illumination is practically constant, therefore scintillation is no longer observed. If the aperture of the instrument be contracted by a diaphragm of such size that only the light of a single bright or dark band can enter the instrument, the twinkling reappears. It is possible in this way to actually measure the radius of curvature of the corrugations of the wave-front in the case of star light. Suppose that at a given instant the wave entering the small aperture of the telescope is concave, it will come to a focus at a point slightly nearer the object-glass than the focus of the telescope for objects at infinity. At another instant when the aperture is in a dark band where the wave is convex, the focal point for this wave will be behind the principal focus. As the dark and light bands sweep across the aperture the image of the star will alternately appear sharp and blurred. If the eye-piece is at the focus for the concave wave it will be inside the focus for the convex wave. By pushing the eye-piece in up to a point where it is possible to occasionally catch a glimpse of a sharp image of the star, and then drawing it out to a point outside the focus, for which the same conditions prevail, it will be possible to determine the minimum radius of curvature of the convex and concave portions of the wave-front. Measurements made in this way show that the average radius of curvature is about 6000 metres, although it may sometimes fall as low as 1800 metres, or rise as high as 20,000 metres. Obviously the conditions most favorable for work with astronomical instruments are to be found when the radius of curvature of the corrugations is very large. This means that the waves are approximately plane.

One other point is worthy of mention in this connection, namely, the difference between planets and stars in the matter of twinkling. In the case of planets the light comes from a luminous disk of an appreciable size, every point of which produces a system of shadow-bands of its own. It is true that the inclination to each other of the rays coming from the different portions of the planet to the eye is very slight, but when we consider that they have traversed a distance of, say 6000 metres, in coming to a focus, that is, in forming a bright band, it is easy to see that the light from one side of the

planet may easily produce a system of shadow-bands exactly out of step with those produced by the light of the other side of the disk. The superposition of a large number of shadow systems results in practically uniform illumination and absence of scintillation. This explanation of scintillation, while it accounts for alternations in the intensity of the light, does not account for the peculiar chromatic changes which were first observed by Respighi. If the spectrum of a scintillating star is observed it is seen to be traversed by broad dark bands, parallel to the Fraunhofer lines, which travel from the red to the violet, or from the violet to the red, according as the star is in the west or the east. If the star is in the zenith, the motion of the bands is oscillatory. Respighi believed that the chromatic changes were due to the rotation of the earth. Owing to the dispersion of the air the violet rays will reach the eye over different paths from those traversed by the red rays. In Fig. 89 the

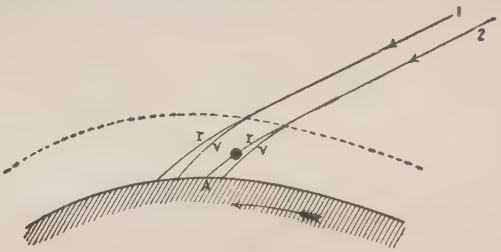


FIG. 89.

dotted arc represents the upper limit of the atmosphere of the earth, which rotates in the direction of the arrow. Let the observer's eye be situated at A. Two parallel rays from the star are designated by 1 and 2. The dispersed rays are designated by r and v. It is clear that the violet rays from 1, and the red rays from 2, enter the eye or the spectroscope at A, if 1 and 2 are taken a little closer together than in the diagram. The violet rays thus traverse paths lying *above* those traversed by the red. Let us now imagine that one of the irregularities in the air previously mentioned, indicated by a black spot, lies in the path of the red rays, and suppose further that this irregularity is of such a nature as to cause the wave-front, originally plane, to become convex. The intensity in the red region will consequently become less, for reasons already given. The rotation of the earth now comes into play, carrying this irregularity up and into the path of the violet rays, causing the minimum of intensity to shift from the red towards the violet end of the spectrum. This is the sequence observed by Respighi for a star in the west, which is the case represented in the diagram. These chromatic changes manifest themselves to the eye as an irregular change in the color of the star.

The effective velocity with which the refracting mass of air is carried across the rays is obviously the difference between the velocity of the surface of the earth and a region high up in the air. Their angular velocity is the same, but the actual velocity of the

upper atmosphere is a little greater, owing to its greater distance from the earth's centre, and a little calculation will show that difference is of the right order of magnitude to account for the chromatic changes observed by Respighi.

We should also have dark bands moving across the spectrum if the upper air currents were in rapid motion, for a wind would cause the refracting regions to sweep across the ray paths as well as the rotation of the earth. Perntner in his *Meteorological Optics* expressed doubts as to the correctness of Respighi's theory, and ascribes the constancy in the direction of motion of the bands in the spectrum to the fact that the upper air currents travel for the most part in the same direction. Probably both factors come into play, and until more data have been obtained it is impossible to say which is the predominant one. As Perntner points out, the prevailing winds in the southern zones are easterly, which should cause the bands to move in the opposite direction along the spectrum. Data obtained by observers in southern latitudes are much to be desired.

The Method of Striae. — A very ingenious and beautiful method was originated by Töpler (*Wied. Ann.*, cxxxi., p. 33) (which he named the "Schlieren-methode") for making visible in a transparent medium those regions in which the refractive index differed but slightly from that of the surrounding regions. By employing as a source of light the instantaneous flash of an electric spark he was able actually to see the spherical sound-waves sent off from another spark which had occurred a moment before. Mach has used the method extensively for studying by photography the air waves given off by sparks, and accompanying rifle bullets in their flight, and an extensive series of photographs were made by the author (*Phil. Mag.*, Aug. 1899, July 1900, May 1901) of sound waves undergoing reflection and refraction, to illustrate some of the fundamental principles of optics. The apparatus for showing these waves can be set up in a few minutes, with very little trouble, and as the experiment is a very beautiful and instructive one it will be described in detail.

Fig. 90.

The general arrangement of the "Schlieren" apparatus is shown in Fig. 90. A good-sized achromatic lens of the finest quality obtainable, and of rather long focus, is the most important part of the device. The object-glass of a small telescope three or four inches in diameter is about right.

This lens is mounted in front of a suitable source of light (in the present case, an electric spark), which should be at such a distance that its image on the other side of the lens is at a distance of about fifteen feet.

The image of the spark, which we will suppose to be straight, horizontal, and very narrow, is about two-thirds covered with a horizontal diaphragm (*a*), and immediately behind this is placed the viewing-telescope. On looking into the telescope we see the field of the lens uniformly illuminated by the light that passes under the diaphragm, since every part of the image of the spark receives light from the whole lens. If the diaphragm be lowered the field will darken, if it be raised the illumination will be increased. In general it is best to have the diaphragm so adjusted that the lens is quite feebly illuminated, though this is not true for photographic work. Let us now suppose that there is a globular mass of air (*b*) in front of the lens of slightly greater optical density than that of the surrounding air. The rays of light going through the upper portion of this denser mass will be bent down, and will form an image of the spark below the diaphragm, allowing more light to enter the telescope from this particular part of the field; consequently, on looking into the instrument, we shall see the upper portion of the globular mass of air brighter than the rest of the field. The rays which traverse the under part of "*b*," however, will be bent up, forming an image of the spark higher up, and wholly covered by the diaphragm, consequently this part of the field will appear black. It will be readily understood, that with the long path between the lens and the image a very slight change in the optical density of any portion of the medium in front of the lens will be sufficient to raise or depress the image above or below the edge of the diaphragm, and will consequently make itself manifest in the telescope.

The importance of using a lens of first-class quality is quite apparent, since variations in the density of the glass of the lens will act in the same way as variations in the density of the medium before it, and produce unequal illumination of the field. It is impossible to find a lens which will give an absolutely even, feeble illumination, but a good achromatic telescope-objective is perfect enough for every purpose. A more complete discussion of the operation of the apparatus will be found in Töpler's original paper in the *Annalen*. The sound-waves, which are regions of condensation, and consequent greater optical density, make themselves apparent in the same way as the globular mass of air already referred to. They must be illuminated by a flash of exceedingly short duration, which must occur while the wave is in the field of view.

Töpler showed that this could be done by starting the sound-wave with an electric spark, and illuminating it with the flash of a second spark occurring a moment later, while the wave was still in the field. A diagram of the apparatus used is shown in Fig. 91. In front of the lens are two brass balls (*a*, *a*), between which the spark of an induction-coil passes, immediately charging the leyden-jar *c*, which discharges across the gap at *e* an instant later. The capacity of the jar is so regulated that the interval between the two sparks is about one ten-thousandth of a second. The field of the lens is thus illuminated by the flash of the second spark before the sound-wave started by the first spark has gone beyond the edge of the lens.

To secure the proper time-interval between the two sparks it is

necessary that the capacity of the jar be quite small. This limits the length and brilliancy of the illuminating spark, and with the device employed by Töpler it was impossible to get enough light to secure photographs of the waves. After some experimenting it was found that if the spark of the jar was passed between two thin pieces of magnesium ribbon pressed between two pieces of thick plate glass, a very marked improvement resulted. With this form of illuminator five or six times as much light could be obtained as by the old method of passing the spark between two brass balls.

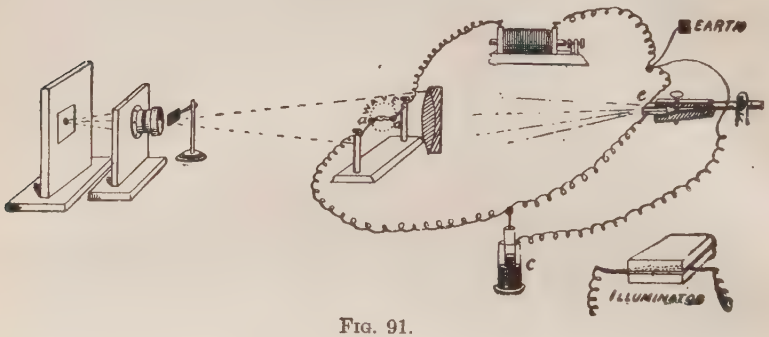


Fig. 91.

The spark is flattened out into a band, and is kept always in the same plane, the light issuing in a thin sheet from between the plates. By this arrangement we secure a light source of considerable length, great intensity, and bounded by straight edges, the three essentials for securing good results. The glass plates, with the ribbon terminals between them, must be clamped in some sort of a holder and directed so that the thin sheet of light strikes the lens: this can be accomplished by darkening the room, fastening a sheet of paper in front of the lens, and then adjusting the plates so that the paper is illuminated as much as possible. The image formed by the lens will be found to have very sharp straight edges, on one of which the edge of the diaphragm can be set in such a manner as to allow but very little light to pass when the intervening medium is homogeneous; a very slight change, however, in any portion may be sufficient to cause the entire amount of light passing through that portion to pass below the diaphragm and enter the telescope.

For photographing the waves the telescope is removed and a photographic objective put in its place. A vertical board is firmly clamped behind this in such a position that the image of the balls, between which the sound-spark passed, would be in focus on a plate held against it. This arrangement is used instead of a camera, because it is necessary to move the plate rapidly during the exposure, to prevent the image of more than one wave being formed on the same place. It was found that simply holding the plate in the hand against the vertical board and advancing it slowly from left to right, at the same time giving it a rapid up-and-down motion, answered every purpose.

The images obtained in this manner show the waves in different stages of development, for the time-interval between the two sparks varies between rather wide limits. This is really an advantage, for on a single plate it is possible to pick out a series showing the successive changes in the form of the wave-front produced by reflection, refraction, etc. Each picture shows the circular field of the telescope-lens with the two brass rods crossing it and supporting in the centre the two balls between which the sound-spark passes. The hot air rising from the spark appears in most of the pictures like a puff of steam above the ball.

A few words regarding the apparatus may be helpful to those wishing to repeat the very beautiful experiments of Töpler. An induction-coil capable of giving a three or four-inch spark is about right, while a good-sized test-tube partly filled with mercury, and standing in a cylinder of mercury, will be found most convenient for a leyden-jar. The balls between which the sound-spark passes should be adjusted so as to obtain almost the maximum spark possible, which will in general be rather less than half as long as the coil will give between its terminals. The best results are obtained when the sparks give off the same crackle found desirable in experiments with Hertz waves. Fresh plates of glass should be put in the illuminator every little while.

It is not at all difficult to get the apparatus to work properly, and doubtless it could be made to work on quite a small scale with a good photographic objective of rather long focus. The objective of a good-sized spy-glass would also give good results. Töpler was, I believe, of the opinion that he got more uniform results with an influence-machine than with the coil. He certainly found the time-interval between the two sparks to be more constant. This, however, is no object in photographic work, for the wide variation is the very thing that makes the pictures interesting.

The refraction of sound in a medium denser than air is shown in Fig. 92, where we have a rectangular tank with sides made of plane-parallel glass, and covered with a collodion film of soap-bubble thickness made by the following method. Ordinary collodion is diluted with about ten parts of ether, poured on a small piece of plate-glass and immediately drained off. As soon as it is quite dry a rectangle is cut with a sharp knife on the film. Töpler's method of removing the film was to place a drop of water on one of the cuts, and allow it to run in by capillarity. The following method will, however, be found better. One end of the plate is lowered into a shallow dish of water, and the plate inclined until the water comes up to one of the cuts. By looking at the reflection of a window in the

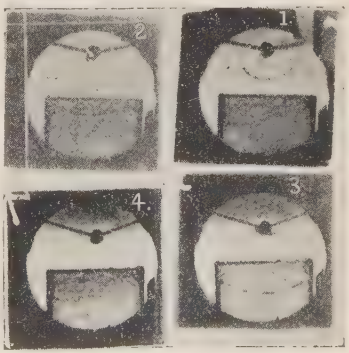


FIG. 92.

water it is possible to see when the film commences to detach itself from the glass. If all goes well it will float off on the surface of the water along the line of the knife-cut and the plate should be slowly lowered (one end resting on the bottom of the dish) until the rectangular piece detaches itself and floats freely on the surface. The edges of the tank are well greased and then lowered carefully upon the film, to which they will adhere. The whole must then be lifted from the water in an oblique direction, when the film will be found covering the tank and exhibiting the most beautiful interference-colors. The tank was filled with carbonic acid and placed under the origin of the sound-wave. On striking the collodion-film the wave was partly reflected and partly transmitted, and it will be seen that the reflected component in the air has moved farther than the transmitted component in the carbonic acid. The spherical wave-front is transformed into one which is parallel to an hyperboloid on entering the denser medium. This is well shown in No. 3 of the series, where the wave in air, moving at higher velocity, has passed out of the field entirely, and there remains only the slower-moving disturbance in the denser gas. In No. 4 the wave is seen returning up through the tank after having struck the bottom.

Photographs were also obtained of the refraction of the wave, when the tank was filled with hydrogen, in which the sound travels faster than in air. The bulging down of the wave-front was very noticeable, though it is not as great as we should expect. These explosive waves travel at a much higher velocity than ordinary sound-waves (nearly double the speed), and it is highly probable that the relative speeds in two different media is not the same as for ordinary sounds, a matter worth investigating.

In examining liquids or solids for striae, or regions of variable refractive index, we can employ a flat gas flame as our source of light, covering the lower part of it with an opaque screen having a straight edge. The lens will form an inverted image of this in front of the objective of the viewing telescope, and all but a strip half a millimeter or so in width is to be cut off from above by a second screen. The object to be examined is placed immediately in front of the lens. A piece of ordinary window glass makes a good object. The heated air rising from the hand can also be seen, and if a tank made of optical glass, filled with warm water, is placed before the lens, a drop lifted out and allowed to fall back can be seen descending through the liquid: the change in the refractive index is obviously due to the cooling by evaporation. Opaque objects placed before the lens appear with brilliantly illuminated margins, the light in this case being diffracted: with the arrangement of screens described only the upper and lower edges appear illuminated, since lateral deflection of the rays is without effect. The method is an extremely useful one, and can be applied to many lines of investigation, and the student should be thoroughly familiar with its possibilities.

Invisibility of Objects. — Opaque substances are seen by the light reflected from their surfaces; transparent substances in part by reflected light and in part by transmitted light. If we analyze

carefully the appearance of a cut-glass decanter stopper we shall find it to be extremely complicated. Each facet reflects the image of some object in the room from its surface, and in addition to this shows some other object by refracted rays which have entered some other facet, these latter being in general more or less spread out into a spectrum by dispersion. If the stopper is wholly or in part made of colored glass, the refracted rays passing through the colored portions are modified by absorption, and affect the appearance. This remarkable complex, we say, looks like a stopper, and unless we try to paint a picture of it, or have our attention drawn to the details, we are apt to regard its appearance as quite simple.

We thus see that reflection, refraction, and absorption all play a part in making objects visible. It is interesting to examine into the conditions under which objects are invisible. If they are immersed in a medium of the same refractive index and dispersion, reflection and refraction disappear; and if they possess in addition the quality of perfect transparency, they will be absolutely invisible, the light rays passing through them without any modification either in intensity or direction. Could a transparent solid be found whose refractive index was the same as that of air, objects made of it would be invisible. The effect of immersing a transparent solid in a medium of similar optical properties is usually illustrated by dipping a glass rod into Canada balsam or oil of cedar, the immersed portion being practically invisible. A still better medium can be made by dissolving chloral hydrate in glycerine by the aid of heat. Only a little glycerine should be taken, as it is necessary to dissolve some eight or ten times its volume of the chloral before the solution acquires the right optical density. A glass rod, if free from bubbles or striae, becomes absolutely invisible when dipped in the liquid, and if withdrawn presents a curious appearance, the end appearing to melt and run freely in drops.

As a matter of fact, transparent objects are only visible by virtue of non-uniform illumination, as is pointed out by Lord Rayleigh in his article on optics in the *Encyclopædia Britannica*. If the illumination were the same on all sides they would be invisible, even if immersed in a medium of very different optical index. A condition approaching uniform illumination might, he says, be attained on the top of a monument in a dense fog. The author has devised a very simple method of showing this curious phenomenon, which, in brief, is to place the object within a hollow globe, the interior surface of which is painted with Balmain's luminous paint, and view the interior through a small hole.

The apparatus can be made in a few minutes in the following manner. A quantity of Canada balsam is boiled down, until a drop placed on cold glass solidifies. The Balmain paint, in the form of a dry powder, is stirred into the hot balsam until the whole has the consistency of thick paint. Two glass evaporating dishes of equal size are carefully cleaned and warmed, and coated on the outside with the hot mixture, which can be flowed over the glass, and by the dexterous manipulation of a small Bunsen flame made

to cover the entire outer surface. Probably two perfectly plain hemispherical finger-bowls could be used instead of the evaporating dishes. As soon as the coating has become hard a small hole is cut through it through which the interior is to be viewed. If the lips of the dishes are placed together the interior can be seen through the small opening, but in this case the line of junction, which is always more or less dark, comes opposite the aperture, which is a disadvantageous arrangement.

If the inner surfaces be exposed to bright daylight, sun or electric light, and the apparatus taken into a dark room, a crystal ball or the cut-glass stopper of a decanter placed inside, will be found to be quite invisible when viewed through the small aperture. A uniform blue glow fills the interior of the ball, and only the most careful scrutiny reveals the presence of a solid object within it. One or two of the side facets of the stopper may appear if they happen to reflect or show by refraction any portion of the line of junction of the two hemispheres.

The apparatus would give better results if made on a larger scale, as the eye would not have to be brought so near the object. Two large wooden bowls would answer the purpose admirably. It is of the utmost importance to have a very thick and uniform coating of the paint, as otherwise the illumination is not uniform.

CHAPTER V

DISPERSION

In our treatment of refraction we have assumed a constant retardation of the waves for a medium of given refractive index. We have seen that the velocity of light in the free ether of space is independent of the color or wave-length. Such, however, is not the case in refracting media, for here the waves not only travel slower than in free space, but waves of different length travel with very different velocities. In all such media as air, water, and glass, the long waves travel faster than the short ones; consequently the deviation of the ray, or the angle through which the wave-front turns when encountering the boundary of the medium, depends on the color of the light as well as on the optical density of the medium.

When white light enters a transparent medium, the long red waves forge ahead of the green ones, which in their turn get ahead of the blue. If we imagine an instantaneous flash of white light traversing a refracting medium, we must conceive it as drawn out into a sort of linear spectrum in the medium, that is, the red waves lead the train, the orange, yellow, green, blue, and violet following in succession. The length of this train will increase with the length of the medium traversed. On emerging again into free ether the train will move on without any further alteration in its length.

We can form some idea of the actual magnitudes involved in the following way. Suppose we have a block of perfectly transparent glass (of ref. index 1.52) twelve miles in thickness. Red light will traverse it in $\frac{1}{10000}$ of a second, and on emerging will be about 1.8 miles in advance of the blue light which entered at the same time. If white light were to traverse this mass of glass, the time elapsing between the arrival of the first red and the first blue light at the eye would be less than $\frac{1}{60000}$ of a second. Michelson's determination of the velocity of light in carbon bisulphide showed that the red waves gained on the blue waves during their transit through the tube of liquid. The absence of any change of color in the variable star Algol furnishes direct evidence that the blue and red rays traverse space with the same velocity. In this case the distance is so vast, and the time of transit so long, that the white light coming from the star during one of its periodic increases in brilliancy, would arrive at the earth with its red component so far in advance of the blue that the fact could be easily established by the spectro-photometer or even by the eye.

Inasmuch as the deviation of a ray of light depends on the change of velocity of a wave on going, say, from a rare into a denser medium, we infer that those rays which are deviated the most, namely the violet, suffer the greatest change of velocity or move the slowest.

Later on, when we come to the study of interference, we shall find other evidence that such is the fact.

Newton was the first to systematically study the phenomenon of dispersion. He discovered that ordinary white light was made up of different colors which could be separated from each other by passing the light through a prism. His most complete and convincing experiment may be briefly summed up as follows. The light of the sun was admitted to a darkened room through a small hole in the shutter, and the narrow beam passed through a prism. Instead of a round white image of the sun, there now appeared on the screen a colored band of light or spectrum, made up in reality of an infinite number of differently colored images of the sun superposed, but slightly displaced with reference to one another.

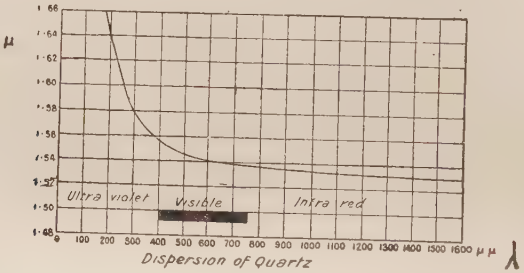


FIG. 93.

A small perforation was made in the screen which allowed light of approximately a single color to pass. This ray was transmitted through a second prism, and was found to form a fairly sharp image of the sun on a second screen, proving that monochromatic light suffers no decomposition or dispersion in the prism. By slightly turning the first prism, the spectrum could be moved so as to allow light of any color to pass through the perforation and be refracted by the second prism. Newton found that the colored image of the sun on the screen changed its position with every change of color, the deviation being greatest when the color was violet, and least when it was red.

The refractive index of a substance varies then with the wave-length of the light employed. To determine the relation between the two we can measure the index of a prism for light of known wave-lengths, *e.g.* the bright lines in the spark spectrum, or the dark lines in the solar spectrum, and plot the results on coordinate paper, taking the refractive indices as ordinates, and the wave-lengths as abscissae.

We shall find that the refractive index increases more rapidly than the wave-length decreases as we approach the violet, the curve having the general form shown in Fig. 93, which is the dispersion curve for quartz. Having plotted such a curve for a given prism, we can determine the wave-length of any other line in the spectrum by determining the refractive index for the line, and finding the corresponding ordinate on the curve. If a prism spectroscope is

to be used for wave-length determinations, it must be calibrated in this manner, for different samples of glass have very different dispersive powers. Were the deviations proportional to the wave-lengths, the curve would be a straight line, and we should have what is known as a normal spectrum. Such a spectrum can be formed by a diffraction grating, but never by a prism. The dispersion curve can be shown experimentally in the following way. Let a vertical normal spectrum, formed by a diffraction grating, be viewed or projected through a prism standing with its refraction edge vertical. The entire spectrum will be deviated by the prism, but the deviation will increase very rapidly as we near the blue, the spectrum being bent into a curve. Figure 94 shows dispersion

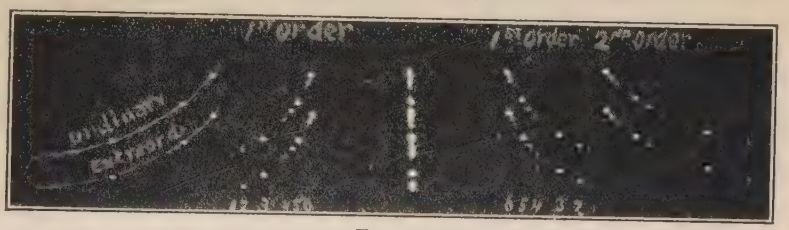


Fig. 94.

curves made in this way. A quartz prism in which the rays travelled perpendicular to the optic axis was mounted in front of the lens of a camera, and a small glass diffraction grating placed behind it. The source of light was an " end-on " helium tube placed at a distance of three meters. Instead of spectrum lines we have small circles of light, corresponding to the different wave-lengths emitted by the glowing helium gas.

The quartz prism is doubly refracting and consequently gives two spectra, and the photograph shows the dispersion curve for each. The complete curves have been drawn in for the second order spectra to the left. The lower is the dispersion curve for the ordinary ray, the upper for the extraordinary. They are incorrectly marked in the figure, which would be the condition for a prism of Iceland spar as we shall see when we come to the subject of double refraction. The two red helium lines are marked 1 and 2. The bright yellow line D_3 is marked 3, and the green, blue, and violet lines 4, 5, and 6. The grating alone would give the spectrum lines (or rather dots) in the positions of these numbers. The straight spectrum is made up of the superposed spectra which the quartz prism gives of the central image formed by the diffraction grating. This will be better understood after we come to the subject of diffraction. This method of " crossed prisms," due to Newton, is of use in studying the remarkable phenomenon of anomalous dispersion, which we shall come to presently.

Newton came to the erroneous conclusion that the dispersion was proportional to the refraction, that is to say that substances of high refractive index had great dispersive powers, or gave wide spectra, while the reverse was true for substances of low refractive index.

While this is apt to be the case, it is not always true, for we find that there are substances the mean refractive indices of which are small, while their dispersive powers are large, and *vice versa*.

Achromatism. — The fact that dispersion is more or less independent of refractive index makes it possible to arrange two prisms of different kinds of glass, with their refracting angles turned in opposite directions, which shall have the power of deviating a ray without spreading it out into a spectrum. One of the prisms almost entirely annuls the dispersion of the other, without entirely annulling the deviation, a thing which Newton considered impossible. Such a combination is known as an achromatic prism. Let us see just how such a system operates.

Flint glass has a much higher dispersive power in proportion to its mean refractive index than crown glass. The refractive indices of the two glasses for red, yellow, and bluish-green light of wavelengths corresponding to the *C*, *D*, and *F* lines in the solar spectrum are as follows:

	C	*D*	*F*
Flint glass	1.630	1.635	1.648
Crown glass	1.527	1.530	1.536

If prisms of small angle are employed we can write the deviations for these colors as proportional to the refractive indices less 1, that is for flint glass the distance from a point on a screen where the direct ray falls, to the points where the red-green and blue rays fall when the prism is put in the path of the light, will be 630, 635, and 648.

The length of the spectrum, or rather the distance between the *C* and *F* lines, is obviously 648 − 630 or 18.

For crown glass the distances will be 527, 530, and 536, and the distance between the *C* and *F* lines will be 536 − 527 or 9. The dispersion of the flint glass is therefore double that of the crown glass. If now we make a crown-glass prism of twice the angle of the flint-glass prism, the distance between the *C* and *F* lines will be the same as with the flint prism, while the distances of the lines from the spot where the direct ray falls will be twice as great as before, or 1054, 1060, and 1072.

Suppose now we place the two prisms together with their refracting angles turned in opposite directions. The crown prism alone would shift the *F* line to a distance of 1072, but the flint prism shifts it back a distance of 648, and its resulting position is 1072 − 648 or 424 from the spot where the direct ray would fall. The *C* line would be deviated by the crown prism to a distance 1054, but the flint one moves it back 630, and its position is 1054 − 630 or 424. The *C* and *F* lines are thus deviated the same amount, and the dispersion is annulled so far as these two colors are concerned. The combination is achromatic for red and greenish-blue light, deviating both to a distance of 424, a trifle less than the deviation produced by the flint prism acting alone. Let us now see if the yellow light falls in the same place. The position of the *D* line will be given by 1060 − 635 or 425, that is, it will be deviated a very little

more than the C and F lines, consequently the combination is not perfectly achromatic. By means of two prisms it is possible to bring any two parts of the spectrum together, the other colors lying a little to the right or left of the superposed portions, forming what is known as the secondary spectrum.

The general rule to follow in the construction of an achromatic prism is as follows. To bring any two lines of the spectrum together, the angles of the two prisms must be so proportioned that the distance between the lines in question is the same for each prism. Were the distances between the other lines the same for both prisms, the combination would be truly achromatic, but such is not the case, owing to the irrationality of dispersion.

Achromatic prisms are of very little practical use, but the principle is of great importance in connection with achromatic lenses.

Partial Achromatization by a Grating. — A diffraction grating gives a normal spectrum, that is, one in which the distances between the lines are proportional to their difference of wave-length. Some very beautiful effects can be seen by viewing a distant gas flame through a flint-glass prism of 60°, and a glass diffraction grating of about 2000 lines to the inch. The grating alone gives two sets of spectra turned in opposite directions. Those on one side of the central image are still further expanded by the prism, while those on the other are more or less closed up. By varying the angle at which the prism is placed, and tilting the grating so as to vary its dispersion, we can achromatize for the middle of the spectrum and obtain a green image of the flame, with a red and blue spectral band extending out from it as a purple flare of light. If we turn the grating so that its direction of dispersion makes a small angle with that of the prism we shall see how this colored image and the purple flare has been produced: the spectrum has been bent back upon itself, and resembles a portion of an ellipse. As we rotate the grating still further, this curve gradually opens out, acquiring the form of the dispersion curve of the prism when the rotation has reached 90°, that is when the prism and grating are crossed. Photographs of the spectrum obtained in this way, with the grating in four different positions, are reproduced in Fig. 95. It is clear that when the curved spectrum is flattened together (which occurs when the lines of the grating are vertical) the red and blue will be superposed, while the middle or green portion will be pretty well concentrated at a single spot, giving the green image of the flame.

FIG. 95.

Direct Vision Prisms. — By referring to the table of refractive

indices for crown and flint glass it is easy to see how a combination of two prisms can give dispersion without deviation, that is, yield an undeviated spectrum. If, instead of giving the crown prism an angle double that of the flint, we make it 1.2 times as great, and make the same calculation as before, we shall find that we have a spectrum the length of which is 7, and the centre of which falls on the spot where the undeviated ray would fall. Such a combination is known as a direct vision prism, and is employed in cases where any considerable deviation would be detrimental, as when compactness of the instrument is desirable.

Achromatic Lenses. — We are now in a position to consider the principle on which the achromatic lens is made. Any lens can be considered as a prism of varying angle, or rather as a solid formed by the rotation of a thin section of a curved prism around its base. Since the distance of the focus of a lens from its centre depends on the deviation of the rays, it follows that the focus will be different for the different colors, the blue rays which are bent the most meeting nearest the lens, and the red, which are bent to a less degree, coming together farther away, an effect known as chromatic aberration. What we require is a combination which will produce an equal deviation, and consequently a common meeting point for rays of all colors. If we can arrange two prisms of crown and flint glass which will give deviation without dispersion, we can in the same way, by employing a double convex lens of crown and a plane concave of flint glass, give exactly the same deviation to two colors widely separated in the spectrum, and very nearly the same deviation to the other colors, with the result that rays of different refrangibility come together at very nearly the same point.

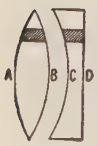


FIG. 96.

Reference to Fig. 96 will make the analogy between the achromatic lens and prism clear. The blackened parts indicate how each portion of the lens combination can be considered as two opposed prisms. We found that in the case of the prism the ratio between the angles was 1 : 2, and applying this to the lens it is easy to see that if the surfaces A, B, and C have the same curvature, the surface D of the flint lens must be plane, since the angle of the elementary prismatic portion of the flint lens must be everywhere $\frac{1}{2}$ that of the opposed elementary crown prism. Just as by employing two prisms we could unite two lines of the spectrum, so by the use of two lenses we can bring rays of any two different colors to the same focus.

Calculation of Achromatic and Direct Vision Prisms. — For prisms of small angle the deviation $\delta=\rho(n-1)$, in which ρ is the prism angle and n the refractive index. The deviation for two definite colors (say lines F and C of the solar spectrum) are given by

$$\delta_F=\rho(n_F-1),$$
$$\delta_C=\rho(n_C-1).$$

Subtracting, $$\delta_F-\delta_C=\rho(n_F-n_C). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The difference $\delta_F-\delta_C$ we can designate as the dispersion angle for the colors in question, and for brevity write it ζ_{FC}. For a second

prism of a different angle and composed of a different glass we have similar equations.

We will take into account three colors corresponding to the lines F, D, and C, for which we have equations:

$$\text{Prism 1.}\quad \delta_D = \rho(n_D - 1), \quad \zeta_{FC} = \rho(n_F - n_C).$$

$$\text{Prism 2.}\quad \delta'_D = \rho'(n'_D - 1), \quad \zeta'_{FC} = \rho'(n'_F - n'_C).$$

Suppose now that these two prisms are opposed. We shall then have the total deviation g of the color D represented by $\delta_D - \delta'_D$, and the dispersion angle ω between the rays F and C by $\zeta_{FC} - \zeta'_{FC}$.

$$g_D = \delta_D - \delta'_D = \rho(n_D - 1) - \rho'(n'_D - 1), \quad . \quad . \quad . \quad . \quad (2)$$

$$\omega_{FC} = \zeta_{FC} - \zeta'_{FC} = \rho(n_F - n_C) - \rho'(n'_F - n'_C). \quad . \quad . \quad . \quad (3)$$

For a *direct vision prism* in which the ray of color D is undeviated, we set $g_D = 0$, and obtain at once from equation (3) the relation

$$\frac{\rho}{\rho'} = \frac{n'_D - 1}{n_D - 1},$$

or the angles must stand in inverse ratio to the refractive indices less 1.

For an achromatic prism which is to deviate the rays of colors F and C the same amount, we write $\omega_{CF} = 0$ and get from our equation the relation

$$\frac{\rho}{\rho'} = \frac{n'_F - n'_C}{n_F - n_C}.$$

We will now investigate the spectrum produced by a direct vision prism which transmits the ray D without deviation. For this we have

$$g_D = 0 = \rho(n_D - 1) - \rho'(n'_D - 1),$$

or

$$\rho' = \rho\frac{(n_D - 1)}{n'_D - 1}.$$

Substituting this value of ρ' in equation (3) and transposing gives us for the angular separation of the rays F and C

$$\omega_{FC} = \rho(n_D - 1)\left[\frac{n_F - n_C}{n_D - 1} - \frac{n'_F - n'_C}{n'_D - 1}\right].$$

The quantity $\frac{n_D - 1}{n_F - n_C} = \nu$ is called the relative dispersion of a substance, and it is usually given in the tables which represent the optical properties of different kinds of glass.

Our equation now becomes

$$\omega_{FC} = \rho(n_D - 1)\left[\frac{1}{\nu} - \frac{1}{\nu'}\right].$$

In a similar manner we can derive an expression for the deviation of the ray D by an achromatic prism which deviates rays F and C

by the same amount. In this case

$$\omega_{FC}=0,$$
$$g = \rho(n_F-n_C)(\nu-\nu').$$

If in addition to deviating the rays F and C by the same amount, rays corresponding to the line A and D are to be united, we have

$$\frac{\rho}{\rho'}=\frac{n'_F-n'_C}{n_F-n_C},$$
$$\frac{\rho}{\rho'}=\frac{n'_D-n'_A}{n_D-n_A},$$
$$\frac{n'_D-n'_A}{n'_F-n'}=\frac{n_D-n_A}{n_F-n_C}.$$

The quantity n_F-n_C is known as the mean dispersion, and quantities such as n_D-n_A or n_F-n_D partial dispersions.

Resolving Power of Prisms. — By the resolving power we mean the ability of the prism to show a line as double when two different wave-lengths are present. This will obviously depend upon two things: the narrowness of the images of the lines and their distance apart. The former depends upon the width of the beam of light, that is the horizontal aperture of the instrument, for as we shall see when we come to the Chapter on Diffraction the image of a very narrow slit broadens by diffraction as the aperture of the lens is reduced, and it always has a finite width. It is moreover accompanied by fainter images on each side, with dark minima between them. Now Lord Rayleigh has shown that if we have two wave-lengths λ_1 and λ_2 which are very nearly equal, to see the slit image double, the image due to λ_2 must be separated from that due to λ_1 by at least the distance of the first minimum bordering the image formed by λ_1. This lies in such a direction that the path difference between the disturbances coming from opposite sides of the aperture, instead of being zero, the condition at the centre of the image, differs by exactly one wave-length.

Though the following treatment by Lord Rayleigh will hardly be understood until after the chapter on Diffraction has been read, it is introduced at this point for future reference.

"Let A_0B_0 (Fig. 97) be a plane wave-surface of the light before it falls upon the prisms, AB the corresponding wave-surface for a particular part of the spectrum after the light has passed the prism, or after it has passed the eye-piece of the observing-telescope. The path of the ray from the wave-surface A_0B_0 to A or B is determined by the condition that the optical distance, represented by $\int \mu ds$, is a minimum; and as AB is by supposition a wave-surface, this optical distance is the same for both points. Thus

$$\int \mu ds \text{ (for } A) = \int \mu ds \text{ (for } B). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

"We have now to consider the behavior of light belonging to a neighboring part of the spectrum. The path of a ray from the wave-surface A_0B_0 to A is changed; but in virtue of the minimum property the change may be neglected in calculating the optical distance, as it influences the result by quantities of the second order only in the change of refrangibility. Accordingly the optical distance from A_0B_0 to A is represented by $\int(\mu+\delta\mu)ds$, the integration being along the path $A_0 \ldots A$; and, similarly, the optical distance between A_0B_0 and B is represented by $\int(\mu+\delta\mu)ds$, where the integration is along the path $B_0 \ldots B$. In virtue of (2) the difference of the optical distance is

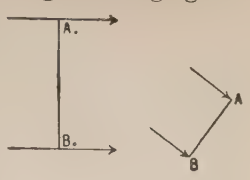


FIG. 97.

$$\int\delta\mu ds \text{ (along } B_0 \ldots B) - \int\delta\mu ds \text{ (along } A_0 \ldots A). \quad . \quad . \quad (3)$$

"The new wave-surface is formed in such a position that the optical distance is constant; and therefore the *dispersion*, or the angle through which the wave-surface is turned by the change in refrangibility, is found simply by dividing (3) by the distance AB. If, as in common flint-glass spectroscopes, there is only one dispersing substance, $\int\delta\mu ds = \delta\mu s$, where s is simply the thickness traversed by the ray. If we call the width of the emergent beam a, the dispersion is represented by $\delta\mu(s_2-s_1)/a$, s_1 and s_2 being the thicknesses traversed by the extreme rays. In a properly constructed instrument s_1 is negligible, and s_2 is the aggregate thickness of the prisms at their thick ends, which we will call t; so that the dispersion (θ) is given by

$$\theta = \frac{t\delta\mu}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

"The condition of resolution of a double line whose components subtend an angle θ is that θ must exceed $\frac{\lambda}{a}$. Hence from (4), in order that a double line may be resolved whose components have indices μ and $\mu+\delta\mu$, it is necessary that t should exceed the value given by the following equation:

$$t = \frac{\lambda}{\delta\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which expresses that the relative retardation of the extreme rays due to the change of refrangibility is the same, namely one wavelength, as that incurred without a change of refrangibility when we pass from the principal direction to that corresponding to the first minimum of illumination.

"That the resolving power of a prismatic spectroscope of given dispersive material is proportional to the total thickness used, with-

out regard to the number of angles, or setting of the prisms, is a most important, perhaps the most important, proposition in connection with this subject. Hitherto in descriptions of spectroscopes far too much stress has been laid upon the amount of dispersion produced by the prisms; but this element by itself tells nothing as to the power of an instrument. It is well known that by a sufficiently close approach to a grazing emergence, the dispersion of a prism of given thickness may be increased without limit; but there is no corresponding gain in resolving-power. So far as resolving-power is concerned, it is a matter of indifference whether dispersion be effected by the prisms or by the telescope."

The expression for the resolving-power of a prism is usually written in the form $\frac{\lambda}{\delta\lambda} = t\frac{\delta\mu}{\delta\lambda}$, which follows at once from (5).

This equation states that two lines of wave-lengths λ and $\lambda+\delta\lambda$ will be just barely separated when the thickness of the prism's base t, multiplied by $\frac{\delta\mu}{\delta\lambda}$, is equal to $\frac{\lambda}{\delta\lambda}$.

As an example we may calculate the thickness of a prism which will just separate the sodium lines. We must first get a value for $\frac{\delta\mu}{\delta\lambda}$.

This we can do by differentiating the dispersion formula

$$\mu = A + \frac{B}{\lambda^2},$$

$$\frac{\delta\mu}{\delta\lambda} = -\frac{2B}{\lambda^3}.$$

The value of B varies with the material of the prism. Let

$B = .984\times10^{-10}$, which is for extra dense flint,

$\lambda = 5.890\times10^{-5}$ cms.,

$\delta\lambda = .006\times10^{-5}$ cms. (difference between D_1 and D_2).

Therefore $t = \frac{\lambda^4}{2B\delta\lambda} = \frac{10^{10}\lambda^4}{1.968\,\delta\lambda} = 1.02$ cms.

The base of the prism must thus be at least a centimeter thick if the sodium lines are to appear separated.

Lord Rayleigh found as a result of a number of experiments that from 1.2 to 1.4 cms. were actually required, depending on the observer.

He also found both from theory and by experiment that a decided improvement in resolving power resulted from stopping the rays which passed through the centre of the aperture.

Christiansen's Experiment (*Wied. Ann.*, Nov. 1884). — While engaged upon some determinations of the refractive indices of white powders by the method of immersing them in liquid mixtures of the same refractive index, Christiansen observed some very remarkable and interesting effects. Owing to the different dispersive powers of the liquid and powder, complete transparency could only be obtained for monochromatic light. If white light was employed

the transmitted light was highly colored, the transmitted color corresponding to the particular wave-length for which the two substances happened to have the same refractive index. Finely powdered glass immersed in a mixture of benzol and bisulphide of carbon was found to exhibit the colors well. The powder must be quite free from dirt, the elimination of which is sometimes very difficult. The author has obtained the best results with the powdered quartz, which can be procured from the large chemical houses. The powder is boiled in nitro-muriatic acid to free it from impurities and thoroughly washed in clean water. It is then dried and placed in a small flask with enough bisulphide of carbon to wet it thoroughly. Benzol is then added a little at a time until the mixture begins to get transparent. It will be found that red light is transmitted first, then yellow, green, and blue in succession as more benzol is added. It is best to stop when the transmitted light is yellow. In the general illumination of a brightly lighted room the colors are not very pronounced, and it is best to employ a distant lamp in a fairly dark room as the source of light. If a permanent preparation is desired, the following method gives good results. A quantity of the quartz powder is introduced into a 100 c.c. flask (not more than $\frac{1}{10}$ of the volume of the flask), the neck of which is then drawn down in a blast until it has a diameter of only a few mms. The liquid previously adjusted in the manner described is then introduced in sufficient quantity to form a rather thick pasty mass, which will stick in a thick layer to the walls of the flask if it is shaken. The flask is then packed in powdered ice and salt and the neck closed by fusion in the flame. The freezing mixture is necessary on account of the inflammability of the vapor and its tension, and it is a good plan to wrap a towel around the beaker containing the flask and cooling mixture in case of explosion. On removing the flask from the ice it will be found to be quite opaque, owing to the change in the refractive index of the liquid. As the temperature rises red light is transmitted first, and by slightly warming the flask in spots by momentary contact with a flame or even with the fingers all colors of the rainbow may be made to appear simultaneously, the whole appearing like a great opal. The reason of these beautiful temperature changes will be readily understood by reference to Fig. 98.

Suppose A to be a linear source of light which is deviated to the right and spread out into two spectra by prisms of the same angle, the one composed of quartz, the other of the liquid mixture. The refractive indices having the same value for yellow light, the yellow of one spectrum will fall immediately above the yellow of the other. Since, however, the liquid has a much higher dispersion it will yield a longer spectrum and the other colors will not be in coincidence, or in other words the refractive indices are different for all the other

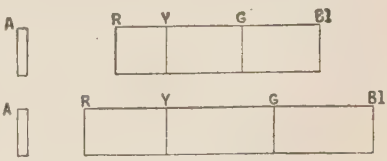


Fig. 98.

colors. The color transmitted will obviously be the one for which we have coincidence in the above schematic diagram. The other colors will be more or less scattered by irregular refractions and reflections. Suppose now we heat the mixture, the increase of temperature will cause the liquid to expand and its refractive index to decrease, while the effect upon the quartz is comparatively slight. This will mean a shift of the lower spectrum in the diagram towards the left, the green regions of the two spectra coming into coincidence, while a further increase of temperature will bring the blue regions together. The effect of the warming is thus to shift the region of transmission down the spectrum towards the blue.

In general, unless the thickness of the heterogeneous medium is considerable, the light which is not directly transmitted emerges to some extent as diffused light. The color of this diffused light is complementary to the transmitted, and the green image of a lamp flame seen through a thin layer of the paste is surrounded by a purple halo. The colors seen when equal volumes of glycerine and turpentine are shaken together into an emulsion are of similar nature, though erroneously attributed to interference in some text-books. The opalescent precipitate obtained by the addition of hydrofluorsilicic acid to a solution of potassium chloride has been found by the author to be another case, the color of the transmitted light changing in a most beautiful manner upon the addition of water, which diminishes the refractive index of the liquid, precisely as the rise of temperature did in the case of the benzol mixture. Fuller particulars regarding these curious mixtures will be found in the original papers of Christiansen, and in an interesting paper by Lord Rayleigh (*Phil. Mag.*, xx. 358, 1885).

Determination of the Dispersion of a Substance in the Form of a Powder. — If a transparent substance in the form of a powder be mixed with a liquid of the same refractive index, the whole becomes optically homogeneous, and the opacity resulting from the irregular reflection and refraction of the particles disappears. Owing to the irrationality of dispersion it is not possible to obtain a liquid of exactly the same refractive index and dispersion, the mixture being optically homogeneous for a single color only: this color is transmitted as we have seen, while the other colors are scattered, and to a greater or less extent refused transmission. Suppose we wish to determine the dispersion of precipitated potassium fluosilicate, which in the solution of KCl, in which it is formed, shows brilliant opalescent colors by transmitted light. Introduce the mixture into a hollow prism and allow it to stand until the precipitate has settled. Place the prism on the table of a spectrometer, cover the upper part of the prism with a card to cut off the light which passes through the clear liquid and examine the transmitted light with a telescope. It will be found to consist of some definite portion of the spectrum, which can be considerably narrowed by shielding all of the prism except the base. Set the cross hair of the eye-piece on the centre of this band, uncover the upper portion of the prism and note the wave-length of the Fraunhofer line which comes nearest to the cross hair. Determine the refractive index of the

liquid for this line in the usual manner, which will be also the refractive index of the powder for the same color. By adding KCl or water we can vary the refractive index of the liquid, making it coincide with that of the powder for the other colors of the spectrum, and in this way the dispersion of the powder can be determined. Compare this with the dispersion of the liquid at such a density, say, that it is optically the equivalent of the powder for green light.

Unless the prism has a thickness of about 10 cms. the transmitted color is very impure in the case of the fluosilicate. In cases such as this more accurate results can be secured by putting the mixture into a long tube closed by glass plates, and making an independent observation of the wave-length of the transmitted light.

The potassium fluosilicate has been found by the author to have the lowest mean refractive index of any known transparent solid, and a dispersion much less than that of water.

Anomalous Dispersion. — In the case of transparent substances the dispersion is said to be normal, that is, the refractive index increases as the wave-length decreases, though the rate of change varies according to the nature of the substance.

In the case of substances which show selective absorption this is not generally the case, the refractive index for the short waves on the blue side of the absorption band being less than the index for the red light on the other side of the band.

This phenomenon has been named anomalous dispersion, but, as we shall see presently, there is nothing anomalous about it, the so-called normal dispersion being nothing more than a special case of the anomalous. Fox Talbot appears to have been the first to notice the peculiar effect, but his discovery was not followed up. In 1860 Le Roux (*Ann. de Chimie et de Physique*, 3d series, vol. xli., page 285, 1861) discovered that a prism containing iodine vapor deviated the red rays more than the blue, the indices at a temperature of 700° C. for the red and violet being 1.0205 and 1.019. Christiansen in 1870 (*Pogg. Ann.*, 1870) detected anomalous dispersion in the case of an alcoholic solution of fuchsine, which is one of the aniline dyes having a strong absorption band in the green. Of the remaining colors, the red, orange, and yellow occur in the same order as in the case of a glass prism. The violet, however, is less refracted than the red, and separated from it by a dark interval. Christiansen's prism was made of two glass plates inclined at an angle of 1 degree, the solution being held between them by capillarity. The subject was next investigated by Kundt, whose papers will be found in *Pogg. Ann.*, 1871, 1872. His observations showed that the phenomenon is to be observed in the case of all bodies which possess what is known as surface color, that is, bodies which selectively reflect certain wave lengths. Ordinary pigments do not belong to this class, the color being produced by absorption, as we shall see in a succeeding chapter. Kundt applied the method of crossed prisms, due originally to Newton, to the investigation of anomalous dispersion. If a spectrum is formed by a glass prism with its refracting edge vertical, and this spectrum is further deviated by a

prism formed of an alcoholic solution of some aniline dye with its refracting edge horizontal, the appearance seen will be similar to that shown in Fig. 99. Kundt established the law that on approaching an absorption band from the red side the refractive index is abnormally increased by the presence of the band, while if the approach is from the blue side the index is abnormally decreased. So great is the difficulty of seeing the effect with the small dispersion obtainable by alcoholic solutions, that the earlier results of Kundt were not at first accepted by some physicists of repute, the effect being attributed to a want of achromatism of the eye. The demonstration by means of crossed prisms, however, removed all doubts regarding the reality of the phenomenon.

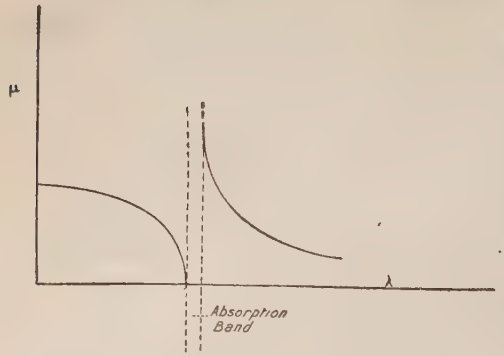


Fig. 99.

Considerable trouble is usually found in repeating Kundt's experiment with fluid prisms.

The phenomenon can be studied to much better advantage by means of prisms formed by squeezing fused cyanine between plates of glass.[1] A certain amount of dexterity is required to make good prisms, which can only be acquired by practice. Small rectangular pieces of thin German plate glass are prepared (measuring about 2×3 cm.), and a thin strip cut from a visiting-card glued along the short side of one. A piece of cyanine[2] about the size of a coarse shot is placed near the opposite side, and the edge of the plate heated over a small flame until the dye fuses, holding another coverstrip in the flame at the same time, in order to have both at about the same temperature. The hot edge of the cover is now to be brought down into the cyanine, and the plate gently lowered until the edge rests on the strip of card. The plates must be at once placed under pressure in a small clamp, where they are to remain until cold. The pressure is to be applied close to the refracting edge of the prism only, as shown in figure. This is very important. Experience is the only guide to the degree of pressure required.

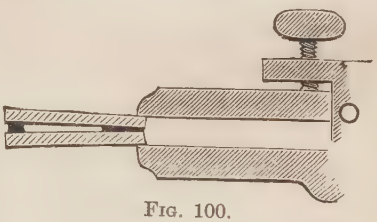


Fig. 100.

[1] Wood, *Phil. Mag.*, June, 1901.

[2] Some preparations of cyanine do not fuse. That used for the preparation of prisms was in the form of needle-like crystals, and was prepared by Grüber.

It will be found that there is a very narrow strip of clear glass at the refracting edge, where the glass plates have come into optical contact. This produces a diffraction-band superposed on the anomalous spectrum, but it is so faint that it is not troublesome. One has only to view a gas flame turned edgewise through the prism, the anomalous spectrum showing colors in the order orange red, blue, green, the latter being the least deviated.

It is usually necessary to turn the prism slightly to get the green part of the spectrum; that is, the incidence should not be normal.

If a prism of this nature is covered with a small diffraction grating, the lines of which are perpendicular to the edge of the prism, the oppositely curved branches of the diffraction spectra appear most beautifully when an arc light is viewed through the combination. If a grating is not available, the cyanine prism can be mounted over a small aperture in a card and combined with a glass prism of low dispersion, or better a water prism, both being mounted on the table of a spectrometer illuminated with sun or arc light.

Other remarkable cases will be described in the Chapter on the Theory of Dispersion.

Anomalous Dispersion in its bearing on Solar Phenomena. — In a communication published in the *Proceedings of the Royal Academy of Sciences*, Amsterdam,[1] W. H. Julius makes the very brilliant suggestion that the "flash spectrum" seen immediately at totality may be due to photosphere light abnormally refracted in the atmosphere of metallic vapors surrounding the sun: in other words, the light of the flash spectrum does not come from the reversing layer at all, but from the photosphere. He shows that the light which will be thus abnormally refracted will be of wave-lengths almost identical with the wave-lengths which the metallic vapors are themselves capable of radiating, that is, it will be light of wave-lengths nearly identical with those of the absorption bands of the vapors. This beautiful theory not only explains the apparent shallowness of the reversing layer, a thing that has always puzzled astrophysicists, but it accounts for the extraordinary brilliancy of the lines.

The theory of Julius supposes the sun to be surrounded by an atmosphere of metallic vapors, the density and refractive index of which decrease with increasing distance from the surface. In this atmosphere the rays of light coming from the photosphere will move in curved paths similar to those of rays in our own atmosphere. The reader should refer back to Schmidt's theory of the solar disk.

The refractive index is, however, very small except for wave-lengths very near those which are absorbed by the vapor, consequently the light most strongly refracted, if it could be sorted out and examined with the spectroscope, would resemble very closely the light emitted by the vapors. Julius shows that this sorting out of the more refrangible rays may account for the bright line spectrum usually attributed to the reversing layer, these rays moving in curved paths in the solar atmosphere, thus reaching us after the photosphere has been hidden by the moon.

[1] See also *Astrophysical Journal*, xii. 195.

This phenomenon, namely the production of a bright line spectrum by the anomalous refraction of light from a white-hot source, was reproduced in the laboratory by the author, and independently by Ebert at about the same time. The conditions supposed by Julius to exist at the surface of the sun were imitated as closely as possible, and a spectrum of bright lines was obtained with light from a source showing a continuous spectrum, by means of anomalous dispersion in an incandescent metallic vapor.

For the reproduction of the phenomenon in the laboratory it is necessary to form an atmosphere of metallic vapor in which the refractive index changes rapidly from layer to layer. This was accomplished by allowing the flame of a Bunsen burner fed with metallic sodium to play against the under side of a white plaster plate. On looking along the surface of the plate it was seen that a dark space existed between the flame and the cold surface, resembling somewhat the dark space surrounding the cathode of a Crookes's tube. It seemed highly probable that, inasmuch as the temperature of the flame was lowered by contact with the plate, the density of the sodium vapor would increase very rapidly from the surface of the plate downward. The under surface of the plaster plate having been thus covered with a non-homogeneous layer of sodium vapor, a spot at the edge of the flame was illuminated with sunlight concentrated by a large mirror. This spot radiated white light in every direction and corresponded to the incandescent photosphere of the sun (Fig. 101). A telescope provided with an objective

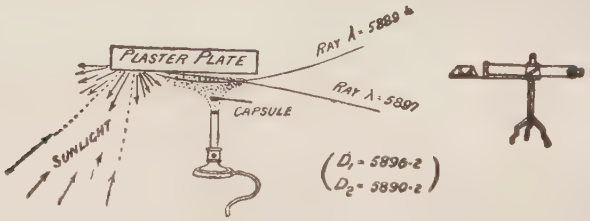


FIG. 101.

direct vision prism was directed toward the white spot and moved into such a position that, owing to the reduction in the width of the source of light by foreshortening, the Fraunhofer lines appeared in the spectrum. This represented the stage of an eclipse when only the thin crescent of the sun is visible. The sodium flame appeared superposed on the spectrum, of course. On moving the spectroscope until it was well inside of the plane of the illuminated surface and feeding the flame with fresh sodium, the solar spectrum vanished and there suddenly blazed out two narrow bright yellow lines, almost exactly in the place of the sodium lines, as is shown in Fig. 102, in which the inverted sodium flame appears on the continuous spectrum. Cutting off the sunlight with a screen caused the instant disappearance of the bright lines. Repeating the experiment it was found that the bright lines came into view on the sides of the sodium lines towards the blue, that is to say, it is light for

which the medium has an abnormally low refractive index that is bent around the edge of the plate and enters the instrument. This is precisely what we should expect, for sodium vapor has a refractive index of less than 1 for waves slightly shorter than D_1 and D_2, as was shown by Julius in his paper. The rays then will be concave upward in a medium in which the refractive index varies, as it has been supposed to vary in the present case. If the sodium vapor is very dense we see only a single bright line bordering D_2, owing to the complete absorption of the light between the lines.

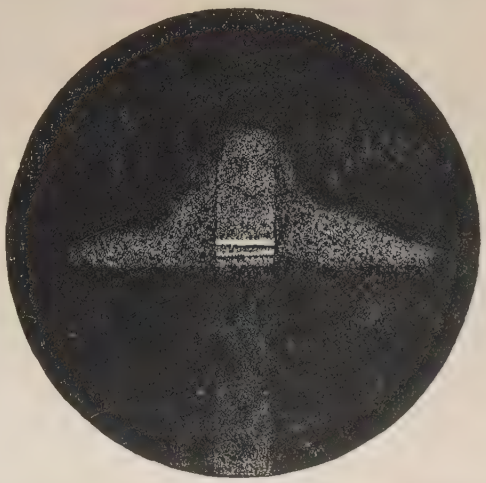


FIG. 102.—FLASH SPECTRUM OF SODIUM PRODUCED BY ANOMALOUS DISPERSION.

A search was next instituted for the light of a wave-length slightly greater than that of the sodium lines. For these waves the vapor has a refractive index greater than 1, consequently the rays will be concave downward in the layer of vapor. If we move our prismatic telescope down in a search for these rays, the solar spectrum will appear and drown out everything, but if we set up a screen (shown in Fig. 103) in such a position as to just cut off the light from the illuminated spot, and feed the flame with sodium, we shall presently see bright lines appear on the side of the sodium lines towards the red. In this case when the vapor is dense we get only a single line bordering D_1. The path of these rays is indicated (on an exaggerated scale) in Fig. 101. The arrangement described is inconvenient in many ways to work with, and was accordingly modified in the following way.

The light of an arc lamp is focussed on a horizontal slit, and a flat metal plate supported so that the plane in which its under surface lies coincides with the plane of the slit. The plate should be an inch or so thick, with a fairly level surface. At a distance of about two metres a telescope provided with a prism (direct vision if possible), arranged so as to give a vertical spectrum, is placed at such a height that the prism barely catches the rays coming from the slit and grazing the surface of the plate (Fig. 103). On looking into the telescope we see a bright continuous spectrum, and the telescope is to be raised until this becomes quite faint. The Bunsen burner beneath the plate is now to be lighted and a bit of sodium, in a small iron capsule, introduced into the centre of the flame. The results obtained are practically identical with those which have

been described. The flash spectrum of potassium has been obtained in a similar manner, consisting of lines in the extreme red, from one to three in number according to the density of the vapor and position of the telescope. Fair results have also been obtained with thallium.

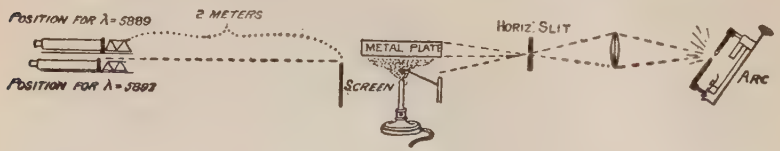


FIG. 103.

Julius applies the anomalous dispersion theory to the prominences as well as to the reversing layer. This phenomenon can also be reproduced in the laboratory. Referring to Fig. 103, we see that its principle is identical with that of the "schlieren" apparatus of Töpler, described in the previous chapter. By arranging a similar apparatus illuminated with arc light, and setting the screens so that the field is dark, most interesting results can be obtained by heating a small capsule containing a bit of metallic sodium in front of the large lens, and placing a large direct vision prism in front of the telescope.

Julius has made quite recently (*Astrophysical J.*, Vol. XXV., No. 2), a very comprehensive study of the ways in which the appearance of an absorption band may be modified by dispersion. The source of light was placed behind a metal tube electrically heated in which sodium vapor was formed. By means of screens with apertures of various shapes the appearance of the D lines could be modified in various ways. Wings could be caused to appear on either side of the lines, as in β and β' (Plate 4) or the continuous spectrum could be caused to disappear entirely, a pair of bright lines only remaining, as in γ and γ'. By using a pair of crossed slits as a screen the curious effects shown in the second column of photographs were obtained, which have suggested to Julius that many of the phenomena seen in the spectra of sunspots, faculae, prominences, etc., may be the result of anomalous dispersion.

Julius has recently described (*Phys. Zeit.*, 1910) some experiments intended to illustrate how dark spots on the solar disk can be produced by refraction. Drops of glycerine suspended in strong brine in a glass trough appeared dark when the trough was placed before an illuminated screen, supposed to represent the sun's disk. He appears to have overlooked the fact that the glycerine drop has a "fish-eye view" of the disk which subtends a solid angle of 96° only, instead of 180°. It seems doubtful if anomalous dispersion can explain as many phenomena as Julius supposes.

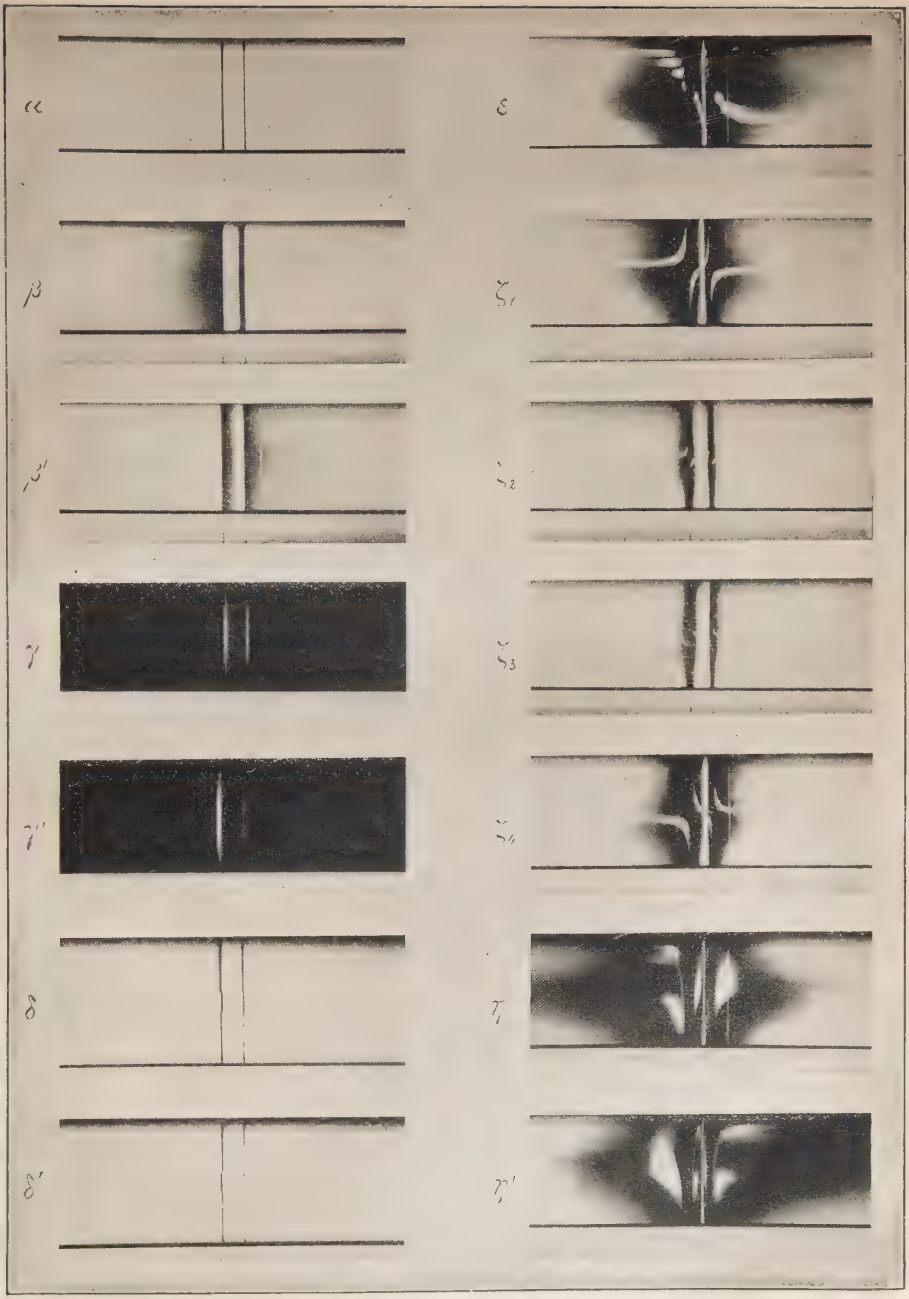


PLATE 4.

CHAPTER VI

INTERFERENCE OF LIGHT

Thus far we have treated single disturbances only, and have not considered the effect at a point when two or more trains of waves act on it simultaneously. We know from observation that two rays of light will cross each other without in any way interfering with one another. The feeble rays from a faintly illuminated object will cross a region traversed by rays of great intensity without being influenced in any way so far as we can see. In this respect then light does not interfere with light. When two light-waves strike the same particle of ether at the same time, its displacement is the algebraic sum of the displacements that would be produced by the waves acting separately. This is known as the principle of superposition. It was stated by Huygens in 1678 as follows. "The displacement, due to a source of small vibrations, is the same whether it acts alone or in conjunction with other sources, provided the displacements are small." This is the fundamental principle which underlies the whole subject of interference. When we consider the effect at a point which is simultaneously acted upon by two separate waves, we have then merely to sum the separate effects.

Thus, if either of the two waves acting alone would cause the ether particle to execute a vibration of unit amplitude, both together will cause it to vibrate over double the path, if the waves are in the same phase, that is, if they both reach the point at the same moment. If they reach the point in opposite phase, that is, half a wave-length apart, the displacements are equal but in opposite directions, the resultant displacement being zero, or in other words the particle does not move.

We must note carefully, however, that the interference is only at this point. The waves have not destroyed each other, for each runs along beyond the point in question precisely the same as if it had not encountered the other. Indeed this must be so, for waves represent energy, and energy cannot be destroyed.

Interference then does not destroy any of the energy, and we shall see later on that whenever we produce a decrease in the illumination at any point by means of interference, we shall produce a corresponding increase at some other point, or the total illumination remains the same. That this is strictly true we shall prove presently.

The intensity of the illumination obviously depends on the amplitude of the vibration, but the relation between them is not at once obvious. We say in general that two candles produce double the illumination that one candle does, that three candles produce

triple the illumination, etc. What is it that we have doubled at the point by lighting the second candle? At first sight it might appear as if we had doubled the amplitude, but we shall show presently that this is not the case. One thing, however, we can be pretty sure of, namely, that we have doubled the amount of energy at the point. Now the energy in wave-motion exists partly as kinetic and partly as potential, that is, we have displaced particles at rest but possessing potential energy in virtue of their displacement from their position of equilibrium, and particles moving across the line of equilibrium which possess kinetic energy only. Other particles on the wave possess both potential and kinetic energy, and it can be shown that the total energy of the wave is equally divided between potential and kinetic. Let us now determine the relation existing between the energy and the amplitude.

Average Kinetic Energy of a Vibrating Particle. — The displacement of a particle at any time t is given by the equation

$$y = a \sin (\omega t - \alpha).$$

Its velocity at any moment then will be $v = \frac{dy}{dt} = a\omega \cos (\omega t - \alpha)$ and its kinetic energy $\frac{1}{2} mv^2$, where m represents the mass of the particle.

The velocity varies from 0 to $a\omega$, as is clear from the above formula, and the mean energy during a complete vibration of periodic time T is

$$\frac{1}{T}\int_0^T \frac{1}{2} mv^2 dt = \frac{ma^2\omega^2}{4T}\int_0^T 2\cos^2(\omega t - \alpha)dt = \frac{ma^2\omega^2}{4T}\int_0^T \left\{1 + \cos 2(\omega t - \alpha)dt\right\}$$

$$= \frac{ma^2\omega^2}{4\pi}\Big|_0^T \left\{t + \frac{1}{2\omega}\sin 2(\omega t - \alpha)\right\} = \frac{1}{4} ma^2\omega^2, \text{ in which } \omega = \frac{2\pi}{T}.$$

The average energy is therefore $\frac{ma^2\omega^2}{4} = \frac{m\pi^2 a^2}{T^2}$,

which can be taken as the measure of the intensity, if we define intensity as the energy in unit volume of the vibrating medium. It can be proved that the total energy is evenly divided between kinetic and potential, and since we have only considered the kinetic energy in the above treatment the total energy will be double the amount calculated. We can also define intensity as the quantity of energy transmitted in unit time across unit cross section of a plane perpendicular to the direction in which the energy is travelling. In this case the velocity of propagation enters as a factor, and we must multiply the quantity calculated above by $v = \frac{\lambda}{T}$, which gives us $\frac{m\pi^2 a^2 \lambda}{T^3}$.

The important thing to notice is that the intensity varies directly as the square of the amplitude, and inversely as the square of the periodic time. The first is of importance in the study of interfer-

ence, the second in considering the laws of radiation which will form the subject of a subsequent chapter.

If we are dealing with two sources of light which emit monochromatic radiations of the same periodic time or wave-length, their intensities are in the ratio of the squares of their amplitudes or

$$\frac{I}{I'} = \frac{a^2}{a'^2}.$$

In comparing the intensities when the periodic times are different, we cannot use the eye, for it is impossible to judge accurately of the equality between two different colors. Moreover the eye cannot directly determine the true intensity, for, as we know, the true intensity or energy of the extreme red end of the spectrum is far greater than that of the yellow, while the eye is more strongly impressed by the latter. In comparing the intensities of two sources which do not emit similar radiations, we must resort to some measuring instrument which reduces them to energy of the same type, for example the thermopile or bolometer, which measures their heating power. Since the intensity of radiation varies as the inverse square of the distance from the source, as can be proved by the most elementary methods, it follows that the amplitude varies inversely as the distance.

Composition of Vibrations. — If we have a point moving in a circular orbit with a uniform velocity, the projection of this point on any diameter of the circle moves with harmonic motion, just as does a particle vibrating under the influence of a force directly proportional to its distance from its position of equilibrium. The point moving in a circle has an acceleration V^2/r (directed towards the centre), where V = the orbital velocity, and r = the radius of the circle. This acceleration can be resolved into two components parallel and at right angles to the given diameter AA'. The one parallel to the diameter is $V^2/r \times x/r$, where x is the distance of P, the projection of the point on the diameter, from the centre of the circle (Fig. 104).

The acceleration of P is then $\frac{V^2}{r^2}.x$, directed always towards the centre, and proportional to its distance from the centre. This acceleration is similar to that which the particles of an elastic body receive when moved out of their position of equilibrium, and we assume the ether particles acted on by a force of a kindred nature. The velocity with which the point P moves on the diameter is $v = V \sin \phi$, where ϕ represents the phase.

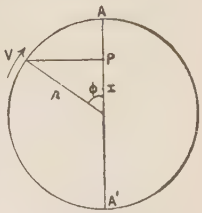


FIG. 104.

Suppose now we require the effect on a point of two harmonic motions of equal periods and different amplitudes and phases. We can represent their motions by constructing two concentric circles with radii proportional to the amplitudes (Fig. 105).

The two harmonic motions will be represented by the projections

on a diameter of two points G and G', which move around these circles with equal angular velocity. P will then represent the position of the particle at a given time as due to the motion represented by G alone, while P' will represent its position at the same time as due to the motion represented by G' alone. If both these motions are impressed simultaneously, the position of the particle will be represented by R, so situated that $RC = PC + P'C$ (by the principle of superposition). The phase difference between the two vibrations is the angle GCG', which of course remains constant. If we complete the parallelogram $GCG'S$, R will represent the projection of S on the diameter, and as the parallelogram turns with G and G', the motion of R, the projection of S, will represent the resultant motion. The diagonal of the parallelogram is evidently the amplitude of the resultant vibration, and its square measures the intensity. Now the square of the diameter of a parallelogram is by Geometry equal to the sum of the squares on two adjacent sides, plus twice the product of the sides into the cosine of the included angle. Consequently if a and a' are the amplitudes of the component vibrations, and e the phase difference between them, the resultant intensity will be

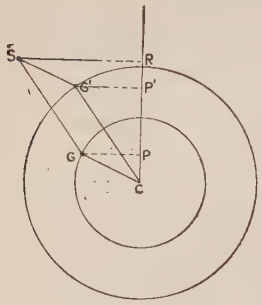


FIG. 105.

$$I = a^2 + a'^2 + 2aa' \cos e.$$

Suppose now that we have two waves of equal length and amplitude, arriving at a point in the same phase. In the above formula a will equal a', and $\cos e$ will equal one, therefore the resultant intensity will be $4a^2$, or quadruple the illumination produced by one wave alone. If the two waves reach the point a quarter of a wavelength apart the phase difference will be 90°, and the illumination $2a^2$, or twice that due to a single wave. If the phase difference is 180°, then $\cos e = -1$ and the illumination becomes zero.

Distribution of Illumination. — If we have two similar sources of light, which are vibrating in unison, the value of e in the expression which we have just deduced will vary from point to point. Let us consider the distribution of illumination along a line, perpendicular to the direction in which the two sources lie. In this case we will consider that $a = a'$ since the sources are similar, and we will consider the sources as lying on each side of the axis of ordinates. Taking distances along the other line as abscissae, and representing the illuminations as ordinates, we have the illumination due to one source represented by a straight line parallel to the axis of abscissae, the ordinate of which is a^2. With both sources acting together the amplitude will vary from point to point; on the axis of ordinates, where the disturbances arrive in the same phase, we have the amplitude $2a$ and the intensity $4a^2$. We can express our abscissae

in terms of the phase difference. If this is 90°, or the waves arrive a quarter of a wave-length apart, $\cos e = 0$ and the illumination is $2a^2$, or double that due to one source acting alone. For $e = 180$, $\cos e = -1$ and the illumination is zero. Intermediate points can be determined by assigning different values to e, a curve similar to that shown in Fig. 106 being the result.

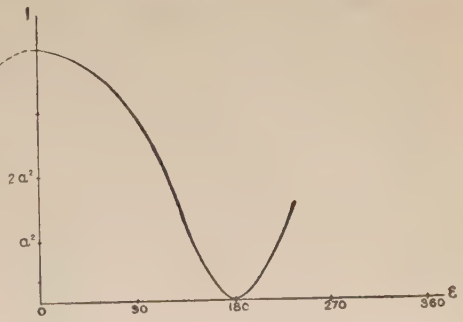


FIG. 106.

If now there be no loss of energy the total illumination must remain the same; we can represent this by the area comprised between the curved line and the axis of abscissae. The total intensity due to the two sources acting without interference, which would be the case if they did not vibrate in unison, would be $2a^2$. This is true of course only when we consider the average illumination for a time which is long in comparison to the time between certain assumed abrupt changes in the phases of the vibrating sources.

If no energy is lost the area between a line parallel to the axis of abscissae of ordinate $2a^2$, and the two ordinates erected at $e = 0$ and $e = 360$ should be equal to the area of the curve within the same limits.

The total illumination along a distance Δx on the axis of abscissae is

$$I_1 = 2a^2 \Delta x,$$

if we assume no interference.

With interference taking place the total illumination is

$I_2 = \int_x^{x+\Delta x} (2a^2 + 2a^2 \cos e)dx$, in which x is the value for which $e = 0$.

Since e is a linear function of x we can write $e = K\Delta x$, in which K is a constant, and if Δx represents the distance from $e = 0$ to $e = 360$

$$K\Delta x = 2\pi.$$

Integrating we have

$$I_2 = 2a^2 \Delta x \int_x^{x+\Delta x} 2a^2 \cos Kx dx,$$

$$I_2 = 2a^2 \Delta x + \frac{2a^2}{K}[\sin K(x + \Delta x) - \sin Kx],$$

$$I_2 = 2a^2 \Delta x = I_1.$$

It must be clearly understood at the outset that to have *permanent* interference, the phase relation between the two sources must

remain constant, or they must be similar; their periodic times of vibration must be the same, and any changes of phase which occur in one must occur also in the other. The only way in which this condition can be attained experimentally is by making one source the image of the other, or by dividing the bundle of rays which issue from a single source into two portions, either by reflection or refraction, and then reuniting them.

Resultant of a Large Number of Disturbances of Arbitrary Phase. — We have seen that when two waves in the same phase act on a point, double the amplitude, and consequently four times the illumination, results. The question now perhaps occurs, why do not two candles produce twice the amplitude of one candle, and consequently four times the illumination? The answer to this will be readily seen if we consider carefully the manner in which any given point is illuminated by a candle. In the flame of a candle there are countless radiating particles, in all possible phases of vibration. The illumination is due to the joint action of them all, and to determine it we must find the *resultant* of a *large* number of vibrations of arbitrary phase. In other words, if we have a great number of particles, each one of which alone would give an amplitude 1 and unit illumination at a given point, what will be the amplitude produced by all of them acting together? If there are n particles, and it so happened that all of them were vibrating in such a manner as to send vibrations in similar phase to the point, the resultant amplitude would be n and the illumination n^2. Another candle with n particles vibrating in the same manner and in unison with the first, acting with the first, would give us an amplitude $2n$ and an illumination $(2n)^2$ or $4n^2$, that is four times the illumination of a single candle. It is obviously impossible, however, for all the particles to send their waves to the point in the same phase, for they are all vibrating independently of each other, and they are, moreover, at different distances from the illuminated point. The amplitude produced cannot then be equal to n. Lord Rayleigh has shown that the average illumination (not amplitude) due to a large number of disturbances of arbitrary phase is simply n times the illumination due to a single one of the disturbances. This was done by the theory of probabilities, and the reader is referred to the original paper for the method, or the earlier edition of this book.

Interference of Light. — Grimaldi, who was the first to observe accurately and describe diffraction, or the bending of light around the edges of obstacles, described as early as 1665 an experiment which he believed proved that darkness could be produced by the addition of light to light. He admitted sunlight into a darkened room through two neighboring pin-holes, and received the light on a white screen. Each pin-hole casts on the screen a circular image of the sun surrounded by a feebly illuminated ring. By placing the screen at such a distance from the pin-holes that the outer rings overlapped, the outer edge of the ring formed by one of the holes being tangent to the outer edge of the sun's image formed by the other, he observed that the edge of the ring was less brilliant in the overlapping

portion than at other places. We shall see presently that interference could not have occurred under these conditions, for two sources of light, in order to produce permanent destructive interference at a given point, must be similar — that is, must be vibrating in unison with similar amplitude and period — and two pin-holes illuminated by sunlight would not fulfil these conditions unless they were less than 0.05 mm. apart, as will be proved later on.

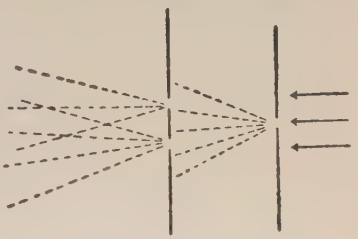


Fig. 107.

A century later this experiment was modified by Young, and true destructive interference of light observed. Young passed the sunlight through a pin-hole, and then received the diverging cone upon two other pin-holes (Fig. 107). From each one of these a divergent cone of light spread out, and where these two cones of light overlapped on a screen, he observed dark and light bands. In this experiment, the two pin-holes lie on the wave-front of the disturbances coming from the first hole, consequently they are always in the same phase. The dark bands are the loci of points situated at distances from the two pin-holes, differing by an odd number of half-waves. The fringes in this experiment, being produced by diffracted light, did not prove that two streams of ordinary light could destroy one another at a point. Diffracted light was not well understood at the time; some modification was supposed to have taken place, and the fringes might be due in some way to this modification.

Fresnel realized the importance of producing two streams of light, capable of interfering and containing no diffracted light. The streams must come from two similar sources, and not pass the edges of any obstacles. This was accomplished by Fresnel by reflecting the rays from a point source of light from two mirrors inclined very slightly towards one another. Two virtual images of the point were thus formed behind the mirrors, separated by a very small distance, depending on the angle between the mirrors. Two mirrors of silvered or black glass, receive light from a point source at S (Fig. 108).

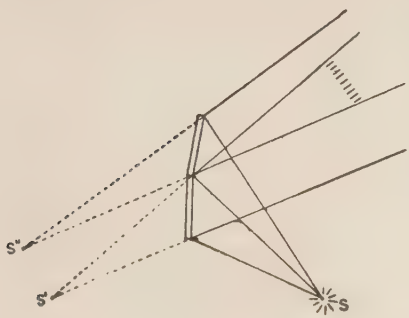


Fig. 108.

The light reflected from the two mirrors comes then from two virtual images S' and S'', which lie very close together if the angle between the mirrors is small. We thus have rays coming from the two similar sources, S' and S'', and within the

region where they overlap interference takes place. The light, instead of being uniformly distributed, is collected, as it were, into bright lines with dark spaces between them. The dark bands are the places where the waves from the two sources arrive half a wave-length apart and destroy each other: at the bright bands the waves arrive together, and we have reënforcement. It is evident that as the angle between the mirrors increases, the two virtual sources S' and S'' approach, coalescing when the angle equals 180.

Let us now examine the form and position of the fringes.

Inasmuch as we can consider the virtual sources S' and S'' as if they were real points of light, we will suppose the mirrors removed, and consider the illumination on a screen placed at a distance "a" from the sources. Let AB be a section of the screen (Fig. 109). At P, which is on a line perpendicular to the line joining the sources at its middle point, we shall have a maximum illumination, since P is equidistant from the sources, and the waves starting together reach it in similar phase and reënforce. Going away from P we shall find a point M half a wave-length nearer S_2 than S_1, and here the waves will arrive half a wave-length apart, and destroy one another. If we advance a little further along the line AB we shall reach a point where the path difference is a whole wave-length, and we shall have another maximum. Let us determine the distance of any bright or dark band from P in terms of the distance a, the distance S between the sources and the wave-length of light. Around M as a centre with a radius MS_2, describe an arc cutting MS_1 at C. Since S_1S_2 is small in comparison to a, this arc is approximately a straight line perpendicular to the line MO (O being the point midway between the two sources).

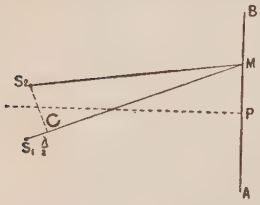


Fig. 109.

S_1S_2 is perpendicular to OP, and therefore the angle $S_1S_2C =$ angle MOP.

If the angles are equal, so also are their circular measures, or $\frac{MP}{OP} = \frac{S_1C}{S_1S_2}$, or calling x the distance of the dark band from the centre of the fringe system, and s the distance between the sources, we have

$$\frac{x}{a} = \frac{\lambda/2}{s}.$$

The general expression then for the position of any bright or dark band will be $x = \frac{a}{s} n \frac{\lambda}{2}$, odd values of n corresponding to dark bands, even values to light.

It is clear from the diagram that the point P will be a maximum for light of any color or wave-length. If the source of light is white this central band will also be white. The positions of other maxima being a function of the wave-length, it follows that the spacing

between the bands will be different for the different colors, consequently there will be an overlapping, and instead of white fringes with dark spaces between we shall have colored fringes, the dark minima being absent except in the immediate vicinity of the central white band.

We will now take up a more complete investigation of the distribution of the maxima and minima in space. The locus of all points equidistant from two points is a plane perpendicular to the middle point of a line joining the points. The first maximum is then a plane lying between the two sources. The second maximum is the locus of all points in space so situated that the differences between their distances from the sources is one wave-length. Points fulfilling this condition lie on a hyperboloid of revolution, the sources being the foci, for by definition an hyperboloid is a surface generated by the movement of a point in such a way, that the difference between its distances from two fixed points is a constant. The locus of the second maximum will be another hyperboloid with a constant difference of 2. The loci of the maxima and minima in space form a system of confocal hyperboloids, and the fringes formed on a screen intercepting them will be hyperbolae.

In the case of our interference experiments the luminous points are so near together, and the screen so far removed, that its intersections with the hyperboloids are approximately straight lines.

Very satisfactory Fresnel mirrors can be made of modern mirror glass, or even of thin plate glass, unsilvered. Silvered glass is preferable owing to its greater reflecting power. The varnish can be dissolved from the silvered surface with alcohol, and the metal film polished. If glass of this description cannot be procured, a piece of thin plate glass can be chemically silvered. Two pieces measuring about 2 cms. along each edge are laid side by side on a second piece of plate glass, the outer edge of one being raised slightly by means of a narrow strip of thin paper. The edges of the plates should be in contact and both should be pressed against the supporting plate. They are then fastened in this position with a little sealing wax. The angle between the plates should be such as to make the reflected images of an illuminated slit (situated at a distance of 40 or 50 cms.) appear about 3 mms. apart. A suitable slit can be made by ruling a line on a piece of the mirror glass with the point of a knife. It should be backed with a bright sodium flame and the mirrors mounted about 30 cms. from it. The dividing line between the mirrors should be adjusted accurately parallel to the reflected images, which lie on either side of it, and the field examined at a distance of 20 or 30 cms. from the mirrors with an eye-piece or pocket magnifying-glass. The eye-piece should be held at the point at which both of the reflected images are seen. If the fringes do not appear at once they can usually be brought into view by readjusting the mirror for parallelism with the slit, the field being watched with the eye-piece. The distance of the nth fringe from the centre of the system is given by

$$x = \frac{a+b}{2\,a\omega} n \frac{\lambda}{2},$$

a and b being the distances of the slit and the plane in which the fringes are seen from the mirrors, and ω the very acute exterior angle between the mirrors. If we measure this angle, which we can do with a spectrometer, and the distance between the fringes, we can determine roughly the wave-length of the sodium light.

The Flow of Energy in a System of Interference Fringes. — The interference minima formed by two similar sources of light form a system of confocal hyperboloids, and the question of the flow of energy in this case, or any similar case, does not appear to have been discussed. Energy is obviously flowing out from both sources at its normal rate, but the direction of flow is perhaps not quite obvious. Suppose the minima equal to zero, which is nearly correct at the centre of the system. Energy evidently cannot cross a plane along which there is no disturbance.

In stationary waves, if the nodes are absolutely at rest, which is the case if the two wave-trains are of equal amplitude, we cannot speak of a flow of energy across them. A node may be considered as having the properties of a perfect reflector, that is to say the point acquires the power of reflecting as a result of the arrival of a wave travelling in the opposite direction. We are thus forced to the conclusion that the flow of energy in the case of the interference fringes must be along the hyperboloids, that is along curved paths. We can show this experimentally by means of ripples in mercury excited by two needles mounted on the prong of a tuning-fork. If we view the mercury surface through a narrow slit opened and closed by the vibrations of another fork slightly out of tune with the first, we see the waves (stroboscopically) creeping slowly along the surface, and following the lines of the hyperboloids. Two questions now naturally occur to us. How does the energy get into the bright fringes, if the dark fringes are supposed to act as barriers? and what is the nature of the wave that is travelling along a bright fringe? In regard to the first question: the dark fringes are never absolutely black, as no one of them is equidistant from both sources. The amplitudes are therefore slightly different, and there will be a flow of energy in the direction of the disturbance having the larger amplitude. Though it may be very slight at any given point, it is ample to account for the flow along the hyperboloid. We can take as an analogous case two parallel sheets of cloth tightly stretched, and very close together. Consider water forcing its way into the space between the two sheets from both sides. A very small flow across unit cross section will give us a large flow across unit section taken perpendicular to the sheets.

We may, however, have a fringe which is absolutely black, for there is nothing to prevent us from considering the sources as vibrating with a difference of phase of 180°. This makes the centre of the system dark, and equal to zero, and it must act as a barrier to the flow of energy from both sources. In other words, the central fringe can be considered as acting as a perfect mirror, and we can regard the fringes as formed by the interference of these reflected waves with the direct. If the flow of energy is along the hyperboloids, it is evident that in the region between the sources

the flow is in a direction nearly perpendicular to the rays. We can watch this flow with the mercury ripples and tuning-forks. (I find that a ring of castor-oil or glycerine poured around the edge of the mercury surface prevents troublesome reflections from the walls, and to a large extent waves due to jars from the table. It is analogous to painting the walls of a room black in optical experiments.) The bars of light perpendicular to the line joining the vibrating sources slide out sideways, each one of them forming one of the waves which travel along the hyperboloid. We can perhaps get a better idea of what happens if we consider what is going on in a bright fringe outside of the region between the sources. What is the type of wave, and is it capable of showing us both of the sources if it alone is allowed to enter the eye? If we regard the dark fringes as absolutely dark, that is as perfect reflectors, we must regard the waves as travelling between them as between two silver walls. The incidence is very oblique, *i.e.* the wave is nearly perpendicular to the reflecting plane; and if we consider the wave as a portion of a sphere with its centre at one of the sources, the wave after reflection from the interference plane will be a portion of a sphere with its centre of curvature at the other source. This process will repeat itself over and over again, a given portion of the wave-front appearing to come first from one source and then from the other. The bright fringe will then contain two groups of wave-fronts inclined to each other at a small angle. These can be seen with the tuning-fork waves, and on looking over some of Mr. Vincent's photographs, published in the *Philosophical Magazine* some time ago, I found one which showed the phenomenon very clearly. An enlargement of a portion of this photograph is reproduced in Fig. 110, the inclined wave-fronts showing especially well above the point marked *X*.

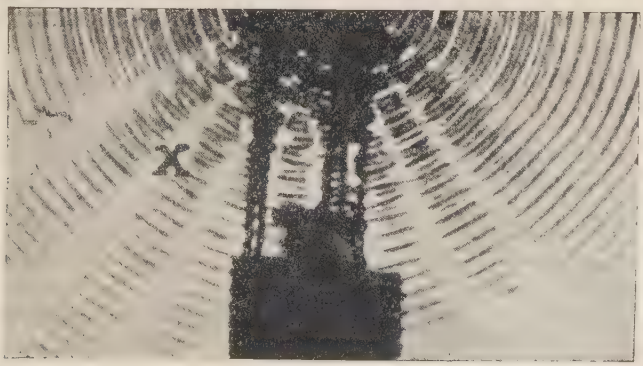


Fig. 110.

Though each bright fringe contains two wave-fronts, we cannot "resolve" the sources with them, for calculation shows that their width will be insufficient. In other words, if we screen off the other bright fringes, passing the waves in the one through a slit, the slit width necessary turns out to be just what is needed to prevent resolution.

In the region between the sources we must regard the same thing as going on, the only difference being that here the incidence is more nearly normal. The waves are stationary on the line joining the sources, but as soon as we get off this line we must regard the stationary waves as oozing out in all directions, the velocity of the oozing increasing with the distance from the line.

The language which I have used here is not very exact, but it is not easy to visualize what is going on, and still harder to put it into words.

Lloyd's Single Mirror. — Even simpler than the Fresnel mirrors is the device employed by Dr. Lloyd. Here the light streams from the source and its reflected image are made to interfere. The experiment is easily repeated with a strip of plate glass thirty or forty centimetres long and three or four wide, mounted in a clamp-stand with its surface vertical. The illuminated slit is placed a little beyond the further end, and one or two millimetres in front of the plane of the surface. If the eye is now brought up to the opposite end, the slit and its reflected image are both seen, and the fringes are easily found at this point with an eye-piece. Dr. Lloyd found that the centre of the system did not lie on the plane of the surface, as might be expected, but was displaced by the width of half a fringe. This is due to the phase change which the light experiences on reflection. As the mirror is turned slowly about a vertical axis, the distance between the fringes changes. With the images close together they are broad and very easily seen; with the images farther apart they are very fine, and only seen with difficulty. This piece of apparatus is the easiest of all to work with, it being almost impossible to miss finding the fringes at the first attempt.

Fresnel's Bi-Prism. — In this experiment the beam of light is divided by refraction by means of a prism of very obtuse angle, as shown in Fig. 111.

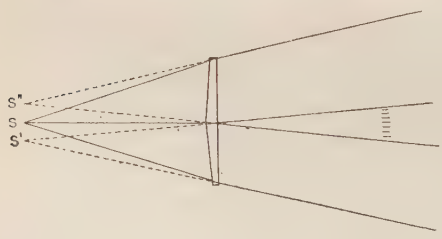


Fig. 111.

The rays, originally emanating from a source at s, after refraction have directions as if they came from the two sources s' and s''. The illuminated slit should be parallel to the edge joining the two opposed prisms.

The wave-length of the light can be approximately determined with the bi-prism.

If a is the distance of the source from the prism, b is the distance of the plane in which the fringes are observed, and c the distance between s_1 and s_2, we have, if we call δ the angle of deviation produced by each half of the prism,

$$c = 2\,a \sin \delta = 2\,a(\mu - 1)\epsilon,$$

in which μ is the refractive index of the glass and ϵ the prism angle.

The distance of the nth fringe from the centre of the system is given by

$$x = \frac{a+b}{c} n \frac{\lambda}{2} = \frac{a+b}{2a(\mu-1)\epsilon} n \frac{\lambda}{2},$$

which shows us that the bi-prism is equivalent to a pair of Fresnel mirrors inclined at an angle $(\mu - 1)\epsilon$. A bi-prism can be easily made in the following manner. Heat a little Canada balsam in a watch glass over a small flame until a drop becomes nearly solid on cooling. Cut two pieces of thin plate glass measuring 1×2 cms., and cement them, with the long edges in contact, to a second piece of plate glass with a little of the balsam, pressing the outer edges into contact with the supporting plate, and allowing the inner edges to be slightly raised (0.5 mm. is about right) by the balsam layer, as shown in Fig. 112. A prism made in this way works almost as well as those supplied by opticians, which are made of a single piece of glass.

FIG. 112.

In using the prism, it should be mounted at a distance of about 40 cms. from the illuminated slit, and the dividing line between the plates made parallel to it. The fringes can be found with the eye-piece in the same way as in the case of the mirrors, a little readjusting of the prism being perhaps necessary. The fringes obtained by all of the devices thus mentioned are modified by diffraction effects, due to the fact that the waves which diverge from the two virtual sources are not complete, but are abruptly cut off at the point of union of the mirrors, or at the obtuse angle of the prism.

Bi-Prism of Variable Angle. — A modification of the bi-prism can be made by cutting off a strip of glass from the *edge* of a sheet of thin plate glass. In polishing the plate the edge is always rounded off a little as a result of the polishing process. If this strip is mounted at a distance of half a metre or so from a slit illuminated by monochromatic light (monochromatic illuminator), and the slit viewed through the portion where the plate begins to "round off," from a distance of 30 cms. or so behind the plate, two images will be seen. An eye-piece will show a beautiful system of interference fringes which crowd closer together as we recede from the edge of the plate. The reason of this is that the rays which are deflected by the edge of the plate, where the prism angle is greatest, come as if from a source at some distance from the real source. The explanation of the dark region between the edge of the plate and the line where the interference fringes begin is very simple and is left to the reader. Diffraction fringes are seen beyond the edge of the shadow; these are much fainter, however.

A photograph of the fringes obtained in this way is reproduced in Fig. 113, together with a diagram of the prism and the rays.

The Corresponding Points of the Sources. — It is clear that continuous interference can result only between streams of light which come from corresponding parts of the two sources. Our slit is

backed by a sodium flame, and even if we make it extremely narrow, the phase of the vibration will by no means be constant across its width. We must remember that the sodium flame contains count-

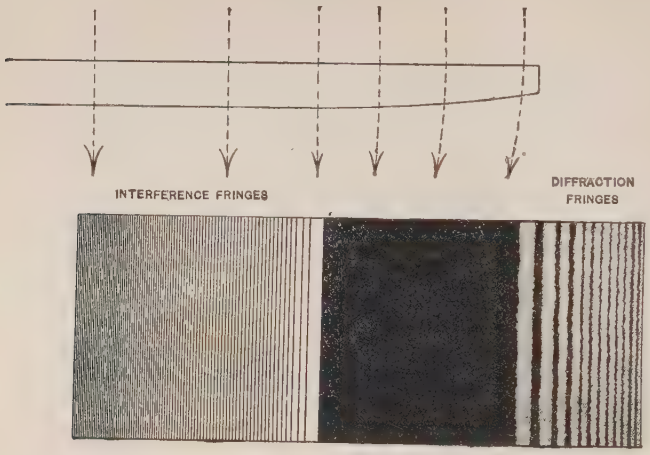


Fig. 113.

less vibrating sources of light, and continuous interference can result only in the case of rays emitted by one of these and its image, or between the two images of the same vibrator. A large number of these sources will be comprised by the width of the slit, consequently interference takes place between streams of light which come from corresponding parts of the images. In the case of the Fresnel mirrors and the bi-prism the corresponding parts lie on the same side of the images, while in the case of Lloyd's single mirror they are on opposite sides. In the latter case the axis of symmetry, or the position of the central fringe, for which the path difference is zero, is the same for all the corresponding points; in the former case it is different for each pair of points. The fringes obtained with Lloyd's mirror are therefore more sharply defined, and a wider slit can be used. This will be clear by reference to Fig. 114, in which A, B, C are corresponding points, and s the axis of symmetry, which in the case of the bi-prism sources is seen to have a different position for each pair of corresponding points.

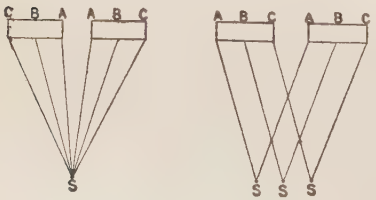


Fig. 114.

Limit to the Number of Fringes. — Very interesting conclusions regarding the vibrations of the molecules in the flame can be drawn from the number of fringes which can be counted. At the first dark fringe it is clear that we have destructive interference between vibrations which left the corresponding points at the same instant.

At the 100th dark fringe we have interference between a vibration from one source with a vibration from the other which left the source 100 T earlier, if T is the time of the vibration. At the 1000th dark fringe we have interference between vibrations which left the sources 1000 T apart. If now we consider that our molecules execute only about 1000 vibrations without any abrupt phase change, it is clear that we cannot have more than 1000 dark fringes; for, with a difference of path greater than 1000 wave-lengths, we shall have a train of waves meeting another train which left the source under different phase conditions, and which may therefore reënforce, instead of destroy, one another. The number of fringes which can be observed gives us therefore information regarding the number of regular vibrations performed by each molecule before an abrupt change occurs. Fizeau counted as many as 50,000 fringes in the case of sodium light, while improved apparatus and methods of modern times have raised the number to a million in the case of certain kinds of monochromatic light, from which we conclude that under favorable conditions as many as a million vibrations can be performed before any abrupt change takes place. We can liken the molecule to a tuning-fork struck at regular intervals with a hammer. At every blow there is an abrupt change of phase. If our fork vibrates 3000 times per second, and we strike it every 2 seconds, we could obtain interference with a path difference of something less than 6000 wave-lengths, while if we struck it but once in 10 seconds the path difference could be increased to nearly 30,000 wave-lengths. In this case the waves would be 4 inches long and the maximum path difference about 2 miles. It is obviously impossible to perform such an experiment, but the analogy is useful.

Shift of the Fringes by Introduction of Thin Transparent Plate. — If a thin plate of some transparent substance is put in the path of one of the interfering streams of light, the optical path will be increased owing to the retardation of the waves in the glass. If the refractive index is μ and the thickness ϵ, the increment of path is $(\mu-1)\epsilon$, in which there are $(\mu-1)\frac{\epsilon}{\lambda}$ waves. Increasing the path by one whole wave-length will cause a bright or dark fringe to shift into the position of its neighbor, therefore in the above case the shift will be n fringe widths; $n=(\mu-1)\frac{\epsilon}{\lambda}$ if we call a fringe width the distance between two bright fringes.

When the fringes are formed with white light the introduction of the plate produces a somewhat more complicated effect. This case will be discussed presently. It is clear that we can determine the refractive index of a thin plate if we know its thickness, and measure the shift of the fringes. The above formula holds only for monochromatic light, and with light of this description the fringes are similar in appearance, and the shift cannot be determined if it exceeds one fringe width, unless it can be produced gradually, as by introducing a gas slowly into a tube, the ends of which are closed with glass plates, and watching the drift of the fringes. More will be said on this subject after we

have taken up the subject of the interference of white light. If we require the actual distance through which the central fringe is shifted we can easily deduce the expression $x=(\mu-1)\epsilon\frac{a}{s}$, in which s is the distance between the sources and a the distance of the screen.

Interference Fringes with White Light. — We have thus far considered our sources as sending out light of a single wave-length only. If we illuminate the slit with white light and examine the fringe system, we find that only a few rainbow colored bands are visible. The cause of this is at once apparent. The formulae which we have deduced for the distances between the fringes show us that this distance is a function of λ the wave-length, which occurs in the numerator of our expression for x, the distance of a given fringe from the centre. It is thus clear that the shorter the wave, the closer together will the fringes lie. If we assume white light to be made up of waves of various lengths, the fringes will be out-of-step at every point save on the axis of symmetry. The central bright fringe will coincide for all the colors, but since the red bands are about twice as broad as the violet, the bands soon get completely out-of-step, and we have practically uniform illumination.

The condition of things is shown in Fig. 115, in which the red fringes are represented by the unbroken line, the violet by the dotted line. The first dark fringe on each side of the central bright one will therefore be tinged deeply with violet. The other colors will produce other systems of bands of intermediate spacing, and it is

Fig. 115.

clear that at points a short distance from the centre, we shall have maximum illumination for a large number of wave-lengths regularly distributed throughout the spectrum. The resultant illumination cannot be distinguished from white light by the unaided eye, and the field therefore appears uniformly illuminated. Something resembling interference is taking place, however, in this region just as before, as we can readily prove by substituting a small spectroscope for the eye-piece, when the spectrum will be found to be crossed by dark bands corresponding to the wave-lengths, for which the position of the slit of the instrument is a position of zero illumination. We can get a better idea of the state of affairs if we consider what happens if we place the slit of the spectroscope on the central bright band and then move it slowly out into the fringe system. At the central bright band we have all colors present, and consequently see a continuous spectrum. On moving the instrument the slit enters presently into the first dark fringe for violet, and the violet of the spectrum disappears. As we move the slit along the other colors disappear in turn, a dark band moving up the spectrum. By the time that we reach the first dark band for red, we are again in a region of maximum illumination for violet, which, therefore, ap-

pears again as the dark band in the spectrum nears the red end. It is clear that owing to the difference of spacing of the fringes, the dark bands will enter the spectrum at the violet end more rapidly than they leave it at the red end; they will consequently accumulate in the spectrum, the number increasing as we move the spectroscope farther and farther away from the central band. The experiment can be easily performed by means of Lloyd's single mirror, illuminating the slit with sun or lamp light, and substituting a small pocket spectroscope for the eye-piece. The instrument should be mounted in a clamp-stand and pointed towards the double source, the slit being close to the edge of the plate. This way of looking at the case is not strictly correct, as we shall see later on. It would be allowable however if white light really consisted of a mixture of all possible colors.

We will now deduce an expression for the number of bands in the spectrum. Take a point in the fringe system corresponding to the nth maximum for λ_1. The path difference will be

$$\delta = 2n\frac{\lambda_1}{2}.$$

Now let the wave-length decrease to λ_2, such that the same path difference is represented by

$$\delta = (2n+1)\frac{\lambda_2}{2}$$

This new value λ_2 represents the wave-length for which the point is a minimum.

Writing
$$2n\frac{\lambda_1}{2} = (2n+1)\frac{\lambda_2}{2},$$

$$2n\lambda_1 = 2n\lambda_2 + \lambda_2, \text{ or adding } \lambda_1 \text{ to each side,}$$

$$2n\lambda_1 - 2n\lambda_2 - \lambda_2 + \lambda_1 = \lambda_1,$$

$$(\lambda_1 - \lambda_2)(2n+1) = \lambda_1,$$

$$\lambda_1 - \lambda_2 = \frac{\lambda_1}{2n+1}.$$

This expression shows us that the change in wave-length $\lambda_1 - \lambda_2$ which is necessary to change the point from a maximum to a minimum is equal to the wave-length divided by $2n+1$. If n is large, *i.e.* if we are far out in the system, the necessary change will be very small. For example, let $\lambda_1 = .0005$, and suppose that we are at the 50th fringe, then

$$.0005 - \lambda_2 = \frac{.0005}{101}, \quad \lambda_2 = .000495.$$

The point will therefore be a maximum or minimum for a large number of wave-lengths, within the range of the visible spectrum. Suppose now that we are at an unknown point in the fringe system and wish to determine the path difference.

Let λ_1, λ_2, λ_3, etc., correspond to the wave-lengths (going from red to violet) of the dark bands in the spectrum.

$$\delta = (2n+1)\frac{\lambda_1}{2} = (2n+3)\frac{\lambda_2}{2} = (2n+2p-1)\frac{\lambda_p}{2}.$$

We count the number of bands between two widely separated Fraunhofer lines (if we are using sunlight). This number is p in the above equation. λ_1 and λ_p are the wave-lengths of the lines between which we have p dark bands; then

$$\delta = (2n+1)\frac{\lambda_1}{2} = (2n+2p-1)\frac{\lambda_p}{2}.$$

Suppose $p = 50$, and $\lambda_1 = 6399$ (C line) and $\lambda_p = 3967$ (H line).

Substituting these values, we find that $n = 80$, which shows us that we are dealing with the 80th minimum for each color. From this we can calculate the path difference, which we find to be 259 half wave-lengths for the violet and 161 for the red.

Fizeau and Foucault were enabled in this way to detect interference with a path difference of 7000 waves. This result has been interpreted by many writers as indicating that the elementary components of white light must consist of periodic wave-trains, several thousand regular vibrations being executed without abrupt change of phase. Lord Rayleigh has shown, however, that we can infer nothing whatsoever about the regularity of the vibrations of the source in this case, the limit of the number of bands seen in the spectroscope depending solely on its resolving power. The interference, in point of fact, does not take place until after the light has passed through the spectroscope. We shall study the case more in detail when we come to the subject of white light.

A Simple Interference Refractometer. — A very simple and inexpensive interference apparatus, which can be used for measuring the refractive index and dispersion of a gas, and for showing and measuring anomalous dispersion at the absorption bands of a gas, can be made of a slit and a long focus lens. This type of apparatus was used by Fizeau in determining the velocity of light in a rapidly moving medium (see Chapter on Relative Motion of Matter and Ether), and it has since been employed by Lord Rayleigh in determining the refractive indices of the rare gases of the atmosphere.

As usually employed its chief disadvantage lies in the fact that the fringes are very narrow, and invisible except when viewed through a very high power eye-piece.

This trouble I have found can be overcome by the use of long focus lenses, which lessens the convergence of the interfering rays. A lens of about 2 metres focus (spectacle lens) is covered by a screen furnished with two vertical slits a millimetre or two in width, and separated by a distance of from .5 to 3 cms. depending on the required separation of the interfering beams. If we are to measure the refractive index of a gas, we use two parallel tubes, with their ends closed by pieces of thin plate glass cut from the same piece,

and the beams must be far enough apart to enable us to pass one down each tube.

The slits can be made by pasting strips of black paper across an aperture cut in a card. Widening the slit decreases the width of the band in which the fringes appear (since the diffraction is then less), but increases their illumination. The illuminated slit should be well made, as it is necessary to make it very narrow. The slit of a small spectroscope can be used, and it should be mounted at a distance of 4 metres from the lens. An image of the sun or the crater of the arc should be focussed on the slit, and the lens with its two apertures placed in such a position that it receives the light from the slit. Hold a card behind the lens, and move back until the two lines of light transmitted by the slits fuse together into a single image at the conjugate focus. This is the position for the eye-piece, and we shall find the image traversed by a beautiful set of vertical interference fringes. Their distance apart will increase, as the distance between the two slits is made less, and their sharpness will increase as the first slit is narrowed.

If it is desired to receive the fringe system on the slit of a spectroscope, the three slits should be mounted in a horizontal position and the slit of the spectroscope substituted for the eye-piece. The spectrum will then be found to be traversed by horizontal dark bands which are closer together in the blue than in the red. This is the arrangement commonly used for the study of anomalous dispersion of gases by the interference method.

The shift of the fringes by the introduction of a transparent plate can be shown by holding a very thin flake of mica over one of the slits. If the flake is too thick the fringes will be shifted out of the field and disappear. Or we can put two pieces of thin plate glass, one in front of each slit, and by slowly turning one of them cause the fringes to move, the result depending on the increase in the path through the glass with increasing angle of incidence.

Application of Interference Phenomena to Astronomical Observations. — If the first slit of the interference refractor just described exceeds a certain width the fringes are no longer visible. This is due to the fact that each vertical linear element of the widened slit forms its own set of fringes, and these sets are relatively displaced, the superposition of all producing uniform illumination. The greater the distance between the two slits in front of the lens, the finer must be the front slit, in order to have the fringes appear. It is thus clear that we have a means of measuring the angular diameter subtended by a distant source of light. This method was proposed by Fizeau in 1868 for measuring the diameters of the fixed stars. In 1874 Stéphan made the attempt but was unable to obtain a measurable quantity. Michelson in 1890 succeeded, however, in measuring the diameters of the four moons of Jupiter, by placing two slits, the distance between which could be varied, in front of the object glass of the Lick telescope. The method can also be used for determining the distance between the components of a double star.

Each star produces its own fringe system, the two being dis-

placed with respect to one another through the angle ϵ, which is the angle subtended by the stars. This angle is, of course, independent of the distance between the slits. If now the slits are close together, the fringes are broad, and the angular displacement ϵ of the two systems does not alter their appearance. As we separate the slits, however, the fringes become finer and presently disappear, owing to the fact that the maxima of one system fall upon the minima of the other. If the distance between the slits, when this happens, is s, we have ϵ given by the equation

$$\epsilon = \frac{\lambda/2}{s}.$$

Interference of Waves of Different Lengths. Light-Beats. — When two tuning-forks of slightly different pitch are sounded simultaneously we hear a fluttering sound, the intensity rising and falling. The interference in this case is not continuous in time at a given point in space. If we draw two wave-trains of slightly different wave-length we shall see that they are "in step" and "out-of-step" at periodic intervals. Where they are in step we shall have maximum amplitude, where they are out-of-step we shall have minimum or zero amplitude. As the double wave-trains sweep by a given point it will be in alternation the seat of large and small disturbances.

If we seek for the optical analogy it is easy to see that two sources of monochromatic light, of slightly different period, should give us a moving system of interference fringes, any given point in space being alternately the seat of maximum and minimum illumination. The frequency of the beats being equal to the difference between the two interfering trains, the wave-length of the beat is $\frac{\lambda'\lambda}{\lambda'-\lambda}$.

Light-beats have never been obtained by uniting two streams of light from sources of different color. Righi has, however, performed an experiment in which the frequency of vibration of one of the two streams of light which form a system of fringes can be increased any desired number of times per second by passing the light through a revolving Nicol prism. This experiment will be described in detail in the Chapter on Elliptical Polarization.

It is also worthy of remark that the moving fringes observed in a Michelson interferometer, as the back mirror is advanced, can be regarded as a manifestation of light-beats, the wave-lengths of one of the interfering trains being lessened by reflection from the moving mirror by Doppler's principle. This manner of regarding the phenomenon appears to be due to Mr. A. B. Porter. The analogy has perhaps occurred to others, but my attention was first directed to it by Mr. Porter's note in *Science*, 1905.

The moving system of interference fringes which constitute beats can be most beautifully shown by means of capillary waves on a mercury surface, the disturbances being originated by two tuning-forks of slightly different pitch.

Two forks of the same pitch are thrown slightly out of tune by

frstening small lumps of soft wax to the prongs of one of them. A light bit of wire is fastened to a prong of each fork, and after setting the forks in vibration, the tips of the wires are dipped into the surface of clean mercury. The hyperboloid fringes will be seen to be in motion, sweeping around in a most beautiful manner. Between the wire points they will be observed to travel from one vibrating point towards the other. If the wax lumps are removed, the fringes immediately become motionless. The phenomenon can be projected on a screen to advantage, by reflecting the light down upon the mercury surface, and thence to the screen through a projecting lens, by means of a pair of mirrors or large reflecting prisms.

Achromatic Interference Fringes. — As we have seen, the fringes obtained with Lloyd's mirror and a source illuminated with white light soon blend into a uniformly illuminated field, owing to the fact that the distance between the maxima and minima varies with the color. If by some artifice we can make the widths of the fringes the same, the system will become achromatic, and we can count a large number of fringes even with white light. This can be accomplished by using, as our source, a short spectrum with its blue end towards the reflecting plate. The blue sources will thus be closer together than the red, and if the adjustments are right the blue sources will give fringes of the same width as the red, which are farther apart.

The condition is best realized by employing a diffraction grating and a lens to form the spectrum. A vertical slit is illuminated with sun or arc light, and a glass grating with two to three thousand lines to the inch, combined with a lens of four or five inches focus, is so arranged as to form a series of diffraction spectra on a card mounted in the focal plane of the lens. The card should be perforated with a small hole through which the light of one of the first order spectra is allowed to pass. The Lloyd plate is placed in such a position as to furnish a reflected image of the spectrum, the blue end of which is turned towards the original spectrum, as shown in Fig. 116. The reflecting surface should be pointed exactly at the central image formed by the grating and lens, if perfect achromatization is desired. The fringes are viewed as before with an eye-piece, a little adjusting of the plate being all that is necessary to completely fill the field of view with fine black and white lines.

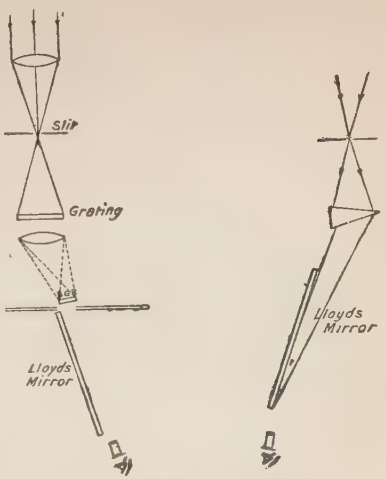


Fig. 116.

If the spectrum is formed by a prism of about 20°, which can be made of plate glass, and filled with water, less perfect achromatization is obtained; still a large number of fringes can be seen.

The spectrum in this case can be virtual, *i.e.* no lens need be used, the prism being mounted between the slit and the mirror, as shown in the second diagram (Fig. 116). If the prism is placed between the mirror and the eye-piece no achromatization results, for in this case the two spectra are not opposed.

Introduction of Thin Transparent Plate. — As we have just seen, a plate of thickness ϵ and refractive index μ shifts the central fringe for monochromatic light through a distance

$$x=(\mu-1)\epsilon\frac{a}{s}.$$

This distance will be different for the different colored systems, since μ varies with the wave-length, and there will therefore be no point at which the waves of all lengths will arrive in the same phase, or, in other words, there will be no strictly achromatic fringe.

There will, however, be a system of colored fringes with a central band which appears nearly achromatic, the determining condition of which is not that the path difference be equal to zero, but that the change in phase with change of λ be a minimum.

Draw a system of fringes with different colored pieces of chalk, adopting the method shown in Fig. 115. Make the distance between the red fringes slightly greater than that between the yellow, the yellow greater than the green, and so on. Suppose now that the introduction of the plate shifts the whole system to the right. Owing to the dispersion of the plate the blue fringes will be shifted through a slightly greater distance than the green, and the green a trifle more than the yellow. It is clear that this difference may close up the rainbow colored band to the right of the central fringe, into an approximately achromatic band, and expand what was originally the white central band into a rainbow band. If we do not take this into account, we shall make an error of one fringe width in measuring the shift of what *appears* to be the central band. If our plate had a higher dispersion with the same average refractive index, the second or third rainbow band might be achromatized and appear to be the centre of the system.

Let the distance of the approximately achromatic fringe from the original centre of the system be x.

The geometrical path difference at this point, disregarding the plate, is $\frac{xs}{a}$, using the same notation as before. The actual optical difference of path is $\frac{xs}{a}-(\mu-1)\epsilon$, since the shift is towards the side on which the plate is introduced, and the original short path is lengthened by the introduction of the plate. Now $(\mu-1)\epsilon$ is a function of λ, and we will write it $f(\lambda)$. The difference of phase at the point in question for any value of λ will be, writing

$$D=(\mu-1)\epsilon=f(\lambda) \text{ and } D'=\frac{xs}{a},$$

$$\Delta=2\pi\frac{D'-f(\lambda)}{\lambda}.$$

The required condition of minimum phase variation with λ is given by differentiating this expression with respect to λ and equating to zero; performing the operation we get, putting $f'(\lambda)=\frac{df(\lambda)}{d\lambda}$

$$\frac{-2\pi f'(\lambda)\lambda-2\pi D'-2\pi f(\lambda)}{\lambda^2}=0, \text{ or } D'=f(\lambda)-\lambda f'(\lambda).$$

The central fringe corresponding to wave-length λ is shifted by the plate to the position of the nth fringe given by

$$n=\frac{(\mu-1)\epsilon}{\lambda}=\frac{f(\lambda)}{\lambda}.$$

By our original supposition regarding the position of the shifted achromatic (approximately) fringe it occupies the position of a fringe of order n' given by

$$n'=\frac{xs}{a\lambda}=\frac{D'}{\lambda}=n-f'(\lambda),$$

and is therefore shifted relatively to the central fringe for monochromatic light of wave-length λ by a number of fringes given by $n'-n=-f'(\lambda)$.

The variation of μ with λ is well expressed in the present case by the equation

$$\mu=A+\frac{B}{\lambda^2},$$

which we shall discuss more in detail when we come to the subject of dispersion.

$$f(\lambda)=(\mu-1)t,$$

$$\frac{df(\lambda)}{d\lambda}=f'(\lambda)=-\frac{2Bt}{\lambda^3},$$

$$n'-n=\frac{2Bt}{\lambda^3},$$

which shows us that the shift of the approximately achromatic fringe obtained with white light, with respect to the central fringe obtained with monochromatic light of wave-length λ, varies inversely as the cube of the wave-length, and directly as the thickness of the plate. We shall have occasion to make use of this formula when we come to the subject of the interferometer.

A remarkable instance of the shifting of the region of fringe visibility far out into the system was observed by the author [1] in studying the dispersion of sodium vapor with the interferometer. The path difference under which it is possible to obtain interference fringes with the D_3 light of a helium tube can be nearly trebled by the

[1] R. W. Wood, "Archromatization of approximately monochromatic interference fringes by a highly dispersive medium," *Phil. Mag.*, September, 1904.

introduction of a small amount of sodium vapor into the path of one of the interfering beams. This development of fringes far out in the system by the dispersive action of the vapor is accompanied by their complete disappearance at the centre of the system, where the difference of path is zero.

The introduction of a medium into the path of one of the interfering beams causes a shift of the fringe system as a whole, and if the medium is dispersing, the shifts will be different for the different colors. The red, green, and blue fringes, which are out-of-step at a given point, may thus be brought into coincidence by the inequality of their respective displacements. In this case, however, since the systems are shifted as a whole, the fringes will be thrown out-of-step at the centre of the system, consequently we have obtained an increased visibility far out in the system at the expense of visibility at the centre. Now the helium light is very near the D lines of sodium, and sodium vapor in this region of the spectrum has a dispersive power so great that a prism of it giving the same deviation as a 60° glass prism (if it could be formed) would separate two lines only $\frac{1}{20}$ as far apart as the D lines, by an amount as great as the distance between the red and the greenish-blue of the spectrum yielded by the glass prism. This enormous dispersive power may well be expected to modify profoundly the appearance of the fringe system produced even with light as monochromatic as that of the D_3 line. That a change is produced depends on the fact that no light is absolutely monochromatic, the finest spectrum lines having an appreciable width. We can thus consider the D_3 light as an extremely short spectrum, and apply the same reasoning as in the case of a thin transparent plate introduced into the path of one of the interfering streams of white light.

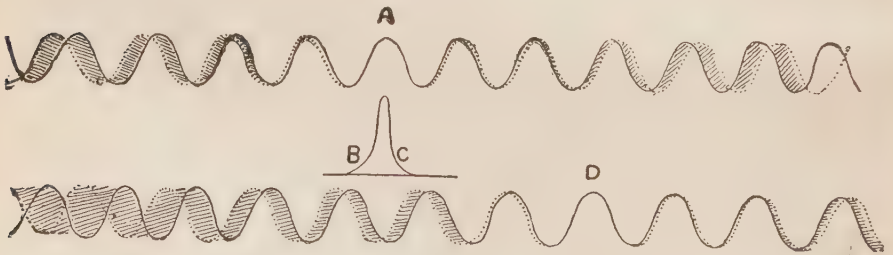


Fig. 117.

The treatment will be better understood after a study of the interferometer and the resolution of spectral lines, but it is given here, on account of the identity of the phenomenon with the displacement of the white centre. The helium fringes under ordinary circumstances disappear when the path difference is between 1.5 and 2 cms., there being no recurrence of visibility by further increment of path difference as in the case of sodium light. We must therefore regard the helium (D_3) line as a single line of finite breadth or a close group of lines. In Fig. 117 let BC represent the intensity curve of the helium light, C being the edge of shorter wave-length. Im-

mediately above we have a schematic representation of the fringe system, with its centre at A. Light from the side B of the D_3 line will produce the fringes indicated by the dotted line, which are farther apart than the fringes formed by the light of shorter wave-length coming from the side C of the line. There will, in addition, be an infinite number of other systems formed by light of wave-lengths intermediate between B and C which are indicated by light shading.

Now suppose sodium vapor to be introduced into one path of the instrument, and the whole system shifted slightly to the left in consequence. Owing to the enormous dispersive power of the vapor, the dotted system (longer λ's) will be shifted more than the other, since the D_3 line lies on the blue side of the sodium absorption-band, and the change in the velocity of the light is greatest for the longest waves, namely, those on the B side of the line. The result of this dispersive action is that the fringes are brought into step at a point D, to the right of the centre, thrown out-of-step at the centre and still more out-of-step to the left of the centre.

The achromatizing action of the sodium vapor is most beautifully shown if we illuminate the interferometer with white light.

Under ordinary conditions only two or three black and white fringes are seen, bordered on each side by perhaps a dozen rainbow colored bands, which fade rapidly into a uniform illumination. If sodium vapor is formed in one of the interferometer paths, the colored fringes rapidly achromatize, and increase in number, breaking up, however, into groups as shown in Fig. 118. As the density of the vapor increases the number of groups increases, each group, however, containing fewer fringes. The position of the centre of the grouped system drifts in the same direction as the point of maximum visibility in the previous experiments.

The explanation of the altered appearance of the fringes in this case is not as simple as in those previously considered. We are dealing with two wide ranges of wave-lengths on opposite sides of the absorption-band. The fringe shifts of the two spectral regions will be in opposite directions, while the drifts of the points of maximum visibility will be in the same direction. Each set will be more or less perfectly achromatized, and in the region in which they overlap we have a periodic visibility, owing to the difference in the widths of the fringes of the two systems.

Fig. 118.

The following treatment, which is rigorous, has been given by Lord Rayleigh.

"The remarkable shift of the bands of helium light when a layer of sodium vapor is interposed in the path of one of the interfering pencils, is of the same nature as the displacement of the white centre found by Airy and Stokes to follow the insertion of a thin plate of

glass. If D denote the thickness of the plate and μ its refractive index, $(\mu-1)D$ is the retardation due to the insertion of the plate, and if R be the relative retardation due to other causes, the whole relative retardation is

$$R+(\mu-1)D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which R and D are supposed to be independent of the wave-length λ, while μ does depend upon it. The order of the band (n) is given by

$$n=\frac{R+(\mu-1)D}{\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the achromatic band in the case of white light, or for the place of greatest distinctness when the bands are formed with light approximately homogeneous, n must be stationary as λ varies, *i.e.*

$$\frac{dn}{d\lambda}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For a small range of wave-length we may write

$$\lambda=\lambda_0+\delta\lambda,$$

so that
$$n=\frac{R+\left(\mu_0+\frac{d\mu}{d\lambda_0}\delta\lambda-1\right)D}{\lambda_0+\delta\lambda}$$

$$=\frac{R+(\mu_0-1)D}{\lambda_0}+\frac{\delta\lambda}{\lambda_0}\left(D\frac{d\mu}{d\lambda_0}-\frac{R+(\mu_0-1)D}{\lambda_0}\right). \quad . \quad . \quad (4)$$

The achromatic band occurs, not when the whole relative retardation (1) vanishes, but when

$$R+(\mu_0-1)D=D\lambda_0\frac{d\mu}{d\lambda_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If D be great enough, there is no limit to the shift that may be caused by the introduction of the dispersive plate.

"As Schuster has especially emphasized, the question here is really one of the *group-velocity*. Approximately homogeneous light consists of a train of waves in which the amplitude and wave-length slowly vary. A *local* peculiarity of amplitude or wave-length travels in a dispersive medium with the *group* and not with the *wave*-velocity; and the relative retardation with which we are concerned is the relative retardation of the groups. From this point of view it is obvious that what is to be made to vanish is not (1), in which μ is the ratio of wave-velocities V_0/V, but that derived from it by replacing μ by U_0/U, or by V_0/U, where U is the group-velocity in the dispersive medium. In vacuum the distinction between U_0 and V_0 disappears, but in the dispersive medium

$$U=\frac{d(kV)}{dk}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)^1$$

[1] *Theory of Sound*, § 191, 1877.

k being the reciprocal of the wave-length in the *medium*. If we denote as usual the wave-length *in vacuo* by λ,

$$k = \frac{2\pi\mu}{\lambda} = \frac{2\pi V_0}{\lambda V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Accordingly

$$\frac{V_0}{U} = \frac{V_0 dk}{d(kV)} = \frac{d(\mu/\lambda)}{d(1/\lambda)} = \mu - \lambda\frac{d\mu}{d\lambda}. \quad . \quad . \quad . \quad . \quad (8)$$

Substituting this for μ in (1), we see that the position of the most distinct band is given by

$$R + \left(\mu - 1 - \lambda\frac{d\mu}{d\lambda}\right)D = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in agreement with (5)."

Distribution of Phase over Small Area illuminated by Source of Finite Size. — In Young's celebrated experiment, where interference was observed between the diffracted rays coming through two pin-holes illuminated by sunlight coming from another small hole, it is clear that the phase of the vibration must be the same at the two pin-holes. If the source of light were infinitely small, the phase relation between the vibrations passing through the two holes would be permanent, even if the holes were widely separated; but if the source has finite size, as is always the case, this will not be true. The reason for this is clear. The vibration at each hole is the resultant of the disturbances coming from the various points of the source, and this will vary with the position of the holes. We will now derive an expression for the maximum distance allowable between the holes, in terms of the size of the source and its distance, or what amounts to the same thing, the area over which we can regard the phase as constant.

Suppose we have a luminous sphere (Fig. 119), the sun for example, with its centre at O. The vibration at P, a point on a sphere concentric with this, is the resultant of all the separate disturbances coming from the source. If we take another point infinitely near P, it is obvious that we shall have the same resultant, while at a point P' the resultant is clearly different owing to the changes in the lengths of the paths over which the component vibrations travel. What we require is the maximum distance between P and P' consistent with uniform phase.

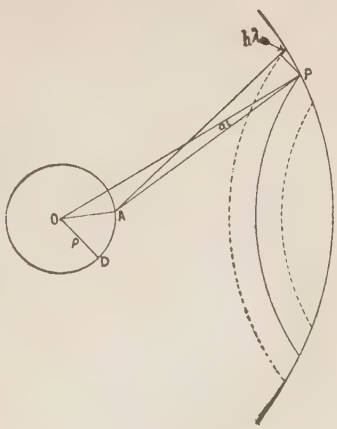


Fig. 119.

The vibrations from a point A on the luminous sphere will reach in the same time a circle described on the outer sphere, passing through P, the plane of which is perpendicular to OA, since all points on it are equidistant from A.

We now construct two other circles on the outer sphere, one just within, the other without the first circle, such that their distances from A differ by only a small fraction of a wave-length from the distance of the first circle. This will give us a narrow circular zone, over which the phase, due to vibrations coming from A, is constant. The width of this zone will be represented by

$$x = \frac{h\lambda}{\sin \alpha},$$

in which α is the angle subtended from P by AO, and x is the half width of the zone; h is a small fraction which should not be over $\frac{1}{4}$.

P, however, receives vibrations from all other points on the luminous hemisphere, and for each one of these we can construct a zone passing through P in exactly the same manner. There will be a small area around P common to all the zones, over which the resultant phase due to all the vibrations will be the same. The largest value which $\sin \alpha$ can have is attained when the luminous point considered is at D, when $\sin \alpha = \frac{\rho}{R}$, in which ρ is the radius of the source and R the radius of the large sphere. Inserting this value in the above equation gives us the semi-diameter of the small area which we are after. The diameter of the area is given by

$$2x = \frac{R\lambda}{2\rho} \text{ if we take } h = \tfrac{1}{4}.$$

Since $\frac{2\rho}{R}$ represents the apparent diameter of the luminous source when viewed from P, we can say in general that the phase can be considered constant over a circular area not greater in diameter than the wave-length of light divided by the apparent diameter of the source. In the case of sunlight $\frac{\rho}{R} = \tan 16' = .005$, and $\lambda = .0005$ mm.

$$2x = \frac{.0005}{.01} = .05 \text{ mm.}$$

In ordinary sunlight, therefore, the phase is the same over an area measuring only .05 mm. in diameter, or in a square millimetre there are 400 different states of vibration. We can easily apply our formula to Young's experiment. Suppose we form an image of the sun with a lens of 5 mm. focus. Its diameter will be .05 mm. and from a distance of 1 metre $\frac{\rho}{R}$ will be .00005.

The diameter of our circle of similar phase will be 5 mms., *i.e.* the two pin-holes should not be over 5 mms. apart if we are to regard them as similar sources, which is the condition which we must fulfil if we wish to obtain interference fringes.

Interference Fringes when the Sources are in Line. — If we consider the hyperboloid fringes formed in space when waves radiate

in all directions from two similar sources, to be cut by a plane perpendicular to the line joining the sources, the maxima and minima form circles which have a common centre on the prolongation of the line joining the sources.

Fringes of this description were obtained by Meslin (*Compt. Rendus*, 1893) by an ingenious modification of the arrangement of

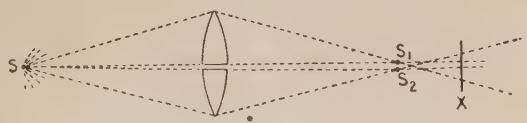


FIG. 120.

the Billet split lens. This instrument consists of a double convex lens, cut in halves, the two portions being slightly separated, so as to form two images s_1, s_2 of the source, the fringes being observed in the plane X (Fig. 120). By displacing one of the halves as shown in Fig. 121 the sources are brought into the line of sight. The

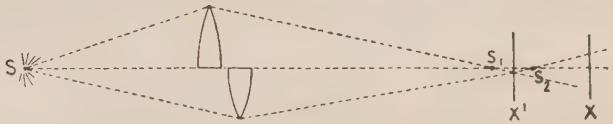


FIG. 121.

fringes will be circles in this case, but they will not be found as before in the plane X, since the beams from the two sources do not overlap in this region, but in the plane X' *between* the sources, where overlapping occurs. This is in reality interference between waves radiating from a source, with waves converging to a similar source.

There are other methods by which two similar sources in line can be obtained.

It would seem at first sight as if the conditions could be easily fulfilled by putting a very minute source of monochromatic light in front of and close to a silvered reflecting surface. A minute electric flame, obtained by passing the discharge of a small induction coil between two metal points charged with sodium, and mounted very close together, would apparently fulfil the conditions. If the experiment is tried, no fringes are seen, however. This is possibly due to the impossibility of getting the source small enough, but more probably a rather peculiar factor comes into play. The light which forms the reflected image leaves the flame in a direction opposite to the stream with which it is to interfere. It is doubtful if we can consider the sources similar in this case. If we were dealing with a single sodium molecule the case would be different, but we must remember that the sodium flame has the power of absorbing precisely the radiations which it emits, consequently we should have more light from the back of the flame in the case of the reflected image, while the direct beam would consist chiefly of light from the front of the flame. Interference between these two portions is

obviously impossible. There are other objections to the arrangement, as a little consideration will show.

If, however, we reflect a point source of light at two parallel surfaces we have beams capable of interfering and producing circular fringes. The reflecting surfaces must be equidistant, *i.e.* parallel, and the incidence normal. The axis of the circular fringe system will then coincide with the ray normal to the surfaces, and an eye-piece cannot be brought into such a position as to show the circular maxima and minima.

If the reflecting surfaces are half-silvered, that is coated with a film of silver of such thickness as to reflect and transmit equal quantities of light, the circular fringes can be seen when an eye-piece is held behind the plates. The distance between the source and its virtual image formed by a double reflection between the plates is twice the distance between the reflecting surfaces. This device is essentially the form of interferometer designed by Fabry and Perot, and will be discussed more in detail presently.

Interference Fringes along Caustics. — In the Chapter on Reflection we have seen that the caustic surfaces are traced by the cusped wave-fronts. Just within the caustic we therefore have two wave-fronts, which travel obliquely with respect to each other, and which came originally from the same source. Let c, c' be the caustic traced by the cusped waves, the crests of which are represented by solid lines, the troughs by dotted lines (Fig. 122). Along the line A we have crests meeting troughs, and a consequent destructive interference, while along the line B we have similar phases, and maximum illumination. It is clear from the diagram that the interference is between a portion of the wave which has passed through a focus with one which is converging to a focus, the condition being similar in some respects to experiments with the Billet split lens, described in the previous section. These fringes are easily seen by concentrating sunlight upon a pin-hole and reflecting the light from an oblique concave mirror. The fringes will be found in the region between the primary and secondary focal lines, *i.e.* along the caustic.

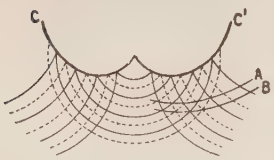


Fig. 122.

Interference of Polarized Light. — The study of the interference of polarized light was taken up by Fresnel and Arago in 1816. Young's explanation of the colors of thin plates of doubly refracting substances in polarized light was not wholly satisfactory to Fresnel. The non-appearance of color in the absence of the polarizer and analyzer had not been accounted for, Young's explanation being simply that the color was produced by interference between the ordinary and extraordinary rays emerging from the thin plate.

Fresnel made as a preliminary experiment the following. Employing a thin crystal of Iceland spar in the same manner as a bi-prism, he looked for interference fringes in the overlapping portion of the two bundles of rays into which the doubly refracting crystal divided the incident light. He had, what amounted to two similar sources of

light, radiating beams polarized at right angles to each other. No fringes were observed. To compensate for the path difference resulting from the difference of retardation between the two rays, Fresnel placed a glass plate of calculated thickness in the path of the least retarded beam. This seemed necessary to Fresnel, because at the time he was unaware of the fact that interference was possible under conditions involving considerable difference of path. The introduction of the retarding plate gave rise however to diffraction fringes, which made it difficult to draw conclusions.

Fresnel accordingly modified the experiment, reflecting the light after its passage through the crystal, from a glass plate of such thickness that the path difference between rays reflected from the front and back surface was the same as the path difference between the ordinary and extraordinary rays emerging from the crystal plate. This arrangement might be expected to show interference between the ordinary ray reflected from the front surface of the glass plate and the extraordinary ray reflected from the back surface, but there was no trace of a fringe system. A still better arrangement was then tried by Fresnel, consisting of two crystals of equal thickness with their principal sections at right angles to one another. The ordinary ray from the first crystal is refracted as an extraordinary ray in the second, and *vice versa*, the result being two beams polarized at right angles to each other with no path difference between them: in this case also no fringes were observed. These experiments established the fact that the two beams of light polarized at right angles to each other, into which a doubly refracting crystal divides ordinary light, are incapable of interfering. Arago then devised an experiment in which the two polarized rays were obtained independently of double refraction. Two parallel slits in close proximity were illuminated as in Young's experiment, and behind each was placed a pile of mica plates at the polarizing angle. By rotating either of the piles the transmitted polarized ray could be set at any angle. It was found that when the planes of polarization were parallel, fringes were produced, but when the planes were at right angles the illumination was uniform.[1]

Fresnel then devised a very beautiful modification of the experiment. A selenite plate was placed in front of the two slits, and a set of fringes produced similar to those produced by the slits alone. Because of the selenite plate we must regard each slit as sending two beams polarized at right angles to each other. The two ordinary beams being polarized parallel to each other interfere and form a fringe system, and the two extraordinary rays, being also polarized parallel, give rise to a second system superposed on the first. If now two beams polarized at right angles were capable of interfering, we should have a set of fringes due to the interference of an ordinary beam from one slit, with an extraordinary beam from the other slit, and since the retardation of the two in the selenite plate is different, there would be a considerable path difference, and the fringe system would be displaced with reference to the first. As a matter of fact two systems would be found, one to the left, the other

[1] Two plates of tourmaline furnish an easier means of repeating this experiment.

to the right of the original system. These extra fringe systems were not found, however.

Fresnel then cut the selenite plate in two between the slits, and turned one half through a right angle. By this device the ordinary ray from one slit was brought into the same plane of polarization as the extraordinary ray from the other: the two sets interfered and produced two systems of fringes displaced to the right and left of the original system, owing to the retardation of the ordinary ray on the extraordinary. These experiments establish the first of the Fresnel-Arago laws that "Two rays polarized parallel will interfere, while two rays polarized at a right angle will not."

Fresnel and Arago then showed by a modification of the last experiment that two rays polarized at right angles, obtained from ordinary light, can be made parallel without thereby acquiring the property of interfering. The two piles of mica plates were placed behind the slits in such positions that the rays were polarized at a right angle, and a doubly refracting crystal was mounted behind them, with its principal section at an angle of 45 degrees to the planes of polarization. This crystal resolved each of the two plane polarized beams into two rectangularly polarized components of equal intensity. At first sight it might appear as if we had here exactly the same conditions that we had in the last experiment; that is, each slit furnishes two equally intense rectangularly polarized beams which would interfere in pairs and produce fringes. Fresnel found, however, that no fringes were formed. Let us see wherein the difference lies.

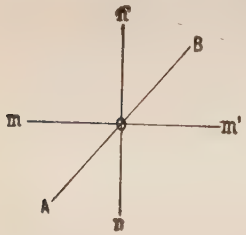


FIG. 123.

We will begin by assuming ordinary light to consist of plane polarized light, the plane of polarization changing constantly with great rapidity. Suppose at a given moment the plane of polarization of the light incident on the slits to be represented by the line AB (Fig. 123): furthermore, let mm' and nn' be the planes of polarization of the beams transmitted by the mica plates. The vibration OB is resolved into the vibrations on' and om'; one pile of plates transmits the former and the other pile the latter component. We thus have beams from the slits polarized at a right angle to each other. We will now bring them into the same plane by means of the doubly refracting crystal, which we will suppose to be tourmaline, since this crystal has the property of absorbing one of the rectangularly polarized components into which it divides a ray. Suppose the crystal so placed that the plane of the transmitted vibration is parallel to the plane of the vibration of the incident light (which we consider plane polarized) for an infinitesimal of time. Call this plane AB as before (Fig. 124). on' will be resolved into components, one of which, od, is transmitted, and om' is also resolved into two components, one of which, oe, is transmitted. These two beams are of equal intensity and polarized in the same plane, and will accordingly produce a set of fringes. And now comes the important point.

The plane of polarization of the incident light is changing with

inconceivable rapidity all the time. In the next infinitesimal of time suppose the plane turned through a right angle to the position $A'B'$. The piles of mica plates resolve this into on' and om at right angles to each other. The tourmaline plate only transmits the components which are parallel to AB, namely, od and od', which do not coincide, but have a phase difference of 180. The two equally intense beams give rise to a set of fringes as before, but a phase difference of 180 exists between them, and the maxima fall in the places occupied by the minima of the previous case. The non-interference of two beams polarized at a right angle, obtained from ordinary light and brought to the same plane of polarization, is then only apparent. What we really have are maxima and minima, which change place as rapidly as the plane of polarization of ordinary light changes, and the result is uniform illumination. Were it possible to take an absolutely instantaneous photograph of the illuminated field we should probably find the fringes.

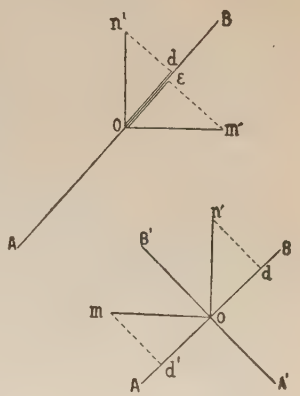


Fig. 124.

If two beams of light polarized at right angles were derived originally from a polarized beam, they will interfere when brought into the same plane of polarization.

This is simply the permanent condition of what in the previous case existed only for an infinitesimal of time, and was experimentally investigated by Fresnel, who found that the position of the maxima in the fringe system depended on whether the plane, into which the rays polarized at right angles were brought, was parallel or at right angles to the original plane, a phase difference of 180 being introduced in the latter case, as we have seen.

The Fresnel-Arago Laws. — These results can be summed up into what are known as the Fresnel-Arago laws.

1. Two rays polarized in the same plane interfere in the same manner as ordinary light.

2. Two rays polarized at right angles do not interfere.

3. Two rays polarized at right angles (obtained from ordinary light), and brought into the same plane of polarization, do not interfere in the ordinary sense.

4. Two rays polarized at right angles (obtained from plane polarized light) interfere when brought into the same plane of polarization.

5. In the latter case, under certain conditions, half a wave-length, corresponding to the phase difference of 180, must be added to the path difference.

The Colors of Thin Plates. — The iridescent colors which are displayed by thin films of transparent substances were first investi-

gated by Boyle. In 1665 Hooke devised the method of producing the colors by means of an air film between two lenses of large radius of curvature. He found that the colors were distributed in concentric rings, showing that they depended on the thickness of the film, and that equal thickness gave always the same color. Hooke explained the production of color as follows. A portion of the light is reflected from the upper surface, and a portion penetrates the film and is reflected from the lower surface. This portion has suffered two refractions and a reflection, and is weakened in consequence. This weaker impulse will reach the eye a moment later than the one coming from the upper surface, and Hooke supposed the sensation of yellow to be due to a weak impulse following a stronger one. If the thickness of the film increases, the weaker impulse will lag behind the stronger until it will unite with the next following stronger, finally lagging behind this sufficiently to once more produce the sensation of yellow. Thus he explained the recurrence of the color with increasing thickness. Hooke's notion was that sensation of color depended on successive impacts on the retina of strong and weak impulses. If the stronger preceded the weaker one color was produced. If *vice versa*, then another color resulted. He was right in explaining the color as produced by the union of the light streams reflected from the two surfaces, and being ignorant of the nature of white light and of wave-length as we speak of it, gave what seemed the simplest and most probable explanation of the regular sequence of the colors.

The subject was more carefully investigated by Newton, who made careful measurements of the colored rings (since known by his name) produced by the air film between a lens and a plate of glass.

It remained for Young, however, to give the true explanation that the rings were due to the interference between the wave-trains reflected from the upper and lower surface of the film.

If we place the curved surface of a plano-convex lens of very long focus (the longer the better, say 2 or 3 metres) on a flat plate of glass, and view the reflected image of a sodium flame, we shall see the point of contact surrounded by dark and bright circles of light, produced by interference of the streams of light reflected from the two surfaces. This gives us a very simple means of obtaining interference under a variable path difference. The diameters of the circles vary according to the same law as that which obtains in the case of the zone-plate, the scale however varies with the wave-length, red light giving us larger rings than yellow or green.

If the source emits two wave-lengths, the bright rings will therefore be in-step at some points, and out-of-step at others. In the case of sodium light we have two wave-lengths the difference between which is $\frac{1}{1000}$ of their actual values. The fringes are therefore exactly out-of-step, or in " Dissonance " at the 500th ring from the centre. At the 1000th ring they are again in-step or in " Consonance." When in dissonance they are quite invisible. The alternate appearance and disappearance of the sodium flame rings cannot be well seen with the lens and plate unless we use a strong magnifying glass to view the fringes.

Their distinctness is much greater if we get rid of the light reflected from the upper surface of the lens, and the lower surface of the plate. This can be done by putting a horizontal slit .5 cm. in width in front of the soda flame. The images reflected from the

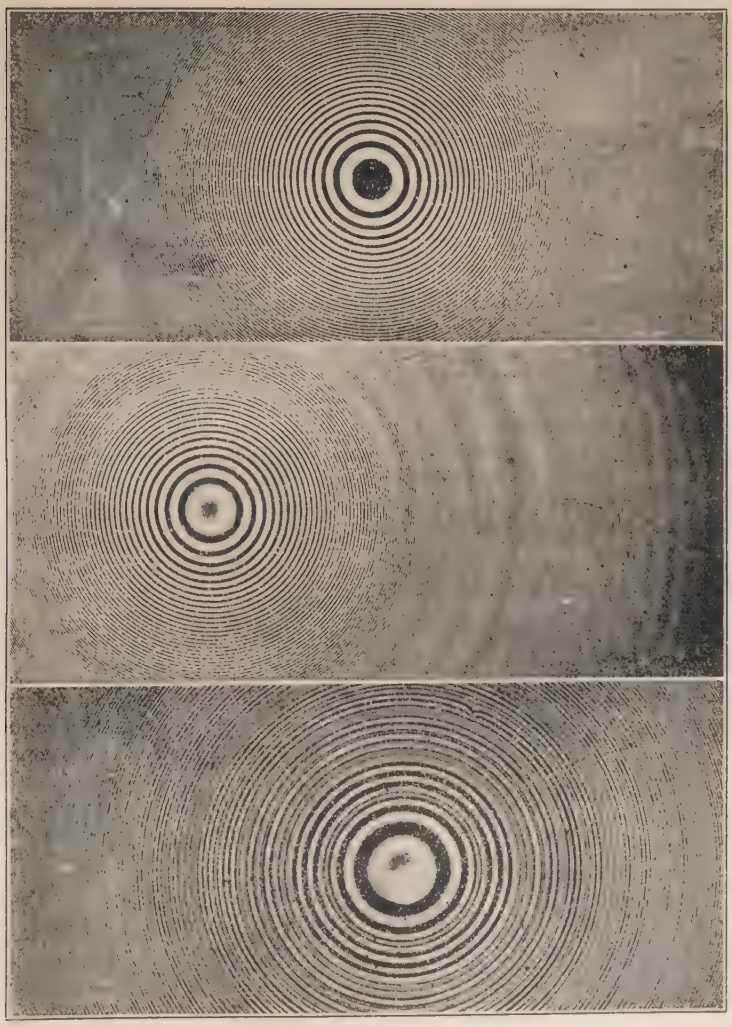


Fig. 125.

different surfaces will appear separated, the rings being seen only in one of them. An easier way of showing Dissonance is to use a lithium and thallium flame. This gives red and green rings where dissonance occurs, and yellow (subjective) at the points where the red and green are superposed. If white light is employed we get colored rings.

The best arrangement of all, however, is to blacken the back surface of the lens with a mixture of glycerine and lampblack, which can be easily washed off, and lay upon the curved surface a thin prism of small angle (say 8 or 10 degrees), fastening it in position with a few drops of melted wax at the corners.

It can now be mounted in a vertical position, and the light from a mercury arc thrown upon it with a condensing lens placed in such a position that the reflected light is focussed upon a photographic objective placed in such a position that it gives an enlargement of several diameters. The rings can be observed with an eye-piece or photographed. A tank filled with a solution of bichromate of potash, used as a ray filter, gives us only the green and the two yellow lines; we can see the dissonance of the green and yellow fringes, and far out in the system the dissonance of the two yellow lines. A photograph of these fringes made by the author is reproduced in Fig. 125. In the upper figure we have the fringes formed by the green mercury light, just below those obtained with the green and the two yellow lines. In this case the green and double yellow lines get out-of-step at intervals. Far out in the system the two yellow lines get out-of-step, and their dissonance with the green line vanishes. Still farther out dissonance again occurs. In the lower figure we have the rings formed by the two yellow, the green and a number of blue and violet lines, producing a remarkable complex of color when seen with the eye-piece.

Newton's Rings. — The thickness of a film of air at any point between a spherical and a plane surface in contact is easily expressed in terms of the distance of the point from the point of tangency, and the radius of curvature of the surface.

This gave Newton the means of accurately determining the color produced by an air film of any thickness. A lens, the radius and curvature of which is known, is placed on a piece of plate glass and viewed by reflected light. Circular colored rings are seen surrounding the point of contact, the colors being most brilliant where the air film is very thin. We wish to determine its thickness ϵ, for example, where the first yellow ring appears. Let the radius of curvature of the lens be R, and the radius of the yellow ring r (Fig. 126). We have

$$r^2 = R^2 - (R-\epsilon)^2 = 2R\epsilon - \epsilon^2 \text{ or } \epsilon = \frac{r^2}{2R},$$

since ϵ^2 is small in comparison to $2R\epsilon$, an expression which shows us that the thickness of the air film is proportional to the square of the radius of the ring. Newton found that with monochromatic light he got alternately bright and dark rings, and that the rings when produced by red light were larger than when produced by blue. With white light, then, we have an infinite number of ring systems superposed, and to the blending of these systems is due the complicated succession of color observed by Newton.

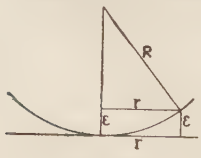


FIG. 126.

It may be remarked here that in viewing the interference phe-

nomena produced by thin plates, the eye must be focussed upon the film.

Let AB be a ray incident nearly normally on the upper surface of the air film (Fig. 127). A portion is refracted to C, where it undergoes a second reflection, emerging from the upper surface after a second refraction at D. It then pursues a direction identical with the first reflected portion of some other ray $A'D$, very close to AB. (If the incidence is absolutely normal, it will coincide, of course, with the first reflected portion of AB.) Suppose the path BCD to be one-half wave-length. Then the waves which have twice traversed the film will be half a wave-length behind those which are reflected from the upper surface, and if the amplitudes are the same there will be destructive interference, and no light will be reflected by the film. In other words, every ray which would ordinarily be reflected from the upper surface will be destroyed by one coming from the under surface. If the film were infinitely thin the path BCD would be zero, and we should expect the waves to agree in phase and reënforce one another; but, as a matter of fact, we find that when the film is very thin exactly the reverse is true, *no light is reflected:* and when the path BCD, which is practically twice the film's thickness, is exactly one-half wave-length, we have the two streams reënforcing each other instead of destroying each other. The explanation of this is that the two reflections take place under different conditions. At the upper surface the reflection is from a dense medium to a rare; at the lower surface, from a rare to a dense. The waves reflected at the rarer medium are reflected without change of phase, those reflected at the denser medium suffer a phase change of 180°. This is, of course, equivalent to a path difference of half a wave-length.

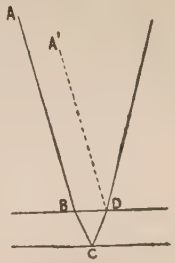


FIG. 127.

When the thickness of the plate is small in comparison with the wave-length, the waves reflected at the lower surface destroy those reflected from the upper surface, by virtue of this sudden change of phase, and no light is reflected.

This explanation was given by Young, who devised a very beautiful experiment in support of it. By using a lens of crown and a plate of flint glass with a film of cassia oil between them, he secured a system in which reflection from the upper and lower surface of the film took place under the same condition, the oil having a refractive index intermediate between that of the crown and flint glass. The ring system formed under these conditions had a white centre, exactly in accordance with his theory.

Under normal incidence we have then the following equations for the thickness e of the film:

$$\left.\begin{aligned} e &= (2n-1)\frac{\lambda}{4} \text{ for a maximum} \\ e &= 2(n-1)\frac{\lambda}{4} \text{ for a minimum} \end{aligned}\right\} \text{ by reflection,}$$

where the reflection takes place under opposite conditions, and

$$\left.\begin{aligned} e &= 2n\frac{\lambda}{4} \text{ for a maximum} \\ e &= (2n+1)\frac{\lambda}{4} \text{ for a minimum} \end{aligned}\right\} \text{ by reflection,}$$

where the conditions are the same as in the case of the cassia oil experiment. In these equations λ is the wave-length in the film. If the film has a refractive index μ, and λ is the wave-length of light used, we must write $\frac{\lambda}{\mu}$ for λ in our equations.

To determine the successive thicknesses of the film which will reflect light of wave-length λ, we give to n values 1, 2, 3, 4, etc., and find for e values corresponding to $\frac{\lambda}{4}$, $3\frac{\lambda}{4}$, $5\frac{\lambda}{4}$, etc. That is, thicknesses corresponding to odd values of the quarter wave-length give maxima, and thicknesses corresponding to $0, 2\frac{\lambda}{4}$, $4\frac{\lambda}{4}$, etc., give minima.

With films of such thickness that no light is reflected, the energy is not lost, but is transmitted; therefore such films have an increased transmitting power for monochromatic light, none being lost by reflection from the first surface.

The above equations show us that as we increase the thickness of the film it alternately reflects and refuses to reflect; therefore in the case of a film enclosed between a lens and a plate, the thickness of which increases as we go out from the point of contact, the locus of points in the film which reflect and which refuse to reflect, are concentric circles.

Newton's Rings in White Light. — The formula for the thickness of the plate required to produce a maximum or minimum, shows us that if we decrease λ we must decrease the thickness of the film; that to produce the first red maximum the thickness of the film must be $\frac{1}{4}$ of the wave-length of the red wave, while to produce the first blue ring, it must be only $\frac{1}{4}$ of the blue wave. The first blue maximum, therefore, lies nearer the centre than the red, and the maxima for the other colors occupy intermediate positions. An inspection of the formula for the maxima $e = (2n-1)\frac{\lambda}{4}$ shows us that with a given thickness, large in comparison to the wave-length, the formula will hold for a number of different values of λ, taking different values as we change λ. Thus a given thickness may fulfil the conditions of the equation for maxima for a large number of different colors. The analogy between this case and that of the fringes produced by the Fresnel mirrors with large path-difference is obvious. Suppose the thickness of the film to be .01 m., we then have $.01 = n\frac{\lambda}{4}$ or $.04 = n\lambda$ for a maximum when the value of n is any odd number. If we give λ its value for red, .0007, we find n to be 57, or we have $n = 57$ for red, corresponding to the 28th maximum.

With violet light of wave-length .0003 we find $n=133$, corresponding to the 66th maximum. Between these two values we shall have 66.28 other maxima for intermediate wave-lengths. Consequently a film measuring .01 mm. in thickness will reflect 38 different parts of the spectrum and refuse to reflect 38 intermediate parts, or if we examine the light reflected from the film with the spectroscope we shall find the spectrum crossed by 38 dark bands.

Refractive Index and Dispersion of a Thin Plate. — It is plain that we have here a means of determining the thickness of a thin film. By examining the light reflected from it with a spectroscope and counting the number of dark bands between any two known points (Fraunhofer lines) in the spectrum, we can, by making the substitutions in the formula, calculate the thickness. In the formula which we have given we have supposed the incidence normal, and λ to be the value of the wave-length in the material of the film. If we are dealing with films of glass we must, of course, reduce the wave-length values to their values in glass.

The complete formula for determining the thickness of a plate of any substance with light reflected at any incidence is

$$e=\frac{n\lambda_1\lambda_2}{2\mu\cos r(\lambda_1-\lambda_2)},$$

in which n is the number of dark bands between wave-lengths λ_1, λ_2, μ the refractive index of the film, and r the angle of refraction.

If we know the thickness we can determine μ at different points of the spectrum, or measure the dispersion.

Influence of Multiple Reflections. — The theory of thin plates as it came from the hands of Young had an imperfection. The portions of the light reflected from the two surfaces are not equal, since the light which suffers reflection at the second surface has already been weakened by reflection at the first. The two portions should therefore never wholly destroy each other as they do when we employ monochromatic light. Poisson showed that we must take into account the multiple reflections which occur within the film. If the retardation of the ray $A'B'C'B$ on the ray AB is $\delta=2e\cos r$, the retardation of consecutive rays incident at B'' B''', etc., are 2δ, 3δ, etc. (Fig. 128). We thus know the phases of the components as they arrive at B, and to calculate their joint effects we must know their amplitudes. A certain percentage of the incident light will be reflected at the glass-air surface, and a certain percentage at the air-glass surface, but we have no right to assume that the fractional part reflected is the same in each case. The following method of ascertaining the relations between the amplitudes of the reflected rays was used by Stokes.

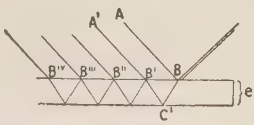


Fig. 128.

Let the amplitude of the incident ray be a, then the amplitude of the reflected ray will be ab, in which b is a fraction, and the amplitude of the refracted ray will be ac, c being in general a fraction

larger than b. By the principle of reversibility, if we send these two rays back along their paths, they should give rise to the original ray, reversed in direction of course, with the original amplitude a. If we reverse BC, however, it will give rise to two rays, one along BA of amplitude ab^2, and one along BE of amplitude abc. In reversing DB we cannot obtain the amplitudes of the reflected and refracted components by multiplying its amplitude by b and c respectively, since the reflection takes place under different conditions. We will therefore designate the amplitudes of the reversed components of BD by acf (along BA) and ace (along BE). If the sum of these components is to represent a ray along BA of amplitude a, and we are to have no ray BE, as must be the case if the reversed rays give rise to the original ray only, the following relations between b, c, e, and f must hold:

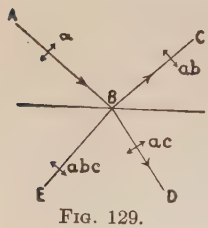


FIG. 129.

$$acf+ab^2=a \text{ and } ace+abc=0.$$

These equations give us

$$cf=1-b^2=1-e \text{ and } b=-e.$$

The latter equation shows us that the amplitude of the ray arising from reflection in passing from the upper to the lower medium is equal to the amplitude of a ray of equal intensity which has suffered reflection in passing from the lower to the upper medium. The fact that the sign of b is opposite to the sign of e indicates moreover that there is a relative phase change of half a period between the ray reflected under opposite conditions. This explains the absence of a ray along BE when we reverse the reflected and refracted components, the components along BE having a phase difference of 180 and destroying one another.

The perfect blackness of the interference fringes when monochromatic light is used follows at once from the above equations. The amplitude of the stream reflected from the first surface is ab. The transmitted amplitude is ac, of which abc is reflected from the lower surface, and $abcf$ emerges into the upper medium. The amplitudes emerging into the upper medium owing to the multiple reflections form a series $abcf + ab^3cf + ab^5cf + \cdots$. Complete interference will occur if the sum is equal to ab. This is seen to be the case, for

$$abcf(1+b^2+b^4, \text{ etc.})=abcf=\frac{1-b^{2n}}{1-b^2}=ab\frac{cf}{1-b^2}=ab,$$

since $cf=1-b^2$, as we have seen above.

Curves of Equal Thickness and Equal Inclination. Haidinger's Fringes. — In the case of fringes formed by the reflection of light from a thin film of variable thickness, the thickness of the film along any fringe is a constant. These fringes are therefore called "curves of equal thickness."

The fringes are located at the film, and the eye should be focussed on the film to see them distinctly.

Another class of fringes was first described by Haidinger, in 1849, and subsequently studied by Michelson in 1882 and by Lummer in 1884. They are produced by interference of light reflected from the surface of a thick plane-parallel plate, when waves of different inclinations fall upon it. To see them as circles the eye must be normal to the plate, and the light reflected down upon it by a transparent plate-glass mirror mounted at an angle of 45° between the eye and the plate. They are located at infinity, *i.e.* formed by the interference of parallel rays, and are invisible unless the eye is focussed for infinity.

Testing Glass Plates for Flatness and Plane-Parallelism. — If we have an optical glass flat, we can test the flatness of any other plate by laying it upon the flat and observing the reflection of a sodium flame from the surfaces. Flatness is indicated by perfectly straight fringes, parallel and equidistant. If we have no flat plate we can use a method due to Lord Rayleigh. The glass plate is mounted in a small dish, which is then filled with clean water until the fluid covers the plate to the depth of about a millimetre. There should be a margin of surface of 2 or 3 cms. between the edges of the plate and the sides of the dish, to avoid capillary troubles. The whole is mounted upon a stand provided with levelling screws, in a locality free from all tremors. A glass plate should be placed over the dish to prevent air currents from disturbing the surface. The fringes formed by the interference between rays reflected from the glass-water surface and the water-air surface are observed. The two surfaces should first be made as nearly parallel as possible, by observing the reflection of a very small and bright point of light, and bringing the two images together by means of the levelling screws. A small mirror should be mounted over the surface of the water which reflects sodium light down upon the surface. The surface should be observed by means of a magnifying glass. At first the fringes will usually be found very close together, but they may be made as broad as desired by levelling. If the water surface is of a certain thickness, they may not appear owing to the fringes formed by D_1 and D_2 being in dissonance. It is safer to use the light of the mercury arc filtered through a green glass, for the green line is very homogeneous. If the fringes are curved the plate is not flat. We must now determine the nature of the surface, that is whether convex or concave, spherical, cylindrical, or saddle-shaped.

If we move our eye so as to increase the angle of incidence, the retardation will decrease, since $\cos i$ decreases in the formula for a

$$2\mu e \cos i = n\lambda$$

fringe of order n which remains constant for a given fringe. The thickness e must therefore increase, and a given fringe move towards a region of greater thickness.

If the surface is convex the fringes will move out towards the

edge as the eye is moved down, if concave they will move away from the edge.

It is however so much easier to work with a glass flat, that the water method should only be attempted to standardize the test plate. Suppose we have a small square plate which we know to be flat.

If we place a larger piece of thin plate glass upon it, and press down upon the four corners with the fingers, the under surface will become concave, and we see Newton's rings, which will close in towards the centre as the eye is moved down.

If the surface is cylindrical, and we can bring the plate into this condition by pressing down upon two opposite edges, the fringes will be straight, but not equidistant, lying closer together along the two edges than at the surface.

A saddle-shaped surface gives fringes shaped like hyperboloids. The thickness along a given fringe is a constant, and the fringes can therefore be regarded in the same way as the contour lines on a map.

It is instructive to try the experiment of improving or figuring a poor surface by means of dilute hydrofluoric acid applied with a soft brush over the regions from which material must be removed. The surface should be tested frequently by laying it upon the test plate. A subsequent polishing with rouge upon a surface of pitch will remove the slight irregularities introduced by the etching process.

To test for plane-parallelism we observe the Haidinger fringes obtained with thick plates when the eye is focussed for infinity.

Mount the plate in a horizontal position and reflect the green light from the mercury arc down upon it from above by means of a piece of plate glass at an angle of 45°. The plate must first be made perpendicular to the line of vision (from above) by observing the reflection of the pupil of the eye in it. The fringes appear as concentric circles when the eye is focussed for infinity. The direction of the incident light and the line of vision should coincide as nearly as possible. Fix the attention upon the central ring, and move the plate slowly in its own plane. If the rings expand, new ones opening out from the center, we are moving towards a region of greater thickness, for the incidence angle i is greater for each successive ring and the formula for a ring of order n is

$$n\lambda = 2\,\mu e \cos i.$$

As i increases $\cos i$ decreases, therefore e must increase, that is the ring moves out. To determine the error, count the rings which develop in passing from edge to edge of the plate. If k represents this number, then $\frac{k}{2}$ is the number of wave-lengths by which the plate departs from parallelism, and $\frac{k}{2\mu}\lambda$ is the actual difference in thickness in millimetres. ($\lambda = .000546$ for the green mercury light.)

Colors of Iridescent Crystals and Opals. — Some very remarkable phenomena connected with the colors of thin films are frequently

exhibited in crystals of chlorate of potash. The cause of these colors was investigated by Stokes, and found to be due to the existence of planes within the crystal at which a periodic "twinning" had occurred. The colors are extremely brilliant and pure, much more so than any exhibited by soap films. An interesting paper by Lord Rayleigh will be found in the *Phil. Mag.*, xxvi., pp. 256–265, 1888. One of the most remarkable facts connected with the phenomenon is that the spectrum of the reflected light is frequently found to consist almost entirely of a comparatively narrow band. The same phenomenon is also exhibited by the fiery opal. One in possession of the author at a certain angle of incidence reflects yellow light which, when examined in the spectroscope, is found to consist of a narrow band not much wider than the distance between the yellow mercury lines. In the case of a *single* thin film, of such thickness that but a single region of the spectrum is reflected, this region is always of considerable breadth. To account for the reflection of light of such a high degree of purity Lord Rayleigh assumes that the reflection takes place at a number of thin laminae sensibly equidistant, the distance between any two being of the order of magnitude of the light-wave. Quoting from his paper, "In order to explain the vigor and purity of the color reflected in certain crystals it is necessary to suppose that there are a considerable number of thin surfaces disposed at approximate equal intervals. At each angle of incidence there would be a particular wave-length for which the phases of the several reflections are in agreement. The selection of light of a particular wave-length would thus take place upon the same principle as in diffraction spectra, and might reach a high degree of perfection." Lord Rayleigh describes an interesting acoustical analogue, the sound of a bird call, giving a pure tone of high pitch, being most copiously reflected from a number of flat equidistant screens made of thin muslin, stretched upon brass rings at a certain distance apart (*Nature*, xl., p. 227, 1889).

This remarkable limitation of the reflected light to a narrow region of the spectrum will be better understood after a study of the diffraction grating. It will be interesting to compare this action of multiple films with the action of the Fabry and Perot Interferometer, which will be described in a subsequent chapter. The colored crystals of chlorate of potash are easily prepared by making a hot saturated solution of the potash, and allowing it to cool slowly in a large flat-bottomed tray. On breaking up the crystalline mass, and shaking the tray gently in front of a window, numerous highly colored flakes will appear, which can be lifted out by means of a small bent spatula, made of thin sheet copper or brass. They should be dried on filter-paper, and mounted in balsam, preferably between two glass prisms of about ten degrees angle. The thin flakes have the remarkable property of reflecting practically all of the incident light of a certain color or colors, while freely transmitting the remainder of the spectrum. The reflected colors are of great spectral purity, the width of the band being sometimes not greater than the distance between the yellow mercury lines. Seen

by transmitted light the flakes often appear delicately tinted, and the spectrum of the transmitted light will be found to be crossed by one or more narrow and intensely black bands.

Stokes observed that many crystals reflected two or even three narrow bands in the visible spectrum, and inferred that, in these cases, there must be several sets of multiple twin planes. It seems curious that he was led to adopt this hypothesis, as the more obvious explanation would be to refer the colors to different orders.

A thin film, or a number of parallel thin films of equal thickness D, reflects at normal incidence light of wave-lengths given by the equation $2D = 0$, $2D = \lambda$, $2D = 3\lambda$, $2D = 4\lambda$, etc., if we consider no phase-change to occur. If $2D = .0006$ mm. the film will reflect red light, and ultra-violet light of wave-lengths .00030 and .00020. If, however, $2D$ is equal to some wave-length in the infra-red region, say at .0012 mm. (1.2μ), it is clear that we shall have more than one reflected color in the visible region, the maxima occurring at wave-lengths .0006, .0004, .0003, .00024, etc. Such a film should appear purple, since it reflects red and violet light. If the first-order color is at $2\,\mu$ the higher orders will fall at $1\,\mu$, $.66\,\mu$, $.50\,\mu$, $.40\,\mu$, etc., and the films will reflect narrow regions in the red, green, and violet.

We can moreover, by measuring the wave-lengths of the bands in the visible and ultra-violet spectrum, calculate the position of the first-order band in the infra-red. An examination of the crystals by transmitted light, derived from a cadmium spark, with a small quartz spectrograph, showed that they were perfectly transparent down to the very end of the spectrum. The following investigations were made by the author.

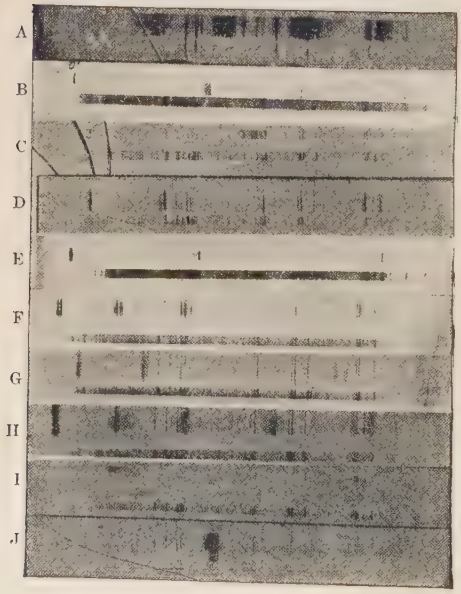


Fig. 130.

A number of the films were mounted in glycerine between two quartz prisms, by which device all light except that reflected from the equidistant laminae is eliminated. The colors appear much more saturated when the films are thus mounted, for the light reflected from their upper and lower surfaces is not destroyed by interference, and dilutes the selectively reflected colors. This circumstance may be due to the fact that the distances of the surfaces from the neighbouring twin planes is not equal to the distances between the parallel strata, or possibly to the large amplitudes of the disturbances reflected from the outer surfaces.

The films, after having been prepared in the manner described above, were mounted in front of the slit of a small quartz spectrograph, and oriented so as to throw the light from a cadmium spark into the instrument.

A series of photographs obtained in this way is reproduced in Fig. 130. In some cases a number of bands appear in the ultra-violet, and in other cases only one or two. It is obvious that the thicker the reflecting strata, the larger will be the number of regions selectively reflected in a given spectral range. The approximate values of the lengths of the reflected waves are given in the following table:

	1st Order Color	Higher Order Colors					
A	1.6 μ	.80	.54	.40	.30	.26	.23
B	.617	.31					
C	.53	.28					
D	1 μ	.53	.28				
E	.64	.325	.22				
F	1.26 μ	.633	.435	.32	.23		
G	1.13	.565	.39	.235			
H	1.23	.614	.42	.326	.266	.226	
I	.44	.23					
J	.60	.30					

In all except *A* and *J* the cadmium-spark spectrum is recorded for comparison.

It will be noticed that certain films (spectra *F* and *G* for example) reflect double bands, which are often as close together as the yellow mercury lines. One film which I examined showed a double line in the red the components of which were just barely resolvable. Its appearance reminded one of the sodium lines when on the point of fusing together owing to the density of the vapor. There seems to be no way of explaining cases of this sort except by assuming a double set of twin planes. In the case above referred to there may have been two thin crystalline plates stuck together. The specimen unfortunately was destroyed so that further examination was impossible.

A search for the predicted bands in the infra-red region was undertaken at my request by Mr. A. H. Pfund. The light from a Nernst lamp was reflected from one of the purple films, into the slit of a reflecting spectrometer furnished with a rock-salt prism and radiometer. This particular film showed a sharp band in the red at wave-length 60, another in the violet at wave-length 40, and others in the ultra-violet. The first-order color should be found at about 1.2 μ. The radiometer gave a deflection of several centimetres, when illuminated with the red band, immediately dropping back to zero as the prism was so turned as to bring the infra-red region upon the slit. At about 1 μ a large deflection was obtained, which at 1.2 μ was "off the scale." The energy curve was then plotted from a large number of readings, the slit having been narrowed in the

meantime. This curve is reproduced in Fig. 131. The width and asymmetry of the infra-red, or first-order band, are due to the fact that the energy curve in this region of the spectrum is very steep and asymmetrical.

A crystal reflecting red, green, and violet bands at wave-lengths 66, 50, and 40, should show two maxima in the infra-red, one at 2 μ, the other at 1 μ.

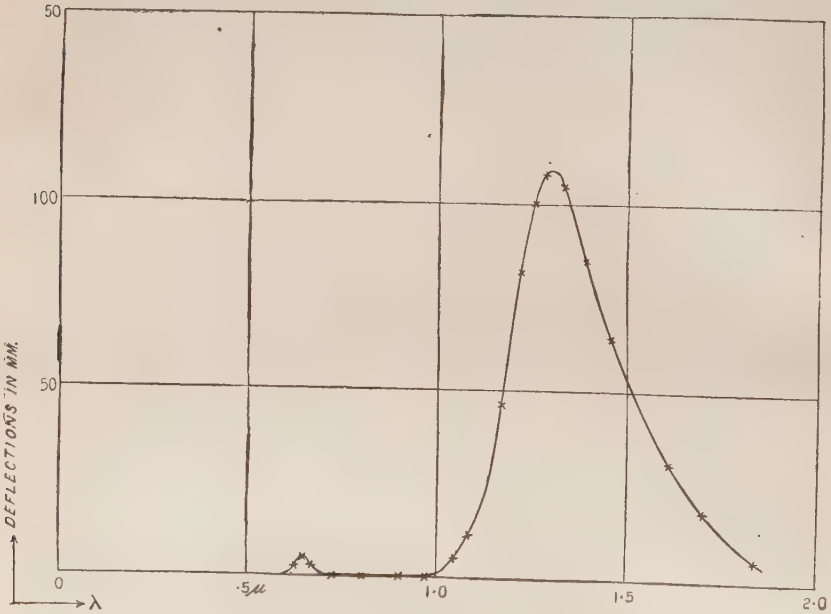


Fig. 131.

It seems probable that large-sized flakes of chlorate of potash may at times prove useful in research work when it is desired to isolate a single spectral line, or cut out a narrow region of a continuous spectrum. The advantage of the flake over the spectroscope would lie in the large amount of light available, which is often of great advantage. The power of the crystals to reflect a comparatively narrow band in the infra-red might sometimes be made use of as well. In making crystallizations on a small scale, I have seldom obtained satisfactory flakes much over .7 cm.^{2}. Larger ones could perhaps be made by working on a large scale.

One flake, measuring about 6 mms. on a side, exhibited total reflection at normal incidence of a region of the spectrum only 10 or 12 Ångström units in width, that is, only double the distance between the *D* lines. The spectrum of the transmitted light exhibited a very black band at the same point, and of the same width. This band was photographed, after having been brought into the vicinity of the *D* lines by suitable inclination of the plate, and the *D* lines themselves impressed on the plate by holding a sodium flame in front of the slit for a few seconds. The photograph is reproduced in

Fig. 132, spectrum "*c*." The position of the *D* lines I have marked on the spectrum immediately above this one. (The marks refer to the lower spectrum only.)

By increasing the angle of incidence, the band can be made to move down the spectrum, widening as it moves. When in the green it appears as in spectrum "*b*" and is accompanied by fainter lateral minima. The narrowness of the reflected region has been shown to be due to multiple twin planes, sensibly equidistant. From the width of the reflected region we can form an estimate of the number of laminae present in the crystal plate. Each lamina gives us by reflection a virtual image of the source, these images being in line, one behind the other, at normal incidence. The action is not unlike that of a diffraction grating when the diffracted ray is at grazing emergence. If we are dealing with a first-order spectrum, 1000 lines are necessary to resolve the *D* lines.

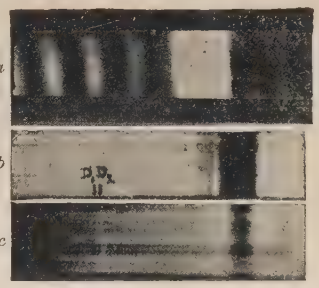


Fig. 132.

If now we compare the width of the band in the photograph with the distance between the *D* lines, it is clear that the crystal plate is very nearly able to separate or resolve the *D* lines. In other words, if we incline the plate a little less, causing the band to move up the spectrum, it will reflect D_2 before it reflects D_1. It was, however, not quite able to do this, but would easily separate lines of twice the separation of the *D*'s. From this we may infer that the number of twin planes in the crystal is somewhere between 500 and 1000, say roughly 700. If we are dealing with a first-order spectrum the path difference between rays reflected from two adjacent twin planes must be equal to the wave-length of the light in the crystal. Assuming no phase-change this makes the thickness of each lamina about .0002 mms., and multiplying this by 700 gives us .14 mm. as the thickness of the crystal plate, which was not very far from the truth.

It is interesting to compare this crystal with the best Lippmann photographs, which show color resulting from the same type of interference.

Mr. H. E. Ives has made the best Lippmann photographs with monochromatic light that I know of. The spectrum of the band reflected from his best plate is shown Fig. 132, "*a*," on the same scale of wave-lengths, *i.e.* taken with the same spectroscope. It has a width about three times as great as the band reflected from the crystal. If I remember rightly, Mr. Ives sectioned a similar film and counted about 250 silver laminae, built up by the stationary waves. This is in good agreement with our estimate of the number in the chlorate crystal.

The intensity of the light reflected from the thin planes will be found to vary in a remarkable manner as the crystal is turned about. The following laws were discovered by Stokes;

(1) If one of the crystalline plates be turned round in its own

plane, without alteration of the angle of incidence, the peculiar reflection vanishes twice in a revolution, viz. when the plane of incidence coincides with the plane of symmetry of the crystal.

(2) As the angle of incidence is increased the reflected light becomes brighter and rises in refrangibility.

(3) The colors are not due to absorption, the transmitted light being strictly complementary to the reflected.

(4) The colored light is not polarized. It is produced indifferently whether the incident light be common light or polarized in any plane, and is seen whether the reflected light be viewed directly or through a Nicol's prism turned in any way.

(5) The spectrum of the reflected light is frequently found to consist almost entirely of a comparatively narrow band. When the angle of incidence is increased, the band moves in the direction of increasing refrangibility, and at the same time increases rapidly in width. In many cases the reflection appears to be almost total.

Effect of a Prism upon Newton's Rings. — It was observed by Newton that, when the colored ring system formed by a lens and a flat plate was viewed through a prism, the number of rings visible was greatly increased on one side of the system, the increase being about twelvefold. If the fringes were equidistant, as is the case with the Fresnel mirrors or the bi-prism, a prismatic shift would bring the fringes of different colors into step at a point far out in the system, but would throw them out-of-step at the centre, so that their appearance would not be much altered. The rings, however, become narrower as we advance out into the system, and if we simplify the problem by supposing that we have only red and blue fringes which are shifted through different distances by the prism, it is not difficult to see how the achromatization results, for the blue arcs, from a portion near the centre, can be made to fit approximately over the red arcs in a more remote region. Now the blue rings are shifted more than the red, consequently the achromatization will occur on the side of centre towards which the shifts have taken place. A full treatment of the subject will be found in Lord Rayleigh's paper on achromatic interference bands (*Phil. Mag.*, 1889). Fringes can often be found by this means on thin glass bulbs, easily made by blowing out a glass tube; sodium light will give fringes without the prism, but nothing can be seen with white light owing to the thickness of the glass.

Achromatization of the Fringes formed by a Thin Reflecting Lamina. — An arrangement was devised by Talbot which yielded achromatic fringes of equal widths. The achromatization which we have just considered depends upon the different widths in the different parts of the system, and cannot be applied to the equidistant fringes obtained with a wedge-shaped film.

To obtain achromatization in this case it is necessary to arrange matters so that the scale of the system is the same for the different colors. Now the scale depends on the angle of the wedge (which is obviously fixed) and the angle of refraction. Under ordinary circumstances the angle of refraction is very nearly the same for the different colors, but if we employ an air film between glass plates,

with the light incident in such a direction that the angle of refraction is nearly 90°, owing to the powerful dispersion the angle will vary with λ, and since the angle is greater for the blue than for the red, the blue fringes may be formed on the same scale as the red, and nearly perfect achromatization result.

To obtain a sufficiently large angle of refraction it is necessary to employ a prism of the form shown in Fig. 133. A right-angle prism answers nearly as well. It should be placed with its hypotenuse on a silvered glass plate or a plate of polished speculum metal. A horizontal slit illuminated with white light is placed in such a position that the light is incident on the second surface at nearly the critical angle. This position can be found by lifting one end of the prism a little and watching the slit image, varying the height of the slit until a number of images appear side by side. On lowering the prism these images will run together. They are virtual images formed by multiple reflections between the prism face and the metal plate. This condition has been overlooked by the majority of previous writers, so far as I have been able to find, and it modifies the appearance of the fringes, as we shall see presently. The colored fringes can now be found with an eye-piece, and by tilting up the edge of the prism which is nearest to the slit it is usually possible to achromatize them at once, 50 or 60 black and white bands appearing in the field. When the prism rests on the plate, the fringes are broad and highly colored. An eye accustomed to the appearance of fringe systems near the central fringe will recognize that there is something peculiar in this case. The appearance is due to the fact that we have a number of virtual sources in line one behind the other. If the slit is illuminated with sodium light, the fringes will present a remarkable appearance. The maxima will appear bright and narrow, with broad minima between them. On *one* side of each maximum a number of fainter maxima will be seen, which gives a corrugated or shaded appearance to the system. A photograph of the system can be obtained by laying an orthochromatic plate on the face of the prism, and exposing it for a couple of hours, taking care to shield it from all light except that which comes through the slit. A picture obtained in this way is reproduced in Fig. 134.

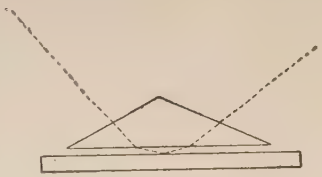


Fig. 133.

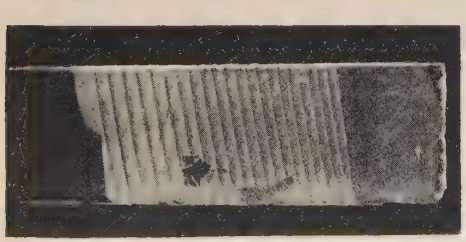


Fig. 134.

The influence of the multiple reflection is to increase the steepness of the intensity curve for the maxima. A similar effect occurs in the case of the Fabry and Perot interferometer. The explanation of

the phenomenon will be postponed until we come to the study of diffraction, and discuss the fringes produced by more than two similar sources.

On the Polarized Fringes produced by the Interference of Two Streams of Light polarized at Right Angles.[1] — In the case of ordinary thin-film interferences the planes of vibration of the disturbances reflected from the two surfaces of the film are parallel.

It is possible, however, to prepare a film which shall fulfil the requirement that the vibrations reflected from its upper surface make any desired angle with those coming from the lower surface. The path-difference between the two streams will vary with the thickness of the film; and if the amplitudes be equal we shall have the vibrations compounding into circular, elliptic, or plane ones, according to their phase-difference.

A thin glass or gelatine film, backed by a metallic reflecting surface, is all that is necessary. The incident sodium light should be polarized at an angle of 45° with the plane of incidence by passage through a Nicol, and the reflected light examined with an analyzing Nicol. The fringes obtained in this way present a most curious appearance, reminding one forcibly of a spectrum line with a fainter component seen in the Fabry and Perot interferometer. Their general appearance is shown in Fig. 134 *a*, which represents the fringes obtained by flowing a plate of speculum metal with a rather dilute solution of gelatine, and allowing it to dry in a slightly inclined position.

The easiest way to get them, however, is to blow out the end of a rather large glass tube into a large thin balloon of tissue glass, picking out a portion, by the light of a sodium flame, which shows fairly straight interference fringes one or two millimetres apart. A small piece of the thin glass is laid, with its slightly convex side down, upon a clean mercury surface, and sodium light, polarized in azimuth 45°, reflected from the surface at an angle of about 60°. On viewing the reflected light through a Nicol, the curious double fringes can be easily found by slowly turning the Nicol. The light will be found to be plane-polarized along the lines 1, 3, 5, 7, though in general the planes of polarization along one set of lines is inclined to the plane of polarization along the alternate lines, as is indicated below the figure, the arrows representing the di-

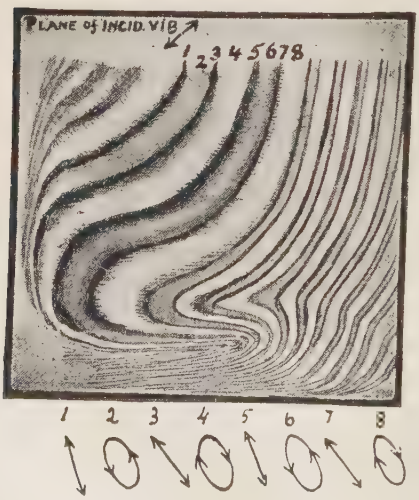


FIG. 134 *a*.

[1] Wood, *Phil. Mag.*, April, 1904.

rection of the vibration (electric vector). Between the lines of plane-polarized light, which appear as dark fringes when the Nicol is so oriented as to extinguish the light, we have either elliptically or circularly-polarized light, as can at once be shown by the introduction of a quarterwave plate, which enables us to extinguish the light along the lines 2, 4, 6, and 8, by suitable adjustment of the mica plate and analyzer. The direction of revolution of the vibration along lines 2 and 6 is opposite to that along lines 4 and 8.

To account for these fringes we must investigate the planes of polarization of the rays reflected from the two surfaces, and then compound them with various phase-differences.

The incident light vibrates in a plane indicated by the arrow at the top of Fig. 134 *a*. The light reflected from the glass surface is of course plane-polarized, vibrating parallel to the surface when the angle of incidence is equal to the polarizing angle. For larger angles of incidence, the plane of the reflected vibration makes an angle with the surface, depending on the magnitude of the reflected component which lies in the plane of incidence. To determine the nature of the vibration coming from the glass-metal surface, it is necessary to get rid of the light reflected from the upper surface of the glass. This can be done by laying a small piece of rather thick plate glass on a plate of speculum metal with a film of benzole between, and allowing a narrow beam of light to fall upon the surface of the glass. The images reflected from the two surfaces appear separated, and can be independently examined with a Nicol. The benzole film practically brings the metal surface into optical contact with the glass. The reflected beams will be found to be plane-polarized, the vibrations being in the directions shown in Fig. 134 *a*, for an angle of incidence near the polarizing angle. As the angle of incidence increases, the planes of the two vibrations both turn towards the vertical, and finally make an angle of 90° with each other, *i.e.* 60° and 30° with the reflecting surface.

If we compound the two components shown in Fig. 135*a* with varying phase-differences, we can account easily for the polarized fringes. In Fig. 135*b* let *BC* be the vibration from the glass and *BA* that from the metal. When the path is zero or a whole number of waves, we have the plane-polarized resultant *BE*. If the path-difference is an odd number of half-waves, we have the plane-polarized resultant *BF*. These two states occur along the lines 1, 5 and 3, 7 respectively. The angles which the planes of vibration make with the reflecting surface should be respectively greater and less than the angle made by the component coming from the metal, as was found to be the case. As we increase the angle of incidence

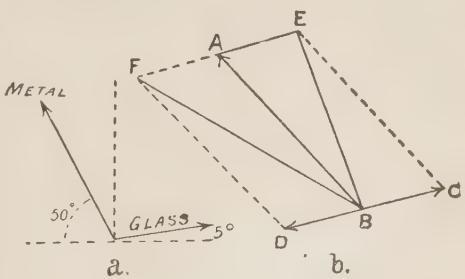


FIG. 135.

the component BC (from glass) increases in magnitude and turns up towards the vertical, the inclinations of the planes of polarization of the two sets of fringes becoming greater.

Between the lines along which the light reflected from the film is plane-polarized, we have lines of elliptically (or in some cases circularly) polarized light. The directions of revolution were determined with the quarter-wave plate, and are as shown in Fig. 134 *a*. Geometrical computations of the elliptic vibrations, resulting from two components such as are shown in Fig. 135 *b*, agreed perfectly with the experimentally determined orbits, both in respect to the directions of the major axes and the directions of rotation. The method used was similar to the one given in Müller-Pouillet's text-book (vol. ii., 1, page 1135), the only modification necessary being the rotation of one of the sets of parallel lines, which represent displacements, through a certain angle, since, in the case with which we are dealing, the two components are not at right angles. By sufficiently increasing the incidence angle, the components AB and BC can, however, be brought to very nearly a right angle. At the same time the intensity of the component from the glass surface has increased to such a degree as to be about equal to the one from the metal, and we have circular polarization along certain lines.

Preparation of Films for the Exhibition of Newton's Colors. — In the case of a thin transparent lamina, such as a soap-film, the amplitudes of the disturbances reflected from the two surfaces are equal, and consequently completely destroy each other when the phase-difference is 180°. Inasmuch, however, as only a small percentage of light is reflected from each surface, the colors, though saturated, are not as intense as is desirable. If a plate of mica is pressed against a pool of molten selenium on a glass plate, and the whole allowed to cool under pressure, on stripping off the mica, films of mica of variable thickness will be left upon the surface of the selenium, which show Newton's colors of great beauty, arranged in mosaics. The patches of equal thickness being sharply bounded by straight lines, present an appearance similar to that of selenite films under the polariscope. The selenium has a much higher refractive index than the mica, consequently the reflection at each surface is the reflection of rays incident from a rare to a denser medium, and the difference of phase is given by the difference of path alone; *i.e.* we do not have the loss of half a wave-length due to reflection under opposite conditions, as would be the case if the mica films were in air.

If the mica is cemented to the plate with sealing wax or any of the common resinous cements, very little trace of the colors is to be seen, owing to the fact that the refractive indices of the two media being so nearly the same, practically no energy is reflected from the boundary. The use of selenium can be avoided by very lightly silvering the surface of the mica, which may then be cemented to the glass with any good laboratory cement, the metallic layer taking the place of the medium of high refractive index. This latter method is the best for the preparation of large mosaics suitable for lantern-projection. In the patches which show no color by reflected light, the interfer-

ence may be detected with a small spectroscope, the spectrum appearing crossed by black bands, corresponding in position to the wave-lengths absent in the reflected light. Still more brilliant films can be prepared by first thickly silvering the mica, cementing the silvered side to the glass plate, and then stripping the mica off. The colors are scarcely visible, owing to the disproportionality between the amplitudes of the two interfering streams of light, but appear as soon as the upper surface of the mica is half-silvered, which can be done by immersing the plate in a silvering solution until the colors reach their maximum brilliancy. To obtain films which show the maximum brilliancy, it is clear that the amplitude of the stream reflected from the first surface must have the largest possible value consistent with the fulfilling of the condition that it be completely destroyed by the disturbance coming from the second surface when the phase-difference is 180°.

In the case of transparent films, the wave-lengths absent in the reflected light appear in excess in the transmitted light, there being no destruction of energy. If the second surface is a perfect reflector, the energy thrown down upon it by interference at the first surface will all be returned. If we consider the upper metallic surface as non-absorbing, and work out the case by the method of multiple reflections, we find that no color will be produced, light of all wave-lengths being reflected with equal facility. The fact that brilliant colors appear, means that light is absorbed at one or both of the silvered surfaces, since this is the only way in which the energy of the absent wave-lengths can disappear. If we assume a certain percentage reflected and a certain percentage absorbed by the thin silver film, we find that the waves for which the phase-difference is 180° are compelled to make more transits through the film than those for which the phase-difference is 360°. It is possible to obtain experimentally a condition in which the former are almost completely absent in the reflected light, while the latter are reflected with scarcely any loss of intensity. To calculate the most favorable conditions, we should require data regarding the percentages reflected and transmitted by films of various thicknesses.

Another method is to substitute a thin film of collodion for the mica, half-silvering the film as before.

A sheet of glass can be silvered chemically, or procured by removing the varnish from the back of a piece of modern mirror-glass with alcohol. The silver film is then flowed with collodion diluted with three or four parts of ether. As soon as the film dries colors appear, contrary to theory. These colors may be quite brilliant, and are due to diffraction, as will be shown presently. If the plate be now immersed in Brashear's silvering-bath, the colors will instantly disappear, owing to the fact that the collodion film and the solution have nearly the same refractive index. As soon as the silver begins to deposit, the colors reappear and increase rapidly in intensity. The bath should be rocked, the process being similar to the development of a negative. A little experience will enable the moment of maximum brilliancy to be correctly judged, when the plate should be immediately removed from the solution, washed, and dried. It is

well to provide the plate with a cover of glass mounted over it at an angle of 20°, the whole forming a prismatic box. The object of inclining the cover is to get rid of the light reflected from it, which would otherwise dilute the interference colors. Plates prepared in this way show a wonderful blaze of color and make excellent preparations for the lantern.

Colors of Frilled Transparent Films on Metallic Surfaces. — We will now consider a remarkable case of interference which appears to be essentially different from any of the cases which have been previously studied. The theory of thin films shows, as Lord Rayleigh points out in his article on "Wave-Theory of Light," that a transparent film on a perfectly reflecting surface shows no interference-colors. As has been already pointed out, a thin film of collodion deposited on a bright surface of silver shows brilliant colors in reflected light. It, moreover, *scatters* light of a color complementary to the color of the directly reflected light. This is apparently due to the fact that the collodion film "frills," the mesh, however, being so small that it can be detected only with the highest powers of the microscope. Commercial ether and collodion should be used. If chemically pure ether obtained by distillation is used, the film does not frill, and no trace of color is exhibited. Still more remarkable is the fact that if sunlight is thrown down upon the plate at normal incidence, brilliant colors are seen at grazing emergence, if a Nicol prism is held before the eye. These colors change to their complementary tints if the Nicol is rotated through 90°, *i.e.* in the scattered light, one half of the spectrum is polarized in one plane, and the remainder in a plane perpendicular to it.

In the cases of the transparent films with the first surface lightly silvered, the second heavily coated, the waves absent in the reflected light are absorbed by the metal, as I have already shown. In the present case these waves are scattered by the granular surface. If a spot on the film which appears purple by reflected light is illuminated with sunlight, it will be found that green light is scattered, not in all directions, but through a range corresponding to the size of the granulation, as in the case of the mixed plates, described in the chapter on Diffraction.

If the light is incident normally, the scattered light comes off through an angular range included between 10° and 30°, and again at an angle of nearly 90°, the latter being strongly polarized. Conversely, if the sunlight be incident at nearly 90°, strongly polarized light is scattered normally. Considerable difficulty has been found in explaining these colors satisfactorily. They appear to be saturated, *i.e.* certain wave-lengths are completely absent in the reflected light, and until the granulation was detected with the microscope it was impossible to make even a satisfactory hypothesis. Even now the polarization effects are difficult to account for.

At first sight it may seem as if the colors could be classed with the phenomena of mixed plates, their brilliancy and saturation reminding one of the appearances produced by laminary retardation. The films, however, show no color by transmitted light when deposited on glass, and the effective doubling of the retardation, by the reflec-

tion back through the film by the metal surface, can hardly account for the observed effects. Moreover, the energy stream reflected from the surface of the collodion appears to be essential, for if we employ light polarized perpendicular to the plane of incidence, and set the plate at the polarizing angle of collodion, so that no reflection occurs except at the metal surface, all trace of color disappears. If the angle of incidence is larger than the polarizing angle, the color of the reflected light changes to its complementary tint when the plane of polarization is made parallel to the plane of incidence. As has been shown in a preceding section, the effects at large angles of incidence involve the interference of two streams of light polarized in planes inclined at 90° to each other, which are complicated enough with monochromatic light and structureless films. For the present only normal incidence will be considered. Though I know of no direct way of proving that, in this case, the light reflected from the collodion surface is an essential factor, there is strong indirect evidence.

If the film is wedge-shaped and sodium light is employed, the dark fringes seen at normal incidence move towards the thick edge of the wedge as the angle of incidence is increased, exactly as they do with thin films of the ordinary type. If the incident light is polarized perpendicularly to the plane of incidence, the fringes gradually fade out, disappearing at the polarizing angle. This indicates that they are produced in the same way at normal incidence as at the polarizing angle, namely, through the agency of light reflected from the surface of the collodion.

If we consider some value of λ, for which the path-difference between the rays reflected from the collodion and metal surfaces amounts to an odd number of half-waves, the color corresponding to this wave-length will be weakened in the reflected beam owing to interference. In the case of transparent thin films the absent color appears in excess in the transmitted light, while in the present case it is thrown back through the film by the metal surface. It is thus clear that the colors which are weakened in the reflected light are made to traverse the frilled film a greater number of times than the colors for which the path-difference is an even number of half-waves. This accounts for the fact that these colors are more strongly scattered by the granulations of the films.

A collodion surface reflects only about 5 per cent of the incident energy; and it is found impossible to account for the strong colors seen in the reflected light, by compounding the feeble stream of light from the collodion with the powerful stream coming from the metal.

It appeared, however, that the observed effects could be accounted for, if the somewhat arbitrary assumption were made that the granulated surface reflected more strongly than a smooth surface. As has been said, the granulations are too small to interfere with the regular reflection of light, the scattering being selective, so to speak, *i.e.* confined to the waves which, owing to interference, are compelled to traverse the film a number of times.

The assumption above referred to appeared to be too arbitrary to make without some experimental evidence, and experiments were

therefore made to determine the effect of the "frilling" of the film on its reflecting power. One of the faces of a 60° prism of crown glass was flowed with collodion of the same dilution as that used in the preparation of the colored films. It showed in reflected light interference-colors, which, however, were very much diluted with white light, owing to the small difference between the refractive indices of the two media. In working with the film on silver it was found that, if the colors did not appear at once, as soon as the film dried, they could be brought out by breathing on the film, the deposit of moisture being advantageous to the formation of the granulations. It was always possible to intensify the colors in this way. The film deposited on the surface of the prism was treated in this way, one-half of it being screened from the deposit of moisture by a plate of glass. As soon as the moisture had evaporated, it was found that the reflecting power of the surface had been greatly increased, the film appearing almost as bright as a half-silvered surface.

The increase in the brilliancy of the reflected light was about threefold, as was shown by covering the unfrilled portion with a sheet of thin glass, which about equalized the intensities. In other words, the frilled collodion surface regularly reflects white light, of an intensity very nearly equal to that of light reflected from three glass surfaces.

On examining the granular surface with polarized light, it was found that the angle of maximum polarization was in the neighborhood of 63°, which would make its refractive index about 1.96. The polarizing angle of the smooth collodion was about 56°, the corresponding refractive index being 1.48.

An attempt was made to determine whether the granulation gave rise to elliptical polarization, the abnormal value of the refractive index suggesting the properties of the surface-films, which play such an important part in the theory of elliptical polarization. No decisive results were obtained, for though the phenomenon was found, it seemed impossible to eliminate the component reflected from the collodion-glass surface, which, as I have shown, may, by interference with the component reflected from the air-collodion surface, give rise to an elliptical vibration.

The interferometer failed to show any change in the refractive index as the result of frilling, which indicates that the effect is confined to the surface. A film deposited on glass of such thickness as to produce a shift of half a fringe width (sodium light) was frilled by moisture, one-half being protected by a glass plate. No shift was found at the line of demarcation, as would have been the case if the refractive index of the film had been raised from 1.48 to 1.96 throughout its entire thickness. The phenomena described in this section are worthy of further investigation.

Stationary Light-Waves. — In all the cases of interference which we have thus far examined, the interfering wave-trains have been moving in the same direction. In acoustics we have cases of interference where the waves are moving in opposite directions. Interference under these conditions gives rise to the so-called stationary waves. If we send a train of waves along an elastic cord, one end of

which is fastened, the waves are reflected from the wall, and running backward along the cord, interfere with the direct waves. The cord is at rest at points half a wave-length apart, vibrating in nodes (Fig. 136). We should expect something of the kind to occur when light is reflected normally from a mirror, and the possibility of such an action was early recognized. Zenker in his *Lehrbuch der Photochromie* explains the colors sometimes exhibited in photographs of the spectrum taken on silver chloride as due to the formation of layers of reduced silver between the nodal points of a system of stationary light-waves.

Fig. 136.

The existence of these stationary light-waves was first demonstrated in 1890 by Otto Wiener.[1] When rays of light are incident normally on a polished mirror the reflected rays travel back over the same course. If the light is monochromatic, we shall then have planes half a wave-length apart, parallel to the plane of the mirror where the ether is at rest. The condition is shown in a crude way in Fig. 137. We require now some device for recognizing the existence of these nodal planes in the medium in front of the mirror. Wiener made use of an exceedingly thin photograph film, the thickness of which was less than one-twentieth of the wave-length of light. Suppose such a film in the position shown by the line AB. The plane of the film now coincides with the first nodal plane. There is no vibration of the ether within the film, and though two powerful streams of light are traversing it, it will be unaffected. It is, however, unnecessary to get the film exactly into one of these planes, a matter which would, of course, be exceedingly difficult. Suppose the film to make a small angle with the mirror occupying the position $A'B'$. It will now cut the nodal planes, some portions of it lying in the loops and other portions lying at the nodes. Those parts of the film which are in the loops will be acted on, the other places will be unaffected. Wiener coated a glass plate with a thin photographic film, placed the film side close to the mirror, at a very small angle, and allowed monochromatic light to pass through the film and suffer reflection from the mirror. On developing the film he found it blackened along lines corresponding to the points where it intersected the loops of the standing waves, while the intervening portions were quite clear. Increasing the angle of course caused the dark lines to become finer, since a greater number of planes were cut in a given distance.

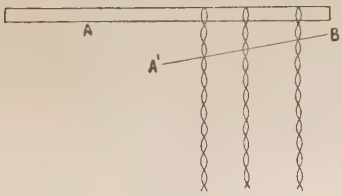


Fig. 137.

Some objections were raised against Wiener's demonstration, founded on the fact that the layer of air between the mirror and

[1] *Wied. Ann.*, 40, page 203, 1890.

the photographic plate would act in the same manner as the air film producing Newton's rings; in other words, the dark bands might be the result of interference between the light reflected from the mirror and the under surface of the photographic film. At first sight these objections seem quite plausible, since the maxima and minima formed in this manner would be identical *in position* with those obtained by Wiener. With a thin film we get a minimum when the thickness of the film is $\frac{\lambda}{4}$ and the next minimum when it is $\frac{3\lambda}{4}$. Thus every time we increase the thickness by half a wave-length we have destructive interference between the light reflected from the upper and lower surfaces. These points correspond to the nodal planes which are half a wave-length apart. The objections were satisfactorily answered by Wiener, who showed that such minima would be produced by the interference of a disturbance of large amplitude coming from the silver, and one of small amplitude coming from the collodion film used for photographing the minima and maxima. Interference-fringes produced under these conditions would be of the same general character as those seen in the transmitted light of this film, *i.e.* the difference in intensity between the bright and dark bands would be very small.

Wiener finally eliminated all such action by putting a layer of benzole between the film and the metallic reflecting surface, thus making the refractive index the same on both sides of the surface of the film, and doing away with the reflected component. The dark and bright bands were formed just as before, proving conclusively that they were caused by stationary waves. The presence of stationary waves can be shown independently of photography by employing a thin fluorescent film in place of the sensitized collodion. This experiment was performed by Drude and Nernst (*Wied. Ann.*, 45, page 460, 1892).

Another very beautiful experiment was performed by the same investigators. One-half of a glass plate was coated with a film of silver, and the whole then coated with a fluorescent film only a small fraction of a wave-length in thickness. When this plate was illuminated with monochromatic light obtained from a spectroscope, it was found that the fluorescence was much stronger over the transparent portions of the plate than over the silvered portions. In the former case the film is traversed by a beam of light of which only a small percentage is reflected back through the film, consequently it fluoresces brightly. In the latter case we have stationary waves, and the film is located at the first node, which lies on the surface of the silver, a position in which fluorescence is impossible, since there is practically no vibratory motion at this point.

Lippmann's Color Photographs. — Photographs in natural color were accidentally obtained by E. Becquerel in 1850, by means of standing light waves, although he was not aware of the part they played. In 1868 Zenker explained the colors, sometimes seen in Becquerel's photographs of the spectrum, as due to standing waves, formed by the reflection of the light from the surface of the silver

plate on which the sensitive film was formed. The silver was reduced in the anti-nodal planes forming a system of reflecting laminae, which showed interference-colors in reflected light in the same way as the crystals of chlorate of potash previously described.

A process of direct color photography, based upon this principle, was originated by Lippmann in 1891. The photographic plate is placed in the camera with the glass side facing the objective, and the sensitive film backed by a reflecting layer of mercury. This of course requires a special form of plate-holder.

A system of stationary waves is formed in the film as shown in Fig. 138, and the silver compound is acted upon only at the antinodes, which form planes parallel to the reflecting surface. On developing and fixing the plate in the usual manner, it is found that the film shows, in reflected light, brilliant colors, similar to the colors which illuminated it. The silver, instead of being reduced in a mass, uniformly distributed throughout the thickness of the film, is laid down in thin laminae, coinciding with the antinodal planes of the stationary light-waves. The distance between the laminae is equal to the half wave-length of the light which formed them, consequently they show the same color by interference in reflected light. The process is not an easy one to carry out, and very few have been successful with it. Especially prepared plates must be used, as the grain of the commercial plates is too coarse to record the minute structure of the wave-system.

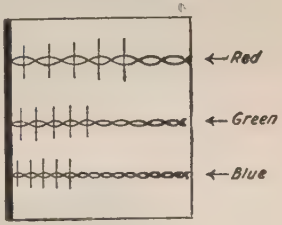


FIG. 138.

A very complete study of the process was made by H. E. Ives, who gives the following formula for the preparation of the plates:

A. Gelatine 1 gramme	B. Gelatine 2 grammes	C. $AgNO_3$ .3 gramme
Water 25 cc.	KBr 0.25 gramme	Water 5 cc.
	Water 50 cc.	

A and B are heated till the gelatine melts, allowed to cool to 40°, C added to A and then A to B slowly with stirring, and the whole filtered. After flowing and setting, the plates are washed for fifteen minutes and allowed to dry. The most satisfactory thickness was obtained by flowing the emulsion on and off glass plates at room temperature. The plates are bathed for 10 minutes in a $\frac{1}{1000}$ solution of isocol in water and dried rapidly to render them color-sensitive. They are exposed in a plate-holder so arranged that mercury can be introduced behind the plate, and in contact with the film. Exposures with $f/3.6$ on sunlit objects range from $1\frac{1}{2}$ to 5 minutes according to sensitizers, etc.

After development and drying, the pictures are made ready for viewing by cementing a thin prism of small angle to the film side to destroy the disturbing surface reflections, and the back of the glass is flowed with asphaltum varnish. The prism is usually cemented by means of Canada balsam.

The plates bathed in isocol lose their color-sensitiveness in four

or five hours, and must therefore be used immediately. If plates of better-keeping quality are required we may introduce 1 cc. each of a $\frac{1}{1000}$ alcoholic solution of a mixture of erythrosine and pinacyanol into the emulsion before pouring it over the plates.

It is of considerable interest to see how nearly we can reproduce a monochromatic color by means of the Lippmann process. If we form the laminae in the film by means of monochromatic light, the light, selectively reflected, will embrace a considerable range of the spectrum unless a very large number of laminae are formed. The case is the same as with the chlorate of potash crystals, which we have already studied.

Mr. Ives made an exhaustive study of this, and found that to obtain a large number of laminae special precautions as to the preparation of the sensitive film and its development must be taken. The film was illuminated by the green line ($\lambda = 5461$) of the mercury arc, and after developing and drying the film, it was used to reflect sunlight into a spectroscope. In this way it was

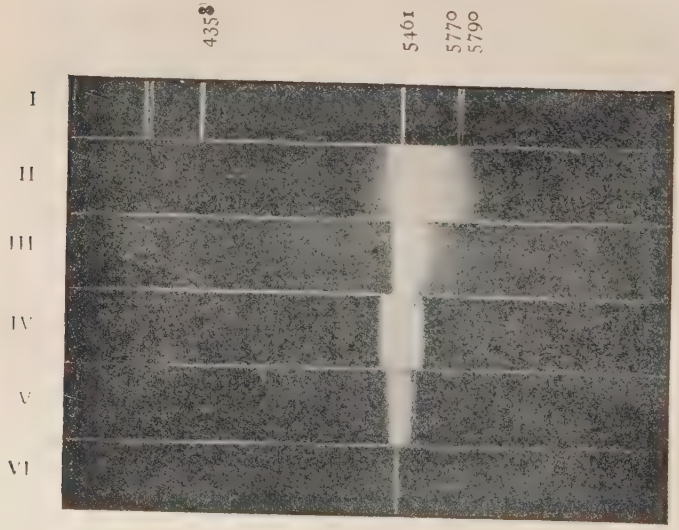


FIG. 139.

I. Mercury vacuum arc.
II. λ 5461 as reproduced by fine-grain film, with pyrogallic acid development.
III. Same from glass side.
IV, V, VI. Hydroquinone-developed bleached films, 50, 150, 250 laminae.

possible to see at once how nearly the film was able to manufacture green monochromatic light. A number of photographs obtained with the spectroscope are reproduced in Fig. 139. As will be seen the green light manufactured from white light in the case of film No. 6 is highly homogeneous. This was the film compared with chlorate of potash crystal previously alluded to. By cutting a section of the film and examining it with a microscope, the laminae thrown down by the standing light-waves can be seen and counted. It was found in this way that as many as 250 laminae could be

formed, if the sensitizing dye was introduced into the emulsion and the plate developed with hydroquinone. In the case of bathed plates the sensitizing action only penetrated a short distance, and with pyrogallic acid development the developing action occurred chiefly near the surface, limiting the possible number of laminæ. A microphotograph of a similar section by Mr. Senior is reproduced in Fig. 140.

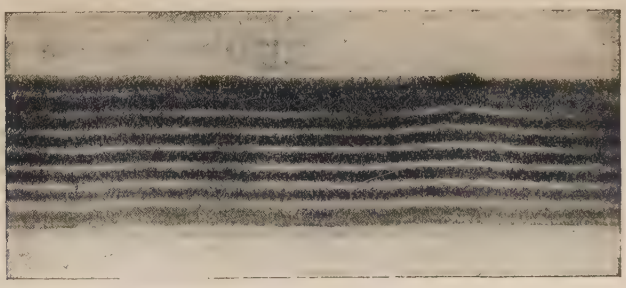


Fig. 140.

In regard to the reproduction of mixed colors Mr. Ives says: "Mixed colors, such as two or more spectral lines, or the broad ill-defined bands of the spectrum given by pigment colors, give standing waves which may be compared to the interference fringes they would give in a Michelson interferometer. That is, we have regions in the film where the different wave-lengths acting reënforce, and other regions where they oppose each other. The visibility curves, therefore, are applicable to the structure of the Lippmann film. Photographs have been published by Lehmann showing that the resultant structure for two radiations agrees with the calculated." Some very interesting results were obtained in the case of films illuminated with light made up of two different wave-lengths. The laminæ were found to be periodically in and out of step, the analogy with "beats" being obvious. The periodic structure produced in this way was coarse enough to be easily seen with low powers, in thin sections of the film. Lehmann has also made some interesting experiments on the production of Lippmann colors of the second and third orders, analogous to the Newton's colors.

At the surface of the film the laminæ due to the two different wave-lengths will be in step, deeper down they will be out of step, and the deposit will be nearly uniform, *i.e.* without laminæ. At a still greater depth they will appear again. The reader is referred to Mr. Ives's paper for a more complete discussion of this part of the subject.

Some very remarkable results were obtained by Neuhauss in the course of some experiments made with a view of determining the action on the Lippmann film of two different spectrum colors. A continuous spectrum was projected upon a plate and an exposure made. The plate was then rotated through 180° and a second

exposure made, the red now falling on the end of the plate which had previously recorded the violet. On developing the plate a number of dark bands parallel to the Fraunhofer lines appeared on the superposed part of the two spectra, as shown in Fig. 142. They are seen only from the glass side, however. Various inadequate or impossible explanations of these dark bands have been given by different writers. The correct explanation was first found by Lehmann, who showed that at points where the reflection fell to a minimum the silver deposits due to the two wave-lengths were out-of-step, and that therefore instead of having laminae we have a more or less homogeneous mass of low reflecting power. The dark bands occur at points where the two sets are in dissonance at the glass surface.

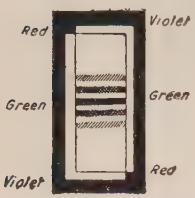


FIG. 142.

CHAPTER VII

THE DIFFRACTION OF LIGHT

The rectilinear propagation of light depends, as we have seen, on the destructive interference within the region of the geometrical shadow, of the elementary waves which we may regard as originating on the wave-front. That this destruction is as complete as it is, depends on the shortness of the wave-length of light, and it is in this respect that sound differs from light. Sound we know does bend around corners, and unless we make our experiments on a very large scale, it is difficult to get any evidence of sound shadows. In the case of light, if we examine with sufficient care the propagation of the radiation by the edges of opaque obstacles, we shall find that it does bend into the region of the shadow to a slight extent, and that the bending is more pronounced for the long waves than for the short. That the bending is greater for the long waves may be shown by the following simple experiment. Sun or arc light is passed through a narrow slit, and a sharp straight edge of metal mounted in the narrow beam at a distance of a metre or so from the slit. If now we bring the pupil of the eye well within the shadow of the edge we shall see the edge strongly illuminated. This light which we see coming from the edge is the diffracted light, and it should be richer in long waves than in short. To show this we have only to view the illuminated edge through a direct vision prism. As we move our eye into the shadow the blue end of the spectrum of the diffracted light disappears first. If any doubt exists as to the reality of the phenomenon it is easy to mount a filament of glass, just outside of the edge, which will give a spectrum of about the same intensity. By a little adjustment the red and yellow of the two spectra can be made the same, when it will be found that one shows the blue and violet distinctly while the other does not.

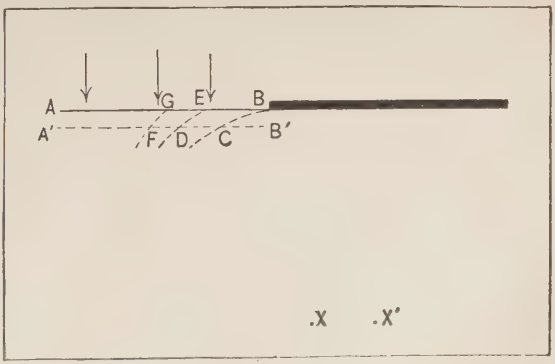


Fig. 143.

That the bending is greater for the long waves and that the intensity falls off rapidly as we pass into the region of the shadow, can be seen from the following considerations. In Fig. 143 let B be the edge of a screen upon which a plane wave is incident. Consider

AB as the trough and $A'B'$ as the crest of this wave. Around the point x, which lies within the shadow, describe arcs CB, DE, and FG. The points C and B are equidistant from x, consequently a plus disturbance from C and a minus disturbance from B will reach x simultaneously and cancel each other. The same will be true for disturbances from D and E, and from F and G.

There remain, however, the disturbances from the portion CB', which are not destroyed, consequently this portion of the wave is effective in illuminating x. Now if the wave is longer, *i.e.* if the distance between AB and $A'B'$ is increased, the length of this uncompensated portion CB' will be greater, and the illumination at x will be increased. If, however, we move to x', farther within the shadow, and describe our arcs around this point, the length of the portion CB will be reduced and the illumination will consequently be less. This method of viewing things shows us also that the type of wave propagated into the region of the shadow is one which appears to start from the edge B of the screen. If we place our eye at x we shall see the edge B shining as if self-luminous, in other words, the diffracted wave is a sphere (or rather a cylinder) having its centre of curvature at B. If the wave-length is made infinitely small, the diffracted wave will vanish, as seems to be the case with X-rays.

This simple bending of the light around the edges of obstacles is termed diffraction, though in the study of the subject we are concerned chiefly with the so-called fringes, or alternate bands of light and shadow which usually accompany diffraction. These fringes always appear just outside of the boundary of the geometrical shadow, and have in reality nothing to do with the bending of light around corners. They are termed diffraction fringes notwithstanding.

The diffraction of light was first observed by Grimaldi about the middle of the 17th century. Admitting sunlight into a darkened room through a very small aperture, he observed that the propagation of the light by the edges of objects, did not obey strictly the laws of geometrical optics. The edges of the shadow were bordered by several rainbow-tinted fringes, while in the case of very small objects similar fringes were found within the geometrical shadow. These fringes should not, however, be spoken of as diffraction fringes, though they owe their origin to diffraction. He also described the branching or crested fringes which appeared within the shadow at the corners of rectangular screens.

Newton repeated and improved upon the experiments of Grimaldi, using light of different colors, and found that the distance between the fringes decreased as the refrangibility of the light increased, and increased as the screen was removed to a greater distance from the object casting a shadow. He explained this phenomenon on his corpuscular theory as due to attractive or repulsive forces, which the edges of the obstacle exerted on the flying corpuscles.

The first attempt to bring the wave-theory to bear upon the subject was made by Young, who regarded the fringes as due to interference between the rays passing close to the edge, and rays

reflected at grazing incidence. This explanation could scarcely be applied to the fringes found within the regions of shadow, and in the case of the external fringes the distances between successive maxima and minima are not such as would occur if the interference took place in the manner imagined by Young. The internal fringes he explained as due to the interference of inflected rays, without attempting to explain how the inflection took place, and in this he was in part correct, for the internal system we can regard as a set of interference-fringes produced by two similar sources of light situated at the edges of the obstacle.

Fresnel made a series of experiments with slits having polished and blackened edges, and showed that the intensity of the fringes was independent of the nature of the edge.

He was the first to give the true explanation of the phenomena, regarding the maxima and minima as the result of the interference of the hypothetical secondary wavelets diverging in all directions from those portions of the wave-front not blocked off by the opaque screen.

The foundations of the theory of diffraction were laid by Fresnel, and though he treated only a few of the simpler cases, his fundamental equations were subsequently applied to, and solved for, much more complicated cases by Schwerd, Knochenhauer, and others.

We shall first examine some of the simpler cases by very elementary methods, and then take up the more rigorous treatments of Fresnel and Fraunhofer.

In commencing our study of diffraction it is best to dispense as much as possible with apparatus. Collimators and telescopes, which are so often employed for the subjective exhibition of diffraction effects, are unnecessary complications, and prevent us from obtaining the clear conception of the actual magnitude of the effects, which we get when we employ simply a brilliant source of light and a white screen, in combination with various obstacles which we place in the path of the light. A narrow beam of sunlight, or the light from an arc lamp, should be focussed with a lens on a pin-hole in a thin sheet of metal mounted over an aperture cut in a large screen of cardboard. The light diverging from the small source thus produced should be received on a screen of white Bristol board at a distance of two or three metres from the first screen, the best distance being found by trial. The room should be made as dark as possible, and if the arc-lamp is used it should be placed either in a box or in an adjoining room, and the light admitted to the room through the pin-hole, which may be placed over the key-hole of the door. Diffraction fringes will now appear around the edges of the obstacles placed midway between the source of light and the screen. Beautiful effects can be obtained by sprinkling a little lycopodium dust in the air, the shadow of each particle being surrounded with colored rings. Even the motes floating in the air produce colored fringes.

Straight Edge. — If an obstacle bounded by a straight edge is placed in the path of the rays we find that a set of rainbow colored

fringes appear on the screen parallel to, and outside of the edge of the geometrical shadow, while within the geometrical shadow the illumination falls off gradually to zero, without showing any maxima and minima. If we examine the fringes carefully we shall see that they are not equidistant, as is the case with two small sources of light, but lie closer together as we recede from the edge.

We will now calculate the illumination at various points in the vicinity of the edge of the geometrical shadow, by dividing the wave-front, or as much of it as gets by the obstacle, into half period elements with respect to the point in question. This method is one commonly employed in text-books, and is given merely as an example of a very elementary treatment. It is applicable only to a linear wave, *i.e.* a thin section of the actual wave which is represented by the circular arc ST in the diagram. In Fig. 144, O is the source of light, AB the diffracting screen, PQ the screen on which the fringes appear; P is the point at which we are to determine the illumination, situated at a distance x from M, the edge of the geometrical shadow. The distance $OA=a$ and $AM=b$. We divide the wave into half-period elements with respect to P, as shown in the figure. We have seen that the effect at P is chiefly due to a few elements on each side of the pole R, and that when the whole wave acts, the action of these elements reduces by interference to a small portion surrounding the pole, approximately equal to the effect which would be produced by one half of the central element. In the case above it is clear that we have the entire half of the wave above the pole, and the portion RA below the pole. If RA contains an even number of half-periods they will destroy each other in pairs, and the illumination at P will be small, being due to RS alone. If RA contains an odd number of elements, the one adjacent to the pole will remain over, after the others have interfered in pairs, and will add its effect to that produced by RS, and the illumination at P will be a maximum. As the point P moves away from M, the illumination will pass through alternate maxima and minima, the positions of which we can calculate as follows:

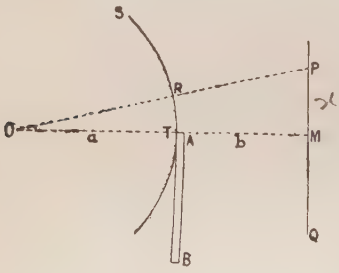


Fig. 144.

$$OP=a+b+\frac{x^2}{2(a+b)} \quad \text{and} \quad AP=b+\frac{x^2}{2b};$$

$$\therefore AP-RP=\frac{x^2}{2}\left(\frac{1}{b}-\frac{1}{a+b}\right).$$

If RA contains an even number of half-period elements $AP-RP$ must equal an even number of half waves, therefore the condition for minimum illumination is

$$\frac{x^2}{2}\frac{a}{b(a+b)}=2n\frac{\lambda}{2},$$

while if RA contains an odd number of elements, the condition for maximum illumination, we have

$$\frac{x^2}{2}\left(\frac{a}{b(a+b)}\right)=(2n+1)\frac{\lambda}{2},$$

which gives us $x=\sqrt{\frac{b(a+b)}{a}2n\lambda}$ for the minima,

$$x=\sqrt{\frac{b(a+b)}{a}(2n+1)\lambda}$$ for the maxima.

This formula shows us that the distance x increases as a diminishes, which was observed by Fresnel.

The maxima are spaced like the Fresnel zones, *i.e.* they lie nearer to each other as we recede from M, the distances being proportional to the square roots of the natural numbers.

We can see this at once, for, as we move P up, we expose the zones below the poles and the successive increments of x necessary to expose zone after zone will be proportional to the decreasing distances between the zones. The illumination within the shadow is due only to a portion of the wave above the pole, since for all points so situated, the pole of the wave is intercepted by the screen. The effect of a wave thus reduces to approximately one-half of the element adjacent to the edge, and as we pass deeper into the region of the shadow, this element is further removed from the zone, and consequently less effective. The illumination thus falls off gradually without passing through maxima and minima.

Circular Disk and Circular Aperture. — The method is better adapted to the discussion of the distribution of the illumination produced by a small circular disk, since if we confine ourselves to the axis of the conical shadow, the disk exactly coincides with zones described on the wave-front with respect to points lying on the axis.

This is the celebrated problem of Poisson, who was led by theoretical considerations to the remarkable conclusion that the illumination along the axis of the shadow of a small circular disk is the same as if the disk were removed, a prediction which was verified experimentally by Arago. It is not very difficult to repeat this experiment, provided it is tried on a large scale. The experiment has been already described in the Chapter on The Rectilinear Propagation of Light.

Applying the Fresnel construction to this case we see that the illumination on the axis is due to the action of the entire wave with the exception of the zone or zones covered by the disk. These reduce by interference to approximately one-half of the effect of the zone bordering the disk. Since the effect of the zones becomes less as we recede from the pole, it is clear that increasing the size of the disk, other things being equal, will reduce the illumination at a given point on the axis. To obtain the most striking results we should so proportion the distance of the screen to the size of the disk, that the latter covers only a very few zones. A small coin at a distance of 4 or 5 metres from the screen shows the effect well. If we make

a small hole in the screen at the centre of the shadow, and bring our eye close up to it, we shall see the rim of the coin brilliantly luminous, which shows that the light within the shadow is propagated as if coming from the edge of the obstacle.

This same effect can be seen within the shadow of a straight edge. It is often seen in nature on a large scale when the sun rises over the tops of tree-covered mountains, the tops of which are situated nearly above the observer, every branch, twig, and leaf shining with a silvery light, the small birds flying about appearing like specks of fire.

Small Circular Aperture. — In this case the illumination along the axis is due solely to the zones lying within the aperture. Suppose the point to be so situated that the aperture contains only the two central zones. The disturbances from these will completely destroy each other at the point, and the illumination will consequently be zero. If we bring the point a little nearer to the aperture the scale of the zones will be reduced, and the aperture will contain say three. The two outer ones will annul each other, and we shall have an illumination due to the outstanding central one. We thus see that the illumination is a maximum or minimum according as the aperture contains an odd or even number of zones.

We shall later on investigate the problem of the circular aperture more completely, the foregoing method being applicable only to the axis.

Fraunhofer Class of Diffraction Phenomena. — The effects which we have been considering belong to the so-called Fresnel class of diffraction phenomena. In this class the source of light and the screen are both at finite distances from the obstacle. The problem of determining the distribution of light and shadow becomes much simpler, if we consider the source of light and the screen removed to infinity. This means that we have plane-waves falling upon the aperture, and the secondary disturbances, which we consider as coming from the plane of the aperture, all start at the same time. In other words, the phase of the vibration is the same at every point of the aperture, which is not the case when the source of light is at a finite distance, the wave-fronts then being spherical. For a screen at infinity we have to consider the interference as taking place between parallel rays. This condition can be reproduced experimentally if we place a lens behind the aperture, which brings every system of parallel diffracted rays to a focus. Effects produced in this way are said to belong to the Fraunhofer class, since they were first studied and discussed by Fraunhofer, in connection with his experiments on the diffraction grating.

We will now consider the diffraction effects produced by a narrow slit and then by a number of parallel equidistant slits, when the incident rays are parallel, and the diffracted rays are brought to a focus by a lens. We shall investigate the subject in an elementary way, and then take up the more complete mathematical treatment.

Narrow Slit. — If the telescope of a spectrometer is directed down the collimator, and the diffracting aperture, in this case a slit, is placed between the two, we have the conditions specified for the

Fraunhofer class, for the light focussed upon the slit of the collimator is made parallel by the lens, which virtually removes the source to infinity, and the parallel diffracted rays are brought to a focus and interfere in the focal plane of the telescope, the maxima and minima resulting therefrom being examined with the eye-piece.

A still simpler way of getting the fringes is to place the slit before the eye, and view a distant lamp through it, the lens of the eye here performing the function of the telescope lens, uniting the parallel diffracted rays upon the retina.

In the following treatment, which is taken from Müller and Pouillet's *Lehrbuch der Physik*, we shall use a very elementary method, and yet arrive at an approximate numerical value for the illumination at various points of the diffraction pattern.

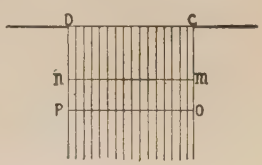


Fig. 145.

Let *DC* (Fig. 145) be the aperture upon which parallel waves are incident in a normal direction. The phase of the vibration will then be the same across the aperture, or along any line parallel to it, such as *p*, *o*. We will consider the normally diffracted ray bundle as divided into 16 elementary ray bundles, which are united by the lens at a point situated at the centre of the diffraction pattern which we are to study. Let the amplitude produced at the focus by one of these elementary bundles be a, then the amplitude produced by all 16 will be $16\,a$, since they all arrive in the same phase, and the intensity of the illumination will be

$$I = 16^2 a^2 = 256\,a^2.$$

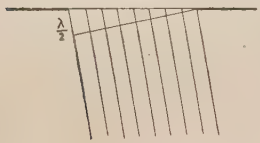


Fig. 146.

Consider next a bundle of parallel diffracted rays which leave the aperture at such an angle that the path-difference between the extreme rays is exactly one-half wave-length, as shown in Fig. 146. These extreme rays will then arrive at the focus with a phase-difference of 180° and destroy each other. The other elementary bundles destroy one another to a certain extent, and we determine the resultant effect in the following way. If the path-difference between the extreme bundles is $\frac{\lambda}{2}$, the path-difference between any two adjacent bundles is $\frac{\lambda}{32}$. For the resultant amplitude of any pair, we have the expression

$$A = a\sqrt{2 + 2\cos 2\pi\frac{x}{\lambda}}, \text{ in which } \frac{x}{\lambda} = \frac{1}{32} \text{ and } \frac{2\pi}{32} = 11^\circ\ 15';$$

$$A = a\sqrt{2 + 2\cos 11^\circ\ 15'} = a\sqrt{3.962}.$$

Now the path-difference between two adjacent double bundles is $\frac{\lambda}{16}$, consequently the amplitude due to two adjacent double bundles

will be

$$B = A\sqrt{2+2\cos(22^\circ\ 30')} = A\sqrt{3.848}.$$

This value B is then the amplitude produced by four adjacent elementary bundles.

In a similar way we find the effect of eight bundles to be

$$C = B\sqrt{2+2\cos 45^\circ} = B\sqrt{3.414},$$

and by combining the two groups of eight each, we get for the final amplitude

$$D = C\sqrt{2+2\cos 90^\circ} = C\sqrt{2}.$$

Substituting for C, B, and A the values found, we get

$$D = a\sqrt{(3.962)(3.848)(3.414)2}.$$

The final intensity $I_1 = D^2 = 104 \cdot a^2 = .406\, I$.

The value found by calculus is

$$I_1 = .4053\, I,$$

which shows that even the crude subdivision which we have employed yields a fairly accurate result.

The intensity at the point at which these diffracted rays come together is therefore about .4 of the intensity at the centre of the pattern.

In a direction such that the extreme rays have a path-difference of λ, we find either by combining two sets of 8 elements each with a path-difference of $\frac{\lambda}{2}$, or by pairing off the 1st and 9th, 2d and 10th, etc., with path-differences of $\frac{\lambda}{2}$, that the resultant illumination is zero.

If the path-difference is $\frac{\lambda}{3}$, we may divide the bundle into three bundles, the path-difference between the extreme rays of each bundle being $\frac{\lambda}{2}$. Two of the three will mutually destroy each other, the resultant illumination being that which would be produced by the third acting alone. The amplitude produced by this third will evidently be $\frac{1}{3}$ of the amplitude produced in the case where the path-difference between the extreme rays was $\frac{\lambda}{2}$, consequently the intensity is

$$I_3 = \tfrac{1}{9}\, I_1 = .045\, I.$$

With a path-difference of $\frac{\lambda}{4}$ between the extreme rays, we again have zero illumination.

The diffraction pattern thus consists of a central bright band which is very intense, bordered by alternate dark and bright bands, the intensity of the latter decreasing very rapidly. As the width

of the slit is made less, the angle of diffraction necessary to give the extreme rays a path-difference of $\frac{\lambda}{2}$ becomes greater and the minima retreat from the centre of the system, the fringes broadening.

This can be seen by holding before the eye a slit made of black paper, the width of which can be varied, and viewing a distant lamp through it: the same experiment can be made with a slit formed between the first and second finger. The fringes produced by diffraction through a single slit were termed by Fraunhofer spectra of the first class.

Two Parallel Slits. — We will now take up the case of diffraction by two similar parallel slits. They produce spectra of the first class in the same position, but we shall find that these maxima are broken up by a new set of minima which run through them. These minima are especially noticeable in the bright central maximum, and are produced by destructive interference between diffracted rays from the first slit and corresponding rays from the second.

Let us assume the slits AB and CD to be so narrow that in the direction represented in Fig. 147 the path-difference between the extreme rays A and B, C and D is $\frac{\lambda}{4}$. Rays diffracted in this direction will then be the rays which bring about the illumination of that portion of the central maximum lying midway between the centre and the point where the illumination is .4 of its value at the centre. If but one slit were acting, the intensity would be about .7. If the slit CD is at such a distance from AB that the path-difference between corresponding rays A and C is $\frac{\lambda}{2}$, the corresponding rays from the two slits will destroy each other and the illumination will be zero.

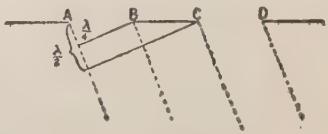


FIG. 147.

It is in this way that the new minima are produced, and though they may be investigated in a manner similar to that employed in the case of a single slit, it will be best to postpone their more complete investigation until we come to the mathematical discussion of the grating.

Transition from Fresnel to Fraunhofer Class. — The gain in the brilliancy of the diffraction effects when we use a lens to bring the parallel diffracted rays to a focus, can be shown by passing gradually from the Fresnel to the Fraunhofer class.

In Fig. 148 we have at the left the diffraction pattern produced by a single slit, a bright central maximum bordered by fainter maxima. These fringes belong to the Fresnel class and can be seen by examining the region behind a narrow slit with an eye-piece; a scratch made with the point of a knife on the film of an old photographic negative answers the purpose. If now we make two slits side by side, say 1 mm. apart, we have the pattern shown in the right-hand side of the figure. In the overlapping portions we have new maxima and minima, much closer together than the others due to the inter-

ference of disturbances coming from the two slits. These maxima and minima do not appear in the bright central maxima A and B, since these are illuminated each by its own slit, in other words, there is no overlapping here, or at least only that over the very faint

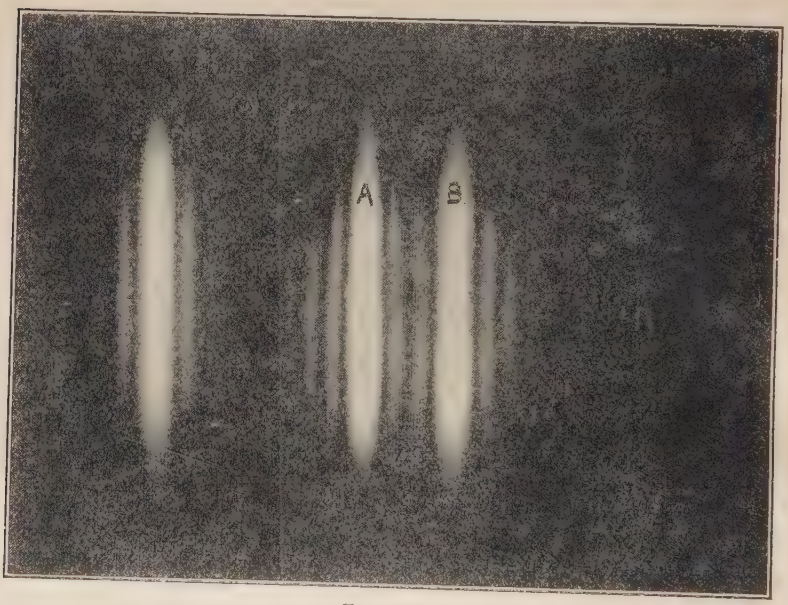


FIG. 148.

third maxima, which are unable to produce visible interference. If now we put a lens behind the two slits, the two patterns A and B will be superposed at the focus, and we shall have the new maxima and minima furrowing the brightest part of the pattern. We can pass gradually from one condition to the other, by holding the eye-piece close to the lens and gradually moving it back to the focus. As we do this A and B, which are at first widely separated gradually approach, fuse together, and immediately fill up with very bright and narrow maxima and minima. This experiment is extremely instructive, and teaches us more about diffraction and the relation of the Fresnel to the Fraunhofer class, than many pages of mathematics. It is apparent that the conditions are the same in this experiment as in the one with the interference refractometer described in the previous chapter.

Graphical Method of Solution of Diffraction Problems. — There is another method of solving problems in diffraction, which, though elementary, depends upon the application of results obtained from the mathematical treatment of the subject. This treatment is for the Fresnel class.

The graphical representation of the resultant of a large number of vibrations, of continuously varying phase and amplitude, was employed by Cornu in the discussion of diffraction problems. The

resultant effect of a number of disturbances of different amplitude and phase can be represented graphically as the closing side of a polygon, the sides of which are proportional in length to the amplitudes produced by the disturbances acting separately, and make angles with a fixed line equal to the phases of the dirturbances. For example, in Fig. 149, let OA, AB, BC, and CD be the amplitudes produced at a point by four disturbances which arrive simultaneously with phases represented by the angles α, β, γ, and δ. The resultant amplitude will be represented by the closing side OD of the polygon, the phase of the resultant being the angle DOX. In diffraction problems we have to determine the effect of a large number of disturbances of different phases coming from the portions of the wave-front which are not cut off by the diffracting screen. We can consider the effect of a complete wave at a point in front of it as made up of a large number of small amplitudes, of variable phase and of constantly decreasing magnitude. Each point on the wave-front will produce at the point in question a disturbance of a certain amplitude and phase. For example, in the figure we may regard the points A, B, and C on the wave-front as producing at the point X the amplitudes OB, BC, and CD above referred to, with phases α, β, and γ. As we recede from the pole of the wave the phase angle of the secondary disturbances at X will increase continuously, owing to the increased distances of the successive points. The effect at X due to the disturbances coming from A, B, and C will be represented by OD, the closing side of the polygon. In reality we have an infinite number of points on the wave-front. Let us consider the first Fresnel zone as divided into eight elements, each one of which produces at X unit amplitude. The effect of all acting simultaneously can be found by employing the construction thus given. The first element, which is next to the pole, will produce the amplitude AB, the second BC, the third CD, etc.; the eighth element will produce the amplitude HI, the phase having turned through 180°, since by Fresnel's construction the edge of the zone is half a wave-length farther away from the illuminated point than the pole. The resultant effect of the first half zone will therefore be represented by AI. (Figure inaccurate; HI should be parallel to AB.) The effects of the second Fresnel zone can be represented by continuing the construction from the point I to the point K, and as the amplitudes due to the successive elements decrease rapidly owing to the obliquity, the broken curve will assume the form of the spiral. If now we consider an infinite number of points on the wave-front, the phase will no longer change abruptly, but will vary continuously in passing from each point to the next. The amplitudes due to the successive ele-

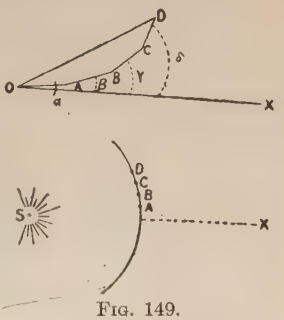


Fig. 149.

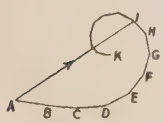


Fig. 150.

ments being very small, the broken curve will now be smoothed out into a continuous one, as shown in Fig. 151.

Cornu's Spiral. — The spiral shown in Fig. 151 was constructed by Cornu from the tables of Fresnel's integrals, which we shall consider subsequently. The effect of each Fresnel zone is represented by a half turn of the spiral, and if we consider the action of the whole wave the spiral will make an infinite number of turns, finally subsiding to asymptotic circles at J and J' of sensibly zero radius. The spiral OJ represents one-half of the complete wave, and the spiral OJ' the other. The line JJ' joining the two asymptotic points represents the action of the complete wave. Any portion of the wave is represented by the corresponding chord of the spiral. The effect, for example, due to the second, third, and fourth zone on the one side of the pole will be found by joining the ends of the second, third, and fourth half turns of the spiral. It is possible by means of this spiral to plot graphically the distribution of light in the diffraction patterns formed when the wave is partially cut off by screens of various types. The spiral will be better understood after we have studied Fresnel's integrals, and it is introduced at this point merely because it furnishes an easy means of solving a number of the simpler problems in diffraction. A larger drawing of the spiral will be found on Plate 11, which should be used in plotting diffraction patterns.

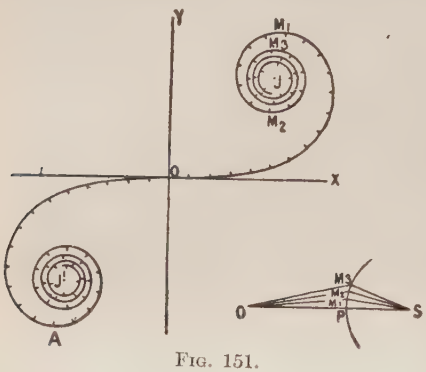


Fig. 151.

We will now consider a number of cases.

Case 1. **Straight Edge.** — The elementary treatment has shown us that in this case we have a system of fringes of decreasing width, outside of the edge of the geometrical shadow, while within the edge the illumination falls off rapidly, without, however, passing through maxima and minima. Consider first the illumination outside of the edge as represented by the spiral.

Since our spiral represents *amplitudes*, we shall find our intensities by squaring the resultant amplitude lines. At the edge of the shadow the intensity will be represented by the square of the distance OJ, since one-half of the complete wave is operative at this point. As we pass out from the edge, the lower part of the spiral begins to operate, and on reaching a distance such that the whole of the first or central Fresnel zone is exposed, the intensity will be found by squaring the line joining the point A with J (Fig. 151); *i.e.* it will be considerably greater than the intensity due to the entire wave, which is represented by the square of JJ'. If we represent the intensities as ordinates, our abscissae must be taken proportional to the distances, measured along the spiral from O, at which the point which we join with J, is located. To facilitate measure-

ments, equal distances have been marked off on the spiral. The first maximum occurs at abscissa 14, the distance from O to the bottom of the spiral.

The intensity as we proceed outward will be represented by the square of the line joining J with a point which travels around the lower half of the spiral. The intensity thus passes through maxima and minima, soon reaching a nearly constant value, owing to the small diameter of the convolutions, the first minimum occurring at abscissa 20.

Within the edge of the shadow we have the intensity represented by the square of the line which joins J with a point travelling from O towards J along the *upper* half of the spiral. This line rapidly shortens, without passing through maxima and minima, consequently the illumination drops rapidly to zero. A curve plotted in this way is shown in Fig. 152.

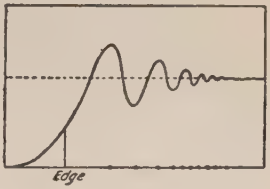


FIG. 152.

CASE 2. **Narrow Slit.** — In this case the amplitude of the vibration is measured by an arc of the spiral, the length of which is proportional to the width of the slit. The intensity will be represented by the square of a line joining the extremities of a constant length of the spiral. Equal distances will be found marked off on the spiral. Suppose the width of the slit and its distance from the screen to be such that it subtends exactly one-half of the central zone. The length of the arc which we are to employ is then equal to that of the first half turn of the spiral, namely, within the geometrical projection of the aperture, the arc will lie partly in the upper and partly in the lower branch of the spiral. At the centre it will be symmetrically placed; that is, with its centre at O. As we proceed from the centre, we push the arc of constant length along the spiral, squaring the line joining its extremities at regular intervals, plotting these values as ordinates, at abscissae corresponding to the distances advanced along the spiral as before.

It is at once apparent that the illumination at the centre of the fringe system may be either a maximum or a minimum, according to the width of the aperture. If the aperture just covers the entire central zone, the illumination will be a maximum, and will have a larger value than that due to the whole wave, while it will be a minimum if the aperture covers two zones.

CASE 3. **Narrow Wire.** — This case is a little more complicated, for the effect of the wire is to cut out a constant arc of the spiral just the reverse of the condition in Case 2. The amplitude is the resultant of the two remaining portions of the spiral, which must be compounded, paying attention to the directions as well as the lengths of the lines joining the extremities of the curves. The direction is always found by measuring from J' to J. This can be seen from the following consideration: The effect due to one-half of the wave is $J'O$, that due to the other half is OJ. The whole wave will produce an amplitude equal to the sum of these vectors. If we take their directions as measured from J' to O, and from O to J, the

amplitudes will be added, and we shall have amplitude $J'J$. If, however, we measure from O to J' and from O to J, the vectors will be opposed, and cancel each other. If this is borne in mind, no difficulty will be found in remembering how to determine the direction in which the amplitude lines point.

Suppose now that the wire cuts off one-half of the central zone, *i.e.* one half turn of the spiral measured from O. At the centre the first elementary distances on each side of O (see Fig. 153, dotted line) will be absent, and the resultant amplitude will be found by compounding $J'b$ with aJ, as shown in the figure. At the edge of the geometrical shadow of the wire, we compound the line $J'O$ with the short line which joins the highest point, say "d," of the upper branch of the spiral with J (as shown in the lower part of the figure), the resultant amplitude being JX. Abscissae are laid off as before, proportional to the distances advanced along the spiral.

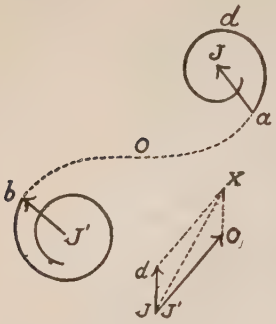


Fig. 153.

The intensity curve should be plotted in this manner for a wire covering say two zones. It will be found interesting to compare it with the curve obtained in the case of a straight edge. Regarding one side of the wire as a straight edge, we see that the exterior fringe system is complicated by the effects due to the exposed portion of the wave beyond the opposite edge.

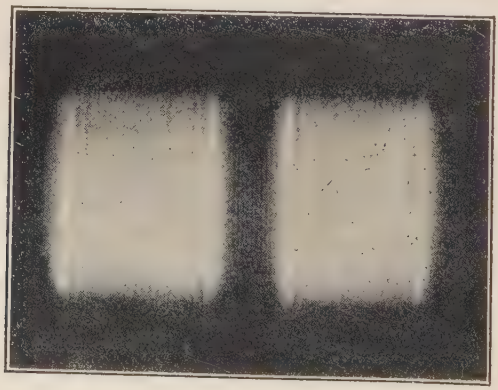


Fig. 154.

In Fig. 154 we have a photograph of the fringes produced by a vertical thick wire stretched across a rectangular aperture. The straight-edge fringes appear outside of the edges of the wire, and bordering the left and right hand edges of the aperture. No fringes border the upper and lower edge, since the source of light was a vertical illuminated slit.

Within the shadow of the wire are seen the fringes produced by the overlapping of the radiations which penetrate within the shadow.

They are equidistant, and we can regard them as interference-fringes produced by two similar sources of light bordering the two edges of the wire.

CASE 4. **Two Parallel Slits.** — The effects in this case are found by compounding the resultants of two arcs, of lengths proportional to the widths of the slits, separated by a distance proportional to the distance between them.

Mathematical Treatment of Fraunhofer Diffraction Phenomena. — If a converging lens is placed behind the aperture upon which plane waves are falling, the lens transforms that portion of the plane [illegible] which gets through the aperture into a concave wave, which, if the laws of geometrical optics were followed, would collapse to a point at the focus.

We have then to determine the effect of a small portion of a spherical concave wave at a plane passing through the centre of curvature of the wave. To do this we first get a general expression in the form of a double integral for the effect of a complete hemispherical wave, and then integrate this expression over the aperture (Fig. 155).

Let the centre of the concave wave be at the origin of three rectangular coordinates. We are to determine the effect of a disturbance starting from the point P with coordinates x, y, z at the point M with coordinates ξ, η, and then the collective effect of all the disturbances coming from the entire wave-front. Call the distance $MP = \rho$ and $dxdy$ the element of the wave at P.

The amplitude at M produced by the secondary disturbances from the area $dxdy$ will be $\kappa dxdy \sin 2\pi\left(\frac{t}{T}-\frac{\rho}{\lambda}\right)$, in which κ is a coefficient depending on the inclination of the surface element $dxdy$ to ρ and on the distance of P from M. Since we limit the area of the wave, by blocking off the greater part we can regard the inclination as the same for all portions considered, and κ therefore becomes a constant.

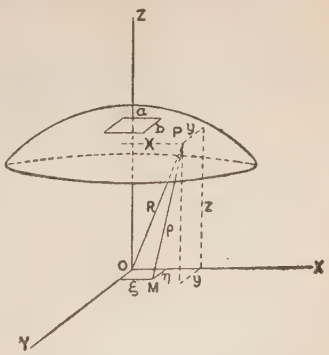


FIG. 155.

The collective effect of all points P at M, taking into account their mutual interference, is

$$a = \int\int \sin 2\pi\left(\frac{t}{T}-\frac{\rho}{\lambda}\right)dxdy$$

(in which a is the displacement).

Let $OP = R$, then

$$R^2 = x^2+y^2+z^2,$$

$$\rho^2 = (x-\xi)^2+(y-\eta)^2+z^2 = R^2-2x\xi-2y\eta+\xi^2+\eta^2,$$

or, if the surface of the wave utilized be confined to the immediate neighborhood of the z axis, approximately

$$\rho = \sqrt{R^2+\xi^2+\eta^2}-\frac{x\xi+y\eta}{\sqrt{R^2+\xi^2+\eta^2}},$$

the variables x and y occurring in the last term.

This gives us for the resultant displacement

$$a = \int\int \sin 2\pi \left(\frac{t}{T} - \frac{\sqrt{R^2+\xi^2+\eta^2}}{\lambda} + \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} \right) dxdy,$$

and since $\sin(a+b) = \sin a \cos b + \cos a \sin b$,

$$a = \sin 2\pi \left(\frac{t}{T} - \frac{\sqrt{R^2+\xi^2+\eta^2}}{\lambda} \right) \int\int \cos 2\pi \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} dxdy$$

$$+ \cos 2\pi \left(\frac{t}{T} - \frac{\sqrt{R^2+\xi^2+\eta^2}}{\lambda} \right) \int\int \sin 2\pi \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} dxdy.$$

Let $$\phi = 2\pi \left(\frac{t}{T} - \frac{\sqrt{R^2+\xi^2+\eta^2}}{\lambda} \right).$$

Let $$A = \int\int \cos 2\pi \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} dxdy.$$

Let $$B = \int\int \sin 2\pi \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} dxdy,$$

then $$a = A \sin\phi + B \cos\phi.$$

Let $\theta = \arctan \frac{B}{A}$, *i.e.* $\frac{B}{A} = \tan\theta$, and let $C = \sqrt{A^2+B^2}$, then if we multiply $A \sin\phi + B\cos\phi$ by $\frac{\sqrt{A^2+B^2}}{C}$ and substitute for the $\frac{A}{C}$ in each term $\cos\theta$ we get at once as the equivalent

$$\sqrt{A^2+B^2}\,(\sin\phi\cos\theta + \cos\phi\sin\theta),$$

or

$$a = C\sin(\phi+\theta),$$

in which C is the amplitude, of which the square represents the intensity of the illumination.

$$I = C^2 = A^2 + B^2.$$

Substituting,

$$I = \left(\int\int \cos 2\pi \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} dxdy \right)^2 + \left(\int\int \sin 2\pi \frac{x\xi+y\eta}{\lambda\sqrt{R^2+\xi^2+\eta^2}} dxdy \right)^2,$$

or, since ξ and η are small in comparison with R,

$$I = \left(\int\int \cos 2\pi \frac{x\xi+y\eta}{R\lambda} dxdy \right)^2 + \left(\int\int \sin 2\pi \frac{x\xi+y\eta}{R\lambda} dxdy \right)^2.$$

Diffraction by a Rectangular Aperture. — Suppose we have a small rectangular aperture of length a and width b, so placed that its sides are parallel to x and y axes, and the z axis passes normally

through its centre. To determine the intensity of the illumination we integrate the above expression between the limits $+\frac{a}{2}$ and $-\frac{a}{2}$, $+\frac{b}{2}$ and $-\frac{b}{2}$.

$$I=\left(\int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_{-\frac{b}{2}}^{+\frac{b}{2}}\cos 2\pi\frac{x\xi+y\eta}{R\lambda}\,dxdy\right)^2+\left(\int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_{-\frac{b}{2}}^{+\frac{b}{2}}\sin 2\pi\frac{x\xi+y\eta}{R\lambda}\,dxdy\right)^2,$$

and for brevity writing $\int^a$ and $\int^b$ in place of the above,

$$\left(\int^a\cos 2\pi\frac{x\xi}{R\lambda}dx\int^b\cos 2\pi\frac{y\eta}{R\lambda}dy-\int^a\sin 2\pi\frac{x\xi}{R\lambda}dx\int^b\sin 2\pi\frac{y\eta}{R\lambda}dy\right)^2$$
$$+\left(\int^a\sin 2\pi\frac{x\xi}{R\lambda}dx\int^b\cos 2\pi\frac{y\eta}{R\lambda}dy\right.$$
$$\left.+\int^a\cos 2\pi\frac{x\xi}{R\lambda}dx\int^b\sin 2\pi\frac{y\eta}{R\lambda}dy\right)^2.$$

If we integrate these terms we find that

$$\int_{-\frac{a}{2}}^{+\frac{a}{2}}\sin 2\pi\frac{x\xi}{R\lambda}dx=0,\quad\int_{-\frac{b}{2}}^{+\frac{b}{2}}\sin 2\pi\frac{y\eta}{R\lambda}dy=0,$$

$$\int_{-\frac{a}{2}}^{+\frac{a}{2}}\cos 2\pi\frac{x\xi}{R\lambda}dx=\frac{R\lambda}{\pi\xi}\sin\pi\frac{a\xi}{R\lambda},$$

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}}\cos 2\pi\frac{y\eta}{R\lambda}dy=\frac{R\lambda}{\pi\eta}\sin\pi\frac{b\eta}{R\lambda},$$

or
$$I=\frac{R^4\lambda^4}{\pi^4\xi^2\eta^2}\sin^2\pi\frac{a\xi}{R\lambda}\sin^2\pi\frac{b\eta}{R\lambda},$$

and finally
$$I=a^2b^2\frac{\sin^2\pi\frac{a\xi}{R\lambda}}{\frac{\pi^2a^2\xi^2}{R^2\lambda^2}}\cdot\frac{\sin^2\pi\frac{b\eta}{R\lambda}}{\frac{\pi^2b^2\eta^2}{R^2\lambda^2}}.$$

We thus see that the intensity at the point x, y is dependent on two variables of the form $\frac{\sin^2 u}{u^2}$.

To find the maxima and minima we differentiate this expression with respect to u and equate to zero.

If
$$S=\frac{\sin^2 u}{u^2},\quad\frac{dS}{du}=\frac{\sin u}{u}\cdot\frac{u\cos u-\sin u}{u^2}=0,$$

which falls into two equations

$$\frac{\sin u}{u}=0 \text{ and } \frac{u\cos u-\sin u}{u^2}=0.$$

The first of these two equations gives the position of the minima, the second that of the maxima.

The second equation takes the form $u\cos u=\sin u$, $u=\tan u$.

This last equation can be solved graphically by plotting the curves $y=x$ and $y=\tan x$ (Fig. 156). The latter equation is represented by a family of curves, having for asymptotes $x=\pm\frac{\pi}{2}$, $x=\pm\frac{3\pi}{2}$ and $x=\pm\frac{5\pi}{2}$, etc. The points at which the former curve cuts the various members of the family satisfy the equation $x=\tan x$ and consequently $u=\tan u$. The first or central maximum occurs at $u=0$, the second between π and $\frac{3}{2}\pi$, the third between 2π and $\frac{5}{2}\pi$, etc. The successive values of u were calculated by Schwerd and found to be

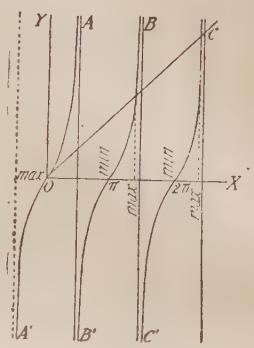


Fig. 156.

0	
$u_1=1.430\,\pi$,	$u_4=4.477\,\pi$,
$u_2=2.459\,\pi$,	$u_5=5.482\,\pi$,
$u_3=3.471\,\pi$,	$u_6=6.484\,\pi$,
	$u_7=7.486\,\pi$.

It is apparent from the diagram that the roots approach the limit $(2n+1)\frac{\pi}{2}$, *i.e.* the points of intersection come nearer to the asymptotes as we recede from the origin, the 7th maximum from the centre being at $7.486\,\pi$ and the asymptote at $7.5\,\pi$. The minima occur where $u=m\pi$, exclusive of the case where $m=0$, which is the value which gives the central maximum. The values of the maxima corresponding to odd multiples of π can be calculated by taking $u=1.5\,\pi$, $2.5\,\pi$, $3.5\,\pi$ ... in the expression $\frac{\sin^2 u}{u^2}$. We find these to be in the ratio $1, \left(\frac{2}{3\pi}\right)^2, \left(\frac{2}{5\pi}\right)^2, \ldots$, *i.e.* if the illumination of the central maximum be taken as unity the illuminations of the succeeding maxima will be $\frac{1}{20}$, $\frac{1}{56}$, $\frac{1}{110}$. We are now in a position to discuss the complete diffraction pattern produced by the rectangular aperture. There will be complete darkness in all places for which one of the factors of the form $\frac{\sin^2 u}{u^2}$ in equation $I=a^2b^2$, etc., has the value 0, or when, m being a whole number, $\frac{\pi a\xi}{R\lambda}=m\pi$ or $\frac{\pi b\eta}{R\lambda}=m\pi$. There will therefore be two systems of dark regions of which the equations are $\xi=\frac{mR\lambda}{a}$ and $\eta=\frac{mR\lambda}{b}$, the former parallel to the y axis, the latter parallel to the x axis. If we bear in mind the fact that we oriented the aperture with its side a parallel to the x axis, and that

ξ is the value in the direction of this same axis, we see that the distances between the parallel interference minima are inversely as the width of the aperture measured in a direction perpendicular to them, and directly as the distance R and the wave-length λ.

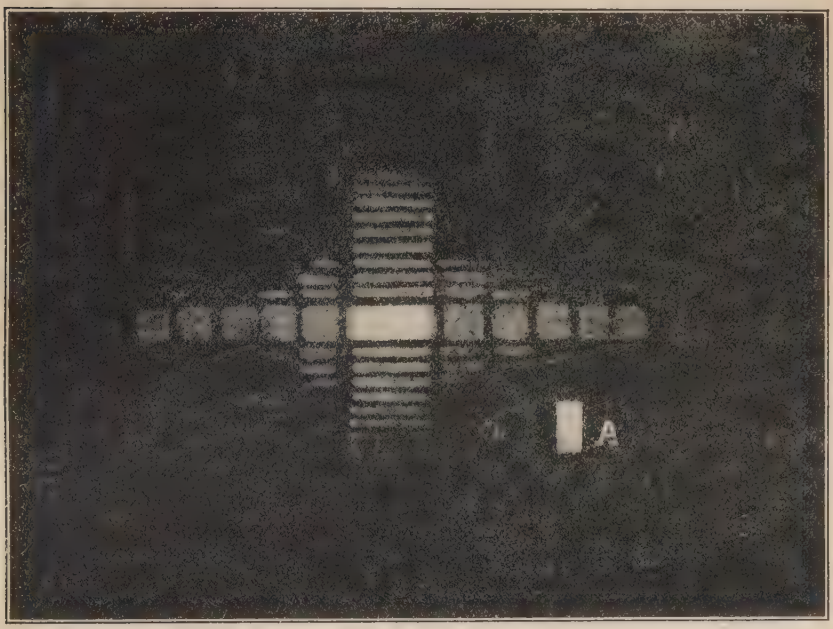


Fig. 157.

The minima form a reticulated meshwork. The central maximum is very bright and in shape similar to the aperture, but rotated through 90° with respect to it. The other maxima are arranged as shown in Fig. 157, in which A represents the aperture. Practically all of the light goes into two bands at right angles to each other, the faint maxima in the angles only being seen when the brighter portions are cut off by a screen. This diffraction pattern can be well seen by covering a lens of about 50 cms. focal length with a screen perforated with an aperture measuring say 2×4 mms., and placing a pinhole illuminated with sun or arc light and an eye-piece at the conjugate foci of the lens, *i.e.* at distances of a metre from the lens. The experiment should be made in a dark room.

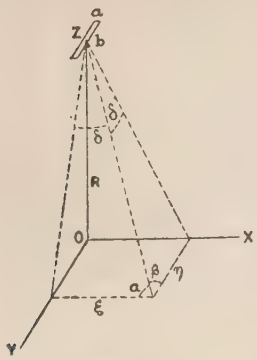


Fig. 158.

A better method is to employ a monochromatic illuminator (see Chapter 1) as a source of light. The eye-piece of a spectroscope should be removed and a small sheet

of metal or cardboard, perforated with a pinhole, placed in the plane in which the spectrum is in focus. If sunlight is focussed upon the slit of the instrument, we have a point source of nearly monochromatic light. It is very important to shut off all extraneous light, and the best method is to arrange the instrument in such a way that the telescope tube projects into an adjoining dark room through a hole in the door or curtain. If we work in an absolutely dark room, the pupil of the eye dilates, and the retina becomes more sensitive, and we can see without difficulty the very faint maxima situated in the angles between the two lines of brighter maxima, if we screen off these with a card. It is instructive to prepare an aperture the dimensions of which can be varied. Fasten two pieces of card to a wooden frame so as to form a slit a millimeter or two in width, and arrange two other pieces in such a way that the length of the slit utilized can be varied from one to ten millimeters. It will be seen that, as the aperture is lengthened, one line of maxima and minima close in and eventually disappear, while the other maxima increase in intensity on account of the larger amount of light transmitted by the aperture. The very faint maxima, which lie in the angles between the lines of brighter ones, are, in a way, analogous to the diagonal spectra seen when two diffraction gratings are crossed, a subject which will be taken up later. If we have an aperture long in proportion to its width, in other words a slit, it produces a single line of maxima and minima. Restricting this aperture by crossing it with another slit spreads out each maximum of the first line into a line of maxima and minima perpendicular to the first line. The fainter ones, referred to above, are thus seen to be somewhat analogous to the "spectra of spectra," which we shall study when we come to the subject of the diffraction grating. The faintness of these diagonal maxima shows us that the destructive interference of the secondary wavelets from the aperture is more complete in the regions which they occupy, though it is not very easy to form a clear picture of why this should be the case, by the elementary methods previously used.

Examples of the lines of maxima and minima produced by small apertures bounded by straight lines are frequently seen in photographs taken at night, in which the electric lights have five or more streaks of light diverging from them. These result from the circumstance that the iris diaphragms of some lenses, when contracted, give a polygonal aperture instead of a circular one. The rays which are seen diverging from the brighter stars, in stellar photographs made with modern reflecting telescopes, result from the diffraction produced by the small rods which support the plate holder, or the auxiliary mirror, and intercept the incident beam of light.

Introduction of Angular Measure. — The investigations which are to follow are much simplified by the introduction of angular measure into the expression for the action of

a concave spherical wave. If R be taken infinitely large the concave wave becomes plane and the x, y plane or screen moves off to infinity.

The elementary diffracted rays which meet at the screen emerge from the aperture parallel to one another. Let α and β represent the angles which these rays make with x and y (Fig. 158). These angles will differ but little from 90° since the angle of diffraction is small.

$$\cos\alpha = \frac{\xi}{\sqrt{R^2+\xi^2+\eta^2}},$$

$$\cos\beta = \frac{\eta}{\sqrt{R^2+\xi^2+\eta^2}}.$$

The expression for the intensity now becomes

$$I = \left(\int\int \cos 2\pi \frac{x\cos\alpha + y\cos\beta}{\lambda} dxdy\right)^2 + \left(\int\int \sin 2\pi \frac{x\cos\alpha + y\cos\beta}{\lambda} dxdy\right)^2.$$

Let δ and $\delta' =$ the small complementary angles which the rays make with the planes xz and yz, and we have

$$I = \left(\int\int \cos 2\pi \frac{x\sin\delta + y\sin\delta'}{\lambda} dxdy\right)^2 + \left(\int\int \sin 2\pi \frac{x\sin\delta + y\sin\delta'}{\lambda} dxdy\right)^2.$$

This expression integrated over a rectangular aperture measuring $a \times b$ gives for the resultant intensity for the point where the rays make angles δ and δ' with the planes xy and xz,

$$I = a^2b^2 \frac{\sin^2\pi\frac{a\sin\delta}{\lambda}}{\pi^2\frac{a^2\sin^2\delta}{\lambda^2}} \cdot \frac{\sin^2\pi\frac{b\sin\delta'}{\lambda}}{\pi^2\frac{b^2\sin^2\delta'}{\lambda^2}},$$

either factor becoming 0 for $\sin\delta = \frac{m\lambda}{a}$ or $\sin\delta' = \frac{m\lambda}{b}$.

Diffraction by a Single Slit. — If we let one dimension of our small rectangular aperture become large, we have the condition of a narrow slit, the diffraction by which we will now investigate as an introduction to the study of the diffraction grating. In this case we can substitute for the luminous point a luminous line, parallel to the slit, without altering the diffraction pattern, a device which enables us to use much more light, though we do not thereby increase the illumination at any given point to any great degree. With a point source of light the diffraction pattern is reduced to a series of maxima and minima, distributed along a line which is perpendicular to the slit. With a linear source of light the maxima are

extended in a direction parallel to the slit, the minima appearing as dark bands.

We will express the intensity by the equation in which the position of the illuminated point is defined by the angle of diffraction δ, and since, if the slit be parallel to the y axis, $(\eta = 0)$, the diffraction will only occur in directions parallel to the x axis, we can put $\delta' = 0$. This makes the second factor equal unity, and we can write the expression for the intensity

$$I = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}} = a^2 \frac{\sin^2 u}{u^2}.$$

It will be seen that b^2 has been omitted. This is because varying b does not change the *distribution* light in the maxima. It, however, affects the intrinsic intensity.

The minima of this function, as we have seen, are given by $u = m\pi$ where m is a whole number (not for $m = 0$, however).

The intensity is zero for all directions δ for which $\sin \delta = \frac{m\lambda}{a}$, or if δ is small the directions for zero illumination

$$\delta = \frac{m\lambda}{a} \quad \text{MINIMA.}$$

The maxima lie in directions given by $\sin \delta = \frac{u_n \lambda}{\pi a}$, in which u_n is one of the roots of the equation $u = \tan u$, or for small values of δ,

$$\delta = \frac{u_n \lambda}{\pi a} \quad \text{MAXIMA.}$$

The maxima are the spectra of the first class which we have already investigated in an elementary way.

If white light is employed, the central maximum is white, the other maxima colored, owing to the fact that their position is a function of the wave-length of the light, the red maxima being farther apart than the blue.

Diffraction by Two Parallel Slits. — This case, which we have already studied by elementary methods, is the next step which is taken in the development of the theory of the diffraction grating.

The width of the slits we will call a and the distance between them d.

The diffracted rays coming in a parallel direction from a slit at angle δ with the normal give a resultant intensity

$$A^2 = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}},$$

in which expression A is the amplitude.

Parallel rays coming from corresponding parts of the two slits have a path-difference of $(a+d)\sin\delta$, and the vibrations when brought together by the lens will have a phase-difference $\frac{2\pi(a+d)\sin\delta}{\lambda}$.

We have seen in the Chapter on Interference that the resultant intensity of two streams of light of amplitude A, with phase-difference e, is $I=2A^2+2A^2\cos\epsilon$.

The resultant intensity in this case is therefore

$$I=2A^2\left(1+\cos 2\pi\frac{(a+d)\sin\delta}{\lambda}\right)=2A^2\,2\cos^2\pi\frac{(a+d)\sin}{\lambda},$$

and substituting A^2 from above,

$$I=4a^2\frac{\sin^2\pi\frac{a\sin\delta}{\lambda}}{\pi^2\frac{a^2\sin^2\delta}{\lambda^2}}\cdot\cos^2\pi\frac{(a+d)\sin\delta}{\lambda}.$$

This expression contains two variable factors, one of form $\frac{\sin^2 u}{u^2}$, which we have already investigated, the other giving equidistant minima equal to zero, given by the equation

$$\sin\delta=\frac{(2n+1)\lambda}{2(a+d)} \quad \text{MINIMA},$$

and maxima given by $\sin\delta=\frac{n\lambda}{a+d}$ MAXIMA,

which expressions simply state that in the first case the rays coming from homologous parts of the two slits meet with a path-difference of an uneven number of half wave-lengths, and in the second case with an even number.

The intensity will be zero when either of the two variable factors is zero, *i.e.* when

$$\sin\delta=m\lambda,$$
$$\sin\delta=\frac{(2n+1)\lambda}{2(a+d)}.$$

The minima given by the first expression are the diffraction minima of a single slit which we have already studied, the second are interference minima resulting from the meeting of homologous rays from the two slits: they are chiefly noticeable in the central maximum of the first class system, and the maxima which lie between them were called by Fraunhofer spectra of the second class. They are the spectra yielded by the diffraction grating. See also the interference refractometer in the Chapter on Interference.

If now we increase the number of slits we shall find that we have in addition spectra of a third class, which, however, practically disappear if the number of slits be very large.

Diffraction by any Number of Parallel Equidistant Slits (Diffraction Grating). — The usual method of treating the diffrac-

tion spectra produced by gratings is so involved, that the student is apt to lose all idea of the physical significance of the expressions. An elaborate formula, involving double integrals, the development of which requires several pages of pure mathematics, and is finally solved by graphical methods, shows that between the principal maxima produced by the grating there are present $(n-2)$ secondary maxima, where n is the number of lines of the grating. As an example we may take the case where the curves $x=n\tan z$; $y=\tan nz$ are plotted, the secondary maxima being given by the points of intersection of the two curves. One is apt to go through this treatment without having the faintest idea as to why secondary maxima are produced at all, though each step is fully understood from a purely mathematical standpoint. The complete expression for a grating of n lines (or slits) of width a, separated by equal distance d, is

$$I=a^2\frac{\sin^2\pi\frac{a\sin\delta}{\lambda}}{\pi^2\frac{a^2\sin^2\delta}{\lambda^2}}\cdot\frac{\sin^2 n\pi\frac{(a+d)\sin\delta}{\lambda}}{\sin^2\pi\frac{(a+d)\sin\delta}{\lambda}}.$$

The following elementary graphical method will be found to account for the secondary maxima, and show as well their number, position, and intensity, for a grating of any number of lines. We shall make use of the well-known method of compounding vibrations, which is employed in the elementary development of Cornu's spiral, and shall show that we have minima equal to zero whenever the amplitude lines form a closed symmetrical figure, or mutually annul each other in pairs. The closed figures are either triangles, squares, regular polygons, or star-shaped figures, which can be plotted in a very simple manner, described later on.

Fraunhofer's treatment shows that a single slit produces maxima and minima, which recede from the centre and broaden as the slit width decreases. These he called spectra of the first class. In the case of the gratings used for optical purposes, the lines are so fine that the central maximum of the first-class spectra occupies the entire field; *i.e.* there are no minima, a single line scattering light of decreasing intensity throughout the entire range between 0° and 90°. In the present treatment we shall consider our lines of this degree of fineness. "Absent spectra," resulting from finite width of the line, and the consequent existence of first class minima, can be separately dealt with.

We will consider parallel rays incident normally upon the grating, the parallel diffracted rays being brought to a focus by a lens.

Each line of the grating acting alone, we will suppose to produce unit amplitude at the focus.

We find the resultant amplitude produced by a number of lines operating together by the well-known device employed in the elementary treatment of Cornu's spiral, the resultant amplitude being the closing side of a polygon, the sides of which (vectors) represent the amplitudes and phases of the vibrations coming from the grat-

ing-lines. We can plot the intensity curve for a three-line grating, by considering phase-differences (P.D.) which increase by 20°. In the normal direction (P.D.0°) the intensity will be 3^2 or 9; in a direction such that we have a P.D. of 90° the intensity will be 1, while with a P.D. of 120° we have a triangle, there is no "closing side" and the intensity is zero. From now on it increases, attaining the value 1 again with a P.D. of 180° when the three vectors are superposed; two of these vectors cancel each other, the illumination being that due to the outstanding one. The various stages are shown in Fig. 159 for different values of P.D. The first order spectrum comes in such a direction that the P.D. is 360° or the path-difference is λ; consequently the point for which P.D. = 180° is midway between the "central image" and the first spectrum, and the diffraction pattern is symmetrical about it. We thus see that there is a secondary maximum at this point.

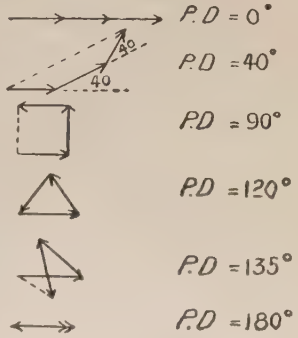


Fig. 159.

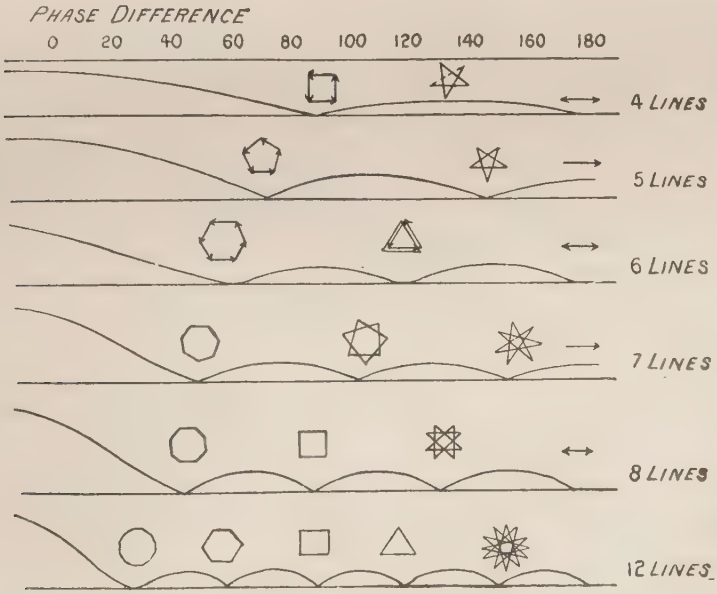


Fig. 159 *a*.

From now on we shall only determine the positions of the minima when more than three lines operate. The complete curve can be calculated in the same manner.

In the case of a four-line grating we have intensity 16 at the centre, zero when the P.D. is 90° or 180°, and unity when the P.D. is 120°. This gives us two secondary maxima between the principal maxima, their intensity being about $\frac{1}{14}$ that of the latter; these maxima occur when the P.D. is 135° and 225°. In Fig. 159 *a*, I have given the positions of the minima and the form of the closed amplitude figure at each, for gratings of 4, 5, 6, 7, 8, and 12 lines. The ordinates of the amplitude curves are not drawn to a scale, of course.

With a five-line grating we get our first minimum when the five amplitude lines form a pentagon, the phase-difference being 72°, and a second when they form a star, the phase-difference in this case being 144.° At the centre (180°) we have intensity one, as in the case of the three-line grating.

It will be noticed that we have a zero value only when the starting and terminal points of our broken line of vectors coincide. With a phase-difference of 180° this will happen when the number of lines in the grating is even. It may also happen when a closed figure such as a triangle, square, or regular polygon is formed.

For a six-line grating we must show the existence of 4 secondary maxima. The illumination will be zero for the 180° phase-difference, also for that of 60° when we have a hexagon, and for 120° when we have two superposed triangles. A star figure cannot be formed of 6 lines compounded as described.

In the case of a seven-line grating we have the zero minima for the regular polygon (P.D. 51°.6) and for two star-shaped figures (P.D. 102° and 154°), giving us five secondary maxima between the two principal maxima. The eight-line grating gives zero when the amplitude lines form an octagon, two superposed squares, an eight-pointed star (P.D. = 135°) and also at the centre of symmetry (P.D. = 180°).

In the case of the twelve-line grating the lines form in succession a twelve-sided polygon, two superposed hexagons, three squares, four triangles, and a twelve-pointed star, the phase-difference in the latter case being 150°.

The following method of ascertaining the number of possible figures will be found useful.

Arrange around a circle as many equidistant dots as there are lines in the grating, and join the dots by straight lines, first skipping one dot, then two, three, four, etc. With twelve dots we get the twelve-pointed star when we skip four dots each time. For a grating of thirteen lines we find it possible to form thirteen-pointed stars in five different ways, between 0° and 180° phase-difference. Each of these gives zero illumination, also the polygon of thirteen sides; consequently we have six minima between 0° and 180°, or eleven secondary maxima. In the case of twelve dots we get but one star, skipping 1, 2, and 3 dots giving the hexagons, squares, and triangles.

We thus see that, in the case of a grating of n lines, we have $(n-2)$ secondary maxima between the principal maxima, the intensity of which can be easily calculated from diagrams similar to those given.

The intensity curves for 1, 2, 3, 5, and 6 slits are shown in Fig. 160, from which we see that by increasing the number of lines of the grating we throw more light into the principal maxima, the secondary maxima decreasing in intensity and becoming more crowded together. We also cause the curves of the principal maxima to become steeper, *i.e.* the light is concentrated more and more nearly into a line. When we have a very large number of lines the principal maxima are exceedingly bright and narrow when monochromatic light is employed, and the secondary maxima disappear entirely. The principal maxima then constitute the narrow spectrum lines seen with the grating.

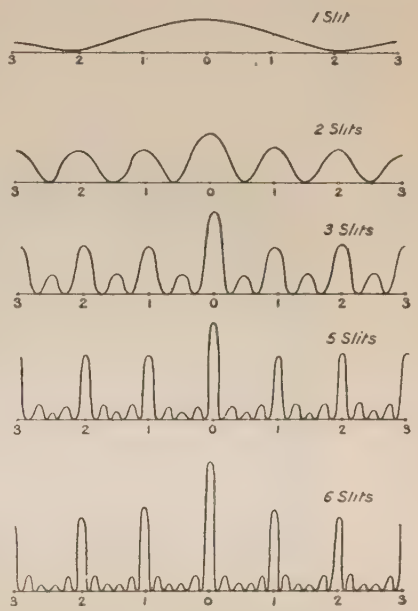


Fig. 160.

The formation of spectra by gratings can be well illustrated by mercury ripples, as was shown by Vincent. If we fill a large shallow basin with mercury, and cause a strip of cardboard attached to a vibrating tuning-fork to touch its surface, plane-waves will be emitted, parallel to the edge of the strip. If, however, we substitute a coarse comb for the continuous strip, and allow the teeth to touch the surface, we shall have a number of systems of plane-waves, corresponding to the central image and lateral spectra. The experiment has been repeated by Reese (*Astro-phys. Journal*, xxiv., p. 48) and illustrated by a photograph similar to the one made by Vincent. It is reproduced in Fig. 161. The author has found that a ring of glycerine run around the edge of the mercury surface completely prevents disturbing reflections from the sides, by damping the waves.

We will now consider the case of the optical grating with many thousand lines, and see what part the secondary maxima, which accompany the spectrum lines, play.

Our broken line now becomes essentially a smooth curve. We have our first minimum when it forms a complete circle, the phase-difference between disturbances from the first and last lines being 360°, or the path-difference λ. The first secondary maximum occurs when the line has wound up into a circle and a half. The ratio of the intensity of the secondary to that of the principal maximum is obviously the ratio of the square of the diameter of the circle of $1\frac{1}{2}$ turns to the square of the total length of the line. This

we easily find by winding up a strip of paper of known length. It will be found to be about 1 : 23.

This shows us that, no matter how many lines we have in the grating, our spectrum lines will always be accompanied by close companions, having at least $\frac{1}{23}$ of their brightness. For an eight-line grating the ratio is not very different, being about $\frac{1}{21}$. It

Fig. 161.

occurs for a phase-difference approximately such that we have a regular pentagon, three sides of which are made of double lines. For a four-line grating it is about $\frac{1}{16}$ (triangle with one side double),[1] and for a three-line grating $\frac{1}{9}$. This last is the maximum value of the ratio.

It seems quite surprising that even for optical gratings the secondary maxima have a brilliancy very nearly one-half of that which obtains in the case of a three-line grating. Their angular distance from the spectrum lines is such as to make the path-difference between disturbances coming from the first and last lines of the grating $\frac{3\lambda}{2}$ more than the path-difference at the line. This angle is obviously that subtended by one and one-half waves, at a distance equal to the width of the ruled surface. The distance be-

[1] This, however, is not the centre or brightest point of the maximum, which is for a P.D. of 135°, the figure having the form shown in Fig. 159 *a*. At this point the intensity is about $\frac{1}{4}$ that of the principal maxima.

tween the secondary maxima is thus seen to depend upon the width of the grating, and not upon the number of lines. At first sight there may appear to be some difficulty about this, since there are $n - 2$ secondary maxima; and we might very naturally expect an increase of n to push them nearer together. This is, however, only true when the " grating space " remains constant, *i.e.* when we add new lines of the same spacing.

Suppose we have a grating of given width with 20 lines which gives 18 secondary maxima. If we interpolate lines between the lines already present, we double the number of secondary maxima, to be sure; but the principal maxima (spectra) of odd order disappear by interference; in other words, they are now twice as far apart as before, the spacing of the secondary maxima remaining the same.

The analogy between the secondary maxima and the fringes produced by a rectangular aperture of the same size as the ruled surface, can be studied to advantage by means of coarse gratings made by ruling four or five lines on a piece of smoked *plate* glass, and making the lower third of the grating clear by wiping out the lines. Sun or arc light filtered through red glass should be used with a small spectrometer, the grating and aperture being covered in succession or used simultaneously.

It is easy to see that, in a spectrum of order m, formed by a grating of n lines, the path-difference between disturbances coming from the first and last line is $mn\lambda$. The first secondary minimum to the right or left of the central maximum will lie in such a direction as to make the path-difference between the extreme rays change by one whole wave-length; that is, the path-difference in this case is $(mn \pm 1)\lambda$. For, as we have seen, this first secondary minimum occurs when our line of vectors turns around into a closed circle, the first and last vectors pointing again in the same direction. This means that the last vector has turned through 360°, which corresponds to a path increment of λ.

We shall use these expressions when we come to the resolving power of gratings; they are of fundamental importance.

Number of Spectra and Intensity Distribution : Absent Spectra. — In the case of gratings formed of opaque lines and transparent intervals the spectra of progressively increasing order lie in such directions that the path-difference between disturbances coming from the corresponding edges of two adjacent apertures is 1, 2, 3, 4, etc., wave-lengths. Referring back to Fig. 148, suppose the direction of the dotted lines to be the direction of the second order spectrum. The path-difference between disturbances coming from A and C is then 2λ. If the widths of the opaque bars are equal to the widths of the apertures, as in the figure, the path-difference in this direction, between the waves coming from A and B will be λ, and the total effect of the disturbances from the aperture will be zero; as we have seen in the elementary treatment of a single slit. If each aperture produces zero illumination, the total effect of all must be zero, or the second order spectrum will be absent. The spectra will in general be absent in the directions corresponding to

the minima of the first class due to a single slit. It is easy to see that if $a=b$, as in the above case, the spectra of even order 2, 4, 6, etc., will disappear. If $a=\frac{b}{2}$, the spectra of order 3, 6, 9, etc., will be absent; if $a=\frac{b}{3}$, spectra 4, 8, 12, etc., will disappear.

The intensity of grating spectra has been treated by Lord Rayleigh ("Theory of Diffraction Gratings," *Phil. Mag.*, xlvii, pp. 81–93, 193–205, 1874; also "Wave-Theory," *Ency. Brit.*). In the case of gratings consisting of opaque and transparent parts of widths d and a, it can be shown that the intensity in any spectrum of order m is given by

$$B_m : B = \frac{1}{m^2\pi^2}\sin^2\frac{am\pi}{a+d},$$

in which B is the intensity of the image produced by the lens in the absence of the grating, and B_m the intensity of the spectrum of order m. Since the sine can never exceed unity, the utmost intensity attainable under the most favorable circumstances is only $\frac{1}{m^2\pi^2}$ of the original light. In the first order, this may amount to $\frac{1}{\pi^2}$ or $\frac{1}{10}$, when the opaque and transparent parts are of equal width. If $d=a$, the formula reduces to

$$B_m : B = \frac{\sin^2\frac{1}{2}m\pi}{m^2\pi^2},$$

and the spectra of even order disappear. This can be shown by means of a grating formed by winding fine copper wire on a brass frame. Two strands should be wound side by side, and pressed close together. One of the ends is then soldered to the frame, and the other wire unwound. The remaining wire is now soldered to the frame, and the wires on one side cut away. It is best to make the frame in such a way that it can be expanded by means of screws, after the winding is completed; in this way the wires are drawn taut. On viewing a slit backed by a sodium flame through such a grating, the 2d, 4th, 6th, etc., spectra will be found wanting. If the grating is turned a little so as to alter the ratio of a to d, these missing spectra at once appear. If the transparent intervals are small in comparison to the total interval $(a + d)$, we have $B^m : B = \{a/(a + d)\}^2$, except for spectra of very high order. In this case the spectra are all of equal intensity. Gratings made by photographing black and white drawings of parallel lines with various ratios of a to d are useful for purposes of illustration. If the width of the lines is small, the points where the spectra are absent are too far removed from the central maximum to appear in the field.

Verification of Results. — The results which we have deduced for the diffraction grating can be easily verified by experiment. A piece of thin plate glass is smoked over a flame, and one edge

moistened with alcohol. The alcohol spreads over the film, and on drying leaves it compact enough to enable us to rule lines through it with a sharp steel point by means of the dividing engine.

If no dividing engine is available it is not difficult to fit up an arrangement by which the plate may be advanced through equal distances by turning a screw provided with a roughly graduated head, while the ruling is done with a needle point mounted on a pivoted arm, arranged to slide back and forth along a fixed line. The distance between the lines should be as nearly as possible equal to the width of the lines. A dozen or so lines will be found sufficient. The plate thus ruled should be provided with a cover, also of thin plate glass (ordinary window glass will not do), to prevent injury to the film, and a movable slide of thin black paper so arranged that the lines can be covered or exposed in succession. If we mount the plate on the table of a spectrometer and illuminate the slit with sunlight which has been passed through a sheet of dense ruby glass, we can verify in succession all of the results which we have deduced.

If all of the slits but one are covered we shall see the broad spectra of the first class, the central one being by far the brightest; on uncovering another slit, this central maximum, as well as the lateral maxima, appear furrowed by narrow dark bands, the bright bands between them being the spectra of the second class. If three lines be uncovered the faint third class maxima appear, one between each pair of second class maxima. As we proceed with the uncovering process we shall see the secondary maxima crowd in between the principal maxima, until, when all the lines are exposed, they are too faint and too close together to be detected.

If a piece of dense cobalt glass, which transmits the extreme red and blue only, be substituted for the ruby glass, a series of red maxima and blue maxima will be seen simultaneously, the latter being closer together than the former.

There is another cause which may operate in causing the disappearance of spectra of certain orders, which we will now investigate.

The Laminary Grating. — If we consider the opaque strips of the grating which we have just studied replaced by transparent strips of such a thickness that some one wave-length in the spectrum suffers a retardation of $\frac{\lambda}{2}$, we have a type of grating which was first studied by Quincke. The mathematical treatment of a grating of this type is somewhat long and involved, and, as the chief peculiarities of the grating can be seen by elementary considerations, had best be omitted. These gratings have the peculiar property of failing to show the central image when light of the specified wave-length is used. If we employ white light, the central image, which is white in the case of the ordinary grating, appears colored owing to the absence of the wave-lengths in the immediate vicinity of the one for which the retardation by the thin lamina is $\frac{\lambda}{2}$.

The reason of this is very easy to see. Let BC, DE, and FG be

the retarding lamina (Fig. 162). The central image is formed by the normally diffracted rays, *i.e.* the rays for which $\delta = 0$. In the case of the ordinary grating these disturbances all arrive at the focus of the lens in the same phase, and produce a maximum. In the case of the laminary grating there is a second set of disturbances from the laminae which reach the focus half a wave-length behind the others, or unretarded ones, which they destroy. Thus the disturbance EE' coming through the clear space is destroyed by FF' coming through the lamina, and so on. For waves for which the retardation is not exactly $\frac{\lambda}{2}$ the destructive interference will not be so complete, and these will appear in the central image, though with reduced intensity. The difference in the retardation is of course due to the dispersion of the lamina, and if the latter be somewhat thick, we may have several values of λ distributed along the spectrum, for which the retardation is $(n+1)\frac{\lambda}{2}$. These wave-lengths will consequently be absent, and the intermediate ones, for which the retardation is an even number of half wave-lengths, will be present. The illumination will be greater for these wave-lengths than in the case of the ordinary grating, since twice as many elements are operative in producing it.

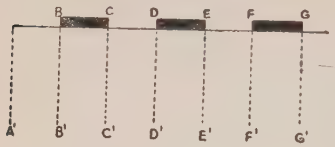


FIG. 162.

We know, however, that luminous vibrations cannot be actually destroyed by interference, consequently the light which fails to make its appearance in the central image must be looked for somewhere else. It is not difficult to see that this light will be found in the lateral spectra. Suppose AA' and CC' to be the rays diffracted in such a direction that their path-difference at the focus is λ, and consider this value of λ the one absent in the central image (Fig. 163). The ray BB', were it not for the retardation, would arrive at the focus with a path-difference of $\frac{\lambda}{2}$ with respect to AA', consequently it would destroy it. If the laminae were removed, the rays traversing the spaces which they occupied would destroy the rays passing through the adjoining spaces, and there would be no illumination in this direction; in other words, the diffraction effects would disappear, the grating having been reduced to a piece of clear glass. But the ray BB', in addition to its $\frac{\lambda}{2}$ path-difference due to the angle δ, has an additional path-difference of $\frac{\lambda}{2}$ due to the retardation in the lamina, consequently it

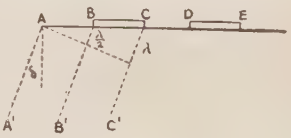


FIG. 163.

will arrive at the focus in condition to reënforce the disturbance along AA'. The spectrum of the first order will consequently be much brighter for this wave-length than if the laminae were opaque; in other words, the light which is absent in the central image appears here.

For some other value of λ it may happen that the disturbances coming from the laminae will be retarded by such an amount that they will destroy the disturbances coming from the clear spaces, in the spectrum of the first order, consequently light of certain colors may be absent in the spectra as well as in the central image.

Gratings of this description may be made by depositing silver on glass by the chemical method (a wedge-shaped film is preferable), and then ruling lines through the film perpendicular to the thin edge of the wedge. By covering the plate with iodine crystals, the silver is transformed into a transparent iodide, and we have a laminary grating, the retardation increasing progressively as we recede from the thin edge. With a grating of this description we can verify the above results. A candle flame viewed through it appears brilliantly colored, the color varying according to the portion of the grating held before the eye, and certain colors will be found to be absent in certain spectra, and present in excess in others.

Similar appearances are sometimes found with photographic copies of gratings ruled on glass made with bichromatized gelatine on glass. The writer has prepared gratings in this way, the laminae of which gave a half-wave retardation for sodium light.

A sodium flame could not be seen *directly* through it at all, though the lateral spectra were very brilliant.

The same thing may happen with a reflection grating if the depth of the groove is $\frac{\lambda}{4}$, and, in fact, most of the gratings ruled on speculum metal show more or less color in the central image. This color is sometimes erroneously referred to oxidation, and is regarded as a bad feature. Quite the reverse is the case, these gratings giving brilliant spectra for obvious reasons.

The Plane Grating. — Diffraction gratings for spectroscopic apparatus are usually ruled on a reflecting surface of speculum metal. If the surface is plane the case is analogous to that of the transparent grating. The formula for the grating we have already deduced in considering the diffraction fringes of the second class, due to two parallel slits. It is $n\lambda = (a + b)(\sin i \pm \sin \theta)$ in which θ is the angle of diffraction, i the angle of incidence, n the order of the spectrum, and $(a+b)$ the grating constant. For normal incidence we have $\sin\theta = \frac{n}{a+b}$. The condition of things is represented in Fig. 164, from which it is apparent that the disturbances which originate at

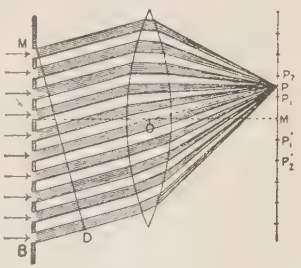


FIG. 164.

the grating elements, when a plane-wave is incident normally upon it, pass through the point P in succession. It is thus apparent that even if but a single wave or pulse struck the grating, we should have a periodic disturbance at P. The grating is thus able to manufacture, as it were, light of a definite wave-length or color, a circumstance which will be considered more fully in the chapter on White Light. The optical paths from what we may call the diffracted wave-front MD to the point P are equal, and since the path-difference between two adjacent diffracted streams for the first order spectrum is λ, the path-difference between the extreme rays is $m\lambda$, m being the number of lines in the grating. As we have seen, if we employ monochromatic light from a slit, made parallel by a lens, we shall have at P a sharp maximum accompanied by very faint maxima, which lose in intensity and crowd closer to the principal maximum as we increase the number of lines in the grating. The path-difference between the extreme rays for the position of the two minima immediately adjacent to the principal maximum is $(m \pm 1)\lambda$, an expression which we shall make use of in considering the resolving power of the grating. If n is the order of the spectrum, the path-difference for the principal maxima and adjacent minima are $mn\lambda$ and $(mn \pm 1)\ \lambda$. The formula for the grating shows us that the position of the diffracted image depends upon the value of λ.

The dispersive power of the grating is represented by

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta},$$

which shows us that the dispersion increases with the order of the spectrum, and that it is inversely proportional to $(a+b)$, the grating constant. For small values of θ, $\cos\theta = 1$ approximately, and the spectrum is normal, *i.e.* equal increments of wave-length correspond to equal increments of θ. In the higher orders of spectra, however, the dispersion increases with θ and therefore with λ, and the spectra are more drawn out at the red than at the violet end, exactly the reverse of what we have in the case of prismatic spectra. On this account the grating should be mounted on the table of the spectrometer so as to stand normal to the observing telescope, for in this position $\theta = 0$, for the centre of the spectrum and the other values of θ are small.

The general formula for a reflecting grating, for any angle of incidence i, is

$$(a+b)(\sin i \pm \sin\theta) = m\lambda.$$

Overlapping Spectra. — The formula for the grating shows us that the spectra of the different orders overlap, for by doubling n and halving λ we have the same value of θ. This overlapping gives no trouble in the visible region, if we limit ourselves to the first or even the second order, but in photographic work it must be taken into account, for the ultra-violet of the second order is superposed on the visible region of the first order, wave-length 25 of the second coinciding with 50 of the first. This overlapping is often of use, as we shall see when we come to the consideration of the concave

grating; if it is desired to eliminate it, color screens or prismatic analysis must be resorted to: glass cuts off practically everything below $\lambda = 32$, consequently a glass lens or plate is all that is necessary when working in the first order spectrum.

Resolving Power of Gratings. — Since the diffraction grating is used largely in place of a prism, for the formation of spectra and the examination of spectrum lines, it is of importance to determine upon what the resolving power of the grating depends. Lord Rayleigh [1] has shown that in order to separate two spectrum lines, the distances between the central maxima of their diffraction images must be at least as great as the distance of the first minima from the central maximum. If this condition is as represented in Fig. 165, the resultant illumination (dotted) being given by summing the ordinates of the two curves, the lines will not appear clearly separated, but the duplicity of the line can be recognized from the slight shading down the centre. Let AB (Fig. 165) be the grating and BC the direction of the diffracted rays, which form, for wave-length λ, the central maximum of the spectrum of the mth order. As we have seen, the central maxima in each spectrum are accompanied by secondary maxima which decrease in intensity, and become crowded together as we increase the number of lines of the grating. The first minimum at π on each side of a central maximum will obviously lie closer to the central maximum of the spectrum line, if we employ a large number of lines in the grating, or more exactly if the grating has a considerable width. In other words, the intensity curve becomes steeper, the light crowding together more nearly into a geometrical line. As an expression for the resolving power we require the change in wave-length necessary to shift the central maximum into the position of the first minimum. For the central maximum for wave-length λ, in a spectrum of order m for a grating of n lines the retardation between the disturbances coming from the edges of the grating is $mn\lambda$ (Fig. 166). The first minimum to the left for this wave-length is represented by the line BD, which will be the direction of the central maximum for some other wave-length slightly greater than λ, say $\lambda + \delta\lambda$. The retardation of the extreme rays of the first minimum is $(mn+1)\lambda$ or $mn(\lambda + \delta\lambda)$, if $\delta\lambda$ is the increment of wave-length which will shift the central maximum

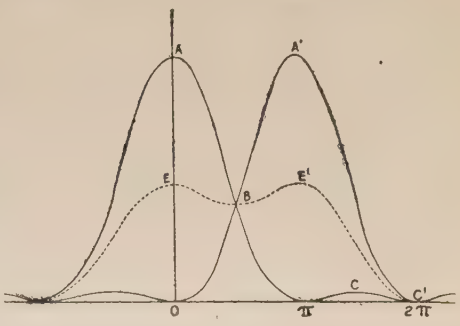


Fig. 165.

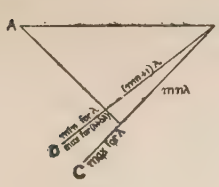


Fig. 166.

[1] "Wave-Theory," *Ency. Brit.*, vol. xxiv.

into the position previously occupied by the first minimum. (See treatment of diffraction grating.) Since these two are equal, we can write

$$mn(\lambda+\delta\lambda)=(mn+1)\lambda, \text{ or } \frac{\delta\lambda}{\lambda}=\frac{1}{mn}.$$

For the D lines of sodium $\frac{\delta\lambda}{\lambda}=\frac{1}{1000}$, so that in order to resolve them in the first order spectrum, we must utilize at least 1000 lines of the grating. They may be separated in the second order spectrum with 500 lines, etc. This treatment is due to Lord Rayleigh.

It is especially to be noticed that the resolving power of the grating does not depend upon the closeness of the ruling, but merely upon the number of lines. Let us take for example a grating one inch in width, ruled with 1000 lines, which in the first order spectrum will barely resolve the sodium lines. Suppose now we interpolate an additional 1000 lines, making them bisect the original spacing. The spectra of odd order will disappear by interference, the energy being thrown into the spectra of even orders, which increase in brilliancy, the gain being fourfold since the amplitudes are doubled.

The resolving power in each spectrum is exactly the same as it was before, since what is now the first order spectrum was previously the second order. If one-half the grating is cut away, leaving 1000 lines in half an inch, the dispersion will not be altered while the brightness and the resolving power are halved. The sodium lines are now just barely resolvable in the first order spectrum. If the grating had been cut in halves before the interpolation of the second ruling, the sodium lines would have been just barely separated in the second order spectrum, 500 lines only operating. This spectrum is identical in position with the first order spectrum in the second case. It is thus clear that with a grating of given size, the resolving power *in a given direction* is quite independent of the number of lines in the grating. With few lines we have a high order spectrum at a given point, and as we increase the number of lines by ruling more in the *same space*, the spectra move out from the central image, and the order of spectrum at the given point becomes less.

If, however, we decrease the order at the given point by compressing the grating, that is, decreasing the grating space without increasing the number of lines, then the resolving power at the given point becomes less. The advantage of ruling the lines close together is twofold. In the first place, for a given aperture of telescope and collimator, we can bring more lines to bear, and consequently increase the resolving power; in the second place, we concentrate the light into fewer spectra, and obtain increased brilliancy. As Lord Rayleigh remarks, "There is clearly no theoretical limit to the resolving power of gratings even in spectra of given order, but it is possible that, as suggested by Rowland, the structure of natural spectra may be too coarse to give opportunity for resolving power much higher than those now in use. However this may be, it would be possible with the aid of grating of given

resolving power to construct artificially of white light mixtures of slightly different wave-length whose resolution or otherwise would discriminate between powers inferior and superior to the given one." We can easily investigate the relation between resolving power and number of lines. Illuminate the slit of the spectrometer with the light of a sodium flame, viewing the spectrum by means of a small grating, the aperture of which can be contracted in the horizontal direction by means of vertical opaque screens. If we are working in the spectrum of the first order, with a Rowland grating of 14,000 lines to the inch, it will be found that the *D* lines run together when the width of the aperture is about one-fourteenth of an inch. Turning the telescope so as to view the lines in the second order, we shall find that they are clearly resolvable. If a grating with coarser ruling is used, the minimum width of the aperture consistent with resolution will be increased in a corresponding degree.

Focal Properties of Gratings: Zone-Plates. — If the distance between the lines of a plane grating, instead of being constant, varies according to the same law which governs the spacing of the Fresnel-Huygens zones, the grating will give focussed spectra without the aid of mirrors or lenses.

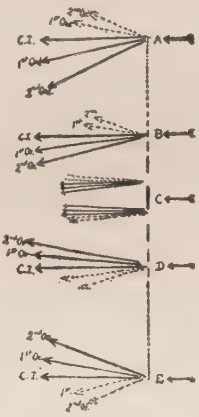


Fig. 166*a*.

It is obvious that the diffraction angle for a spectrum of given order must be, for each element of the grating, such as to cause the diffracted rays to meet at a point, instead of remaining parallel to one another.

We may regard the zone-plate, described in the chapter on the Rectilinear Propagation of Light, as a circular diffraction grating of variable spacing.

In Fig. 166*a* let *ABCDE* represent a section of the zone-plate through which parallel rays of light are passing. At the points *A* and *E*, where the elements are close together, we have the diffracted rays corresponding to spectra of different orders leaving at the angles designated by arrows. It is clear that the rays for the first order spectra will meet at a point at some distance from *AE*, and the second order rays at a point nearer to the plate.

At *B* and *D*, the distance between the diffracting lines being greater, the diffraction angles are less and the rays from these elements meet at the same points as those from *A* and *E*. We thus see that the different foci of the zone-plate merely represent spectra of different orders. The diffracted rays indicated by dotted arrows meet only if they are produced backwards behind the zone-plate. These foci are of course virtual, and the plate therefore acts as a concave, as well as a convex, lens. Now we know that

the relative intensities of spectra of different orders depends upon the ratio of the width of the opaque to that of the transparent portions of the grating. If $a = b$, the spectra of even order disappear.

In this connection the zone-plate represented in Plate II is especially interesting, as the ratio $a : b$ is not constant, and if the eye is placed at the second-order focus circular zonal regions, in which $a = b$, will appear less brilliantly illuminated. They will never appear black, however, as some of the first-order rays from them are entering the eye as well.

In the condition that $a = b$, any actual zone on the plate, as seen from the second-order focus point, contains *two* of the Fresnel-Huygens zones, consequently each real zone of the plate produces zero illumination at this point.

Intensity of Grating Spectra. — The intensity of grating spectra can be calculated in the case of gratings made up of opaque and transparent intervals. As gratings of this type are seldom or never used, such calculations are of little practical value. In the case of gratings ruled on speculum metal the distribution of light in the spectra of different orders is very irregular, depending upon the form of the groove. The following method of measuring the intensity, used by the author in determining what percentage of the total incident light appeared in the very bright, first order spectrum of a particular grating, may be of interest, as the measurements are very easily made.

The measuring apparatus, or photometer, consisted of a pair of Nicol prisms (one mounted in a graduated circle), a small piece of silvered glass, and a bright and uniform sodium flame. The silvered glass can be made by dissolving the varnish from the back of a piece of modern mirror, and polishing with rouge. It is mounted vertically at an angle of 45° with the axes of the Nicols, and covers the lower half of the field (Fig. 167). The soda flame is immediately behind the polarizing prism, and the grating stands to one side, as shown in the figure. By turning the grating, the central, or any one of the lateral (spectral) images of the flame can be viewed in the silvered mirror, immediately in contact with the image of the flame seen through the Nicols, and by turning one of them the intensities can be accurately adjusted. We first set the graduated Nicol in the zero position, and then turn the other Nicol to the position of extinction. The intensity of the restored light for a given angle measured from this position, is proportional to the square of the sine of the angle. The central image can be located easily by watching for the reflection of the flame in the unruled portion of the sur-

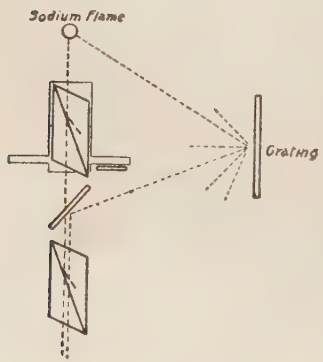


Fig. 167.

face. The results obtained are recorded in the following table, eight spectral images having been measured:

Fourth Spectrum	Third Spectrum	Second Spectrum	First Spectrum	Central Image	First Spectrum	Second Spectrum	Third Spectrum	Fourth Spectrum
0.073	0.057	0.20	0.31	0.16	0.98	0.096	0.032	0.01

The numbers given are the squares of the sines of the angles, and represent the intensities of the images as fractional parts of the light transmitted through the first Nicol. The intensity of the first spectrum on the right is as great as the sum of all the others, together with the central image (0.94), which amounts to saying that half of the total light reflected is found in one spectrum.

It is frequently stated that a Nicol reduces the intensity of unpolarized light by one-half. The reduction is obviously greater than this on account of the reflections at the two oblique surfaces, and to a slight extent by the balsam film. In the present case the surfaces of the prism were slightly dull, and I doubt if the intensity of the transmitted light was much over 40 per cent of the original intensity. Calling the intensity of the soda flame 100, we get the intensities of the spectra by multiplying 40 by the fractions given in the table. The sum of these intensities (eight spectra and central image) is 75.6, which agrees fairly well with Rubens's determination of the reflecting power of speculum metal for yellow (70 per cent). This indicates that the ruling of the surface interferes in no way with the *total* reflection, which is what might be expected. The interesting point is that half of the total light is found in one spectrum. If speculum reflects 70 per cent, this means that we have 35 per cent of the light in the first order spectrum, or about one-third of the original amount.

Intensifying Glass Gratings. — Gratings ruled on glass can be much improved by a method described by the author in 1906. They usually give rather weak spectra, the diamond cuts being deep and narrow, usually much narrower than the clear spaces between them. Dilute hydrofluoric acid widens the cuts and increases the intensity of the spectra often as much as tenfold. The process can be controlled most perfectly, and there is no danger of injuring the grating. Coat the inside of a saucer with hot paraffine and allow it to cool. Fill the saucer with water and pour in two or three ccms. of hydrofluoric acid. Dip one corner of the grating into the dilute acid until 2 mms. of the ruled surface is immersed. Hold it in the fluid for 15 seconds, remove, wash, and wipe it dry. Now hold it at arm's-length against a distant gas flame and bring the spectra in succession across the etched corner. If no brightening is observed repeat the etching process on the same corner for a minute. As soon as the corner shows improvement immerse the whole grating, examining it every half minute or so, until the rest of the surface shows a brilliancy as great as the corner. As long as the corner remains brighter than the rest it is safe to continue the etching. It is a good plan to protect the back with a coating of wax, if

the grating is a valuable one, as the etching process is apt to develop latent scratches.

False Lines due to Periodic Errors or "Ghosts." — If the illumination is sufficiently intense, the bright lines of the spectrum are usually seen accompanied by companion lines, symmetrically placed to the right and left of the principal lines. These fainter companions are termed "ghosts," and are due to periodic errors of ruling. Very elaborate mathematical treatments of their cause have been given by Rowland and others, which hardly fall within the scope of this book. The following simple method of regarding the effects of periodic errors will, however, make the reasons for the occurrence of the spurious lines clear, though we cannot determine the number of the lines or their relative intensities, as can be done by the more rigorous mathematical methods.

Suppose that some irregularity of the ruling occurs at regularly recurring intervals, due to some slight imperfection of the dividing engine. For the sake of simplicity we will suppose that the irregularity consists in the introduction of an extra line at points half a millimetre apart, a fault that would not be likely to occur in practice, of course. The grating would thus be the equivalent of two superposed gratings, one with a fine spacing, namely, that which the machine is ruling, the other with a 0.5 mm. spacing. Consider now the effect of superposing two gratings. If we view a slit illuminated with sodium light through the fine-spaced grating we shall see the central image accompanied by widely separated spectral images. On introducing the second grating, with lines ruled at wide intervals, the central image and the lateral spectral images will all be accompanied by lateral spectra, which lie close together on account of the coarseness of the ruling, and are very faint on account of the small value of the ratio of the width of the line to that of the space between the lines. In other words, the second grating forms spectra not only of the slit (central image), but also of the lateral spectral images.

The errors which actually occur in the process of ruling are, of course, much more complicated. For example, the spacing of the lines may alter gradually in a periodic manner. We can say, however, that, in general, the effect of any periodic irregularity in the ruling will be to produce spectra at angular distances similar to those produced by a grating with a spacing equal to the spacing of the periodic error. If the eye is placed in the spectrum the periodic error can often be seen on the surface of the grating if the source of light gives a discontinuous (line) spectrum.

In practice it is usually not difficult to recognize ghosts, for they are only seen in the case of very strong lines, and have the same appearance regardless of the element, the spectrum of which is under examination. They are often useful, as in cases where the line is greatly broadened by overexposure, its true position being midway between the ghosts. The method of crossed gratings, which is analogous to Newton's method of crossed prisms, enables us to recognize a "ghost" in cases where it is not otherwise possible to do so. As this method has been used in the interpretation of the results

obtained with the interferometer of Lummer and Gehrke, which will be described in the Chapter on Interference Spectroscopes, it may be well to take it up in this connection.

In Fig. 168 let *o* be the source of light (slit of spectroscope, for example) viewed through two superposed gratings, with their lines perpendicular to each other, one of which shows "ghosts." Let *AB* be a first order spectrum produced by this grating, with bright lines 1, 2, 3, and a very bright line 5 accompanied by two ghosts 4 and 6.

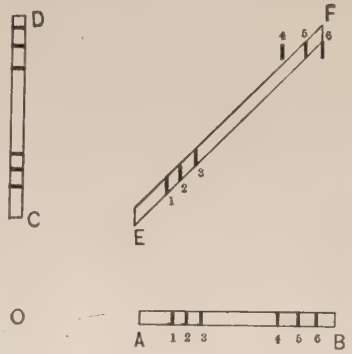


Fig. 168.

The other grating produces the spectrum *CD* of the source and the oblique spectrum *EF*, which we can regard as the spectrum produced by the good grating (lines horizontal) of the spectrum *AB*; produced by the bad grating (lines vertical) that is to say, the different elements of *AB* are deviated by different amounts proportional to their wave-lengths. The lines 1, 2, and 3, being true lines and having different wave-lengths, will lie in the positions 1, 2, 3, on *EF*. Lines 4, 5, and 6, however, have the same wave-length, consequently they will be deviated by equal amounts and occupy positions shown by 4, 5, 6, on *EF*. In other words, the ghosts are thrown out of the spectrum. The same method could be used with reflection gratings.

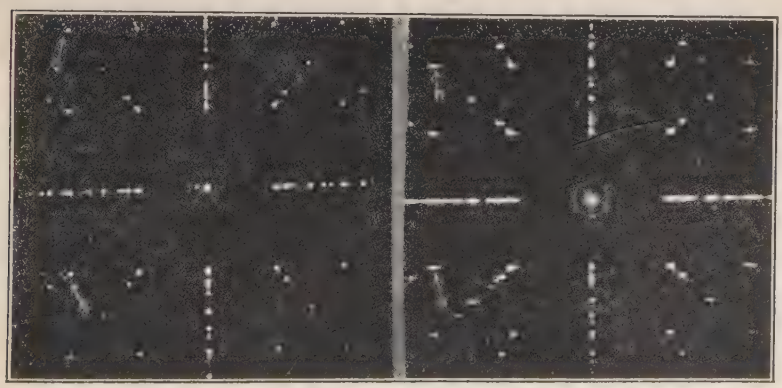


Fig. 169.

In Fig. 169 will be found photographs of the spectra of a helium tube, made by the author with two crossed gratings. The left-hand figure was made with a pair of gratings free from ghosts. The source of light was the small brilliantly illuminated circle of an "end on" tube. The spectra show the helium lines in the form of rows

of dots, and the dots lie along single straight lines in the diagonal spectre. In the right-hand figure one of the gratings has been replaced by a very poor one with a bad periodic error; the horizontal spectra produced by this grating are almost unrecognizable, owing to the numerous ghosts, which produce multiple dots overlying one another. The vertical spectra produced by the other grating are seen to be free from ghosts. In the diagonal spectra the horizontal arrangement of the ghosts at each spectral line (dot) is clearly seen. Compare points indicated by arrows.

The Nature of Optical Images. — All optical images arise by interference since the rectilinear propagation of the rays which form them is the result of destructive interference. In certain cases the part played by interference is very striking, and an experiment has been described by A. B. Porter which illustrates the theory and which is very instructive.

"The experiment consists in passing a parallel beam of monochromatic light through a coarsely ruled, black-line, diffraction grating, and then through a convex lens. On the far side of the lens a system of sharply defined interference fringes is formed which can be seen by aid of an eye-piece, or intercepted on a screen, at any point over a considerable range along the axis. Somewhere in this system of fringes is the geometrical image of the grating, but it is visually quite indistinguishable from any other transverse section of the fringe system. Clearly in this case, the geometrical image is merely that section in which the geometrical condition of similarity to the object is satisfied.

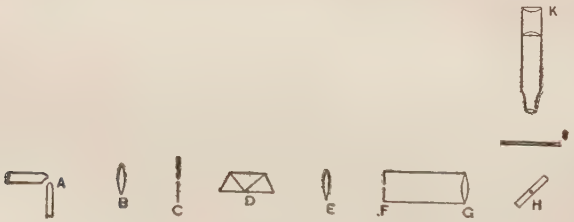


Fig. 170.

"The best arrangement of the experiment is the following. Light from an arc lamp A (Fig. 170) is focussed by means of the lens B upon a narrow slit C. Thence it passes through a direct-vision prism at D, and the spectrum is focussed by the lens E upon the narrow slit of a collimator FG. The parallel beam of monochromatic light thus obtained falls upon the mirror H of a microscope KJ, upon whose stage, at I, the grating is placed in such a position that its ruled lines are parallel to the projections of the two slits C and F. Using a black-line grating of 400 lines to the inch, and having both slits narrowed down to a small fraction of a millimetre so as to secure very homogeneous illumination, the field of view was examined with $\frac{2}{3}$-inch objective and 1-inch eye-piece. The interference fringes appeared in the field of the eye-piece with exquisitely sharp definition throughout the whole range of the coarse adjustment of the

microscope, *i.e.* over a distance of 58 mms., beginning with the front of the objective in contact with the grating and with its focal plane 7 mms. below the ruled surface; and the fringes could be traced through a much greater range by withdrawing the eye-piece and moving it back along the axis. As the microscope is slowly focussed upward, the bands undergo curious changes in appearance, the lines showing sometimes close together and again farther apart, but the definition is almost equally sharp throughout the whole range of adjustment, so that any section of the fringe system is as good an apparent image as any other section. Similar but less perfect effects may be obtained by illuminating the field by means of sodium light passing through a slit a couple of millimetres wide at a distance of one or two metres."

"If the angle of the incidence of the light on the grating is changed by moving the mirror, the whole fringe system shifts to one side or the other except in the focal plane, where it remains stationary. This shows (1) that the focal plane is the plane in which the interference fringes formed by light of all incidences coincide; (2) that, when a broad source is used, the geometrical image is really a superposition of coincident interference patterns; and (3) that the usual absence of a sharp image outside the focal plane is due to the more or less uniform illumination resulting from the overlapping of fringe systems due to light coming from various points in the source. When the grating is illuminated by a parallel beam of white light by means of a collimator with very narrow slit, or, less perfectly, by a distant gas flame turned edgewise, the effects are similar except that outside the focal plane the fringes are colored. Hence (4) the focal plane is also the plane of achromatic interference, *i.e.* the plane in which the fringes due to light of various wave-lengths coincide."

"These experiments show very clearly why it is in general essential to use a condenser to illuminate the field of a microscope in order to obtain a critical image, *i.e.*, an image which comes sharply into and out of focus and which is hence as free as possible from confusion with details of structure lying above and below the focal plane."

Abbé's Diffraction Theory of Microscopic Vision. — The diffraction theory of microscopic vision was proposed about thirty years ago by Ernst Abbé. It may be briefly stated as follows. For the production of a truthful image of an illuminated structure by a lens, it is necessary that the aperture be wide enough to transmit the whole of the diffraction pattern produced by the structure. If but a portion of the diffraction pattern is transmitted, the image will differ from the object, and will correspond to an object the entire diffraction pattern of which is identical with the portion passed by the lens. If the structure is so fine, or the lens aperture so narrow, that no part of the diffraction pattern is transmitted, the structure will be invisible, no matter what magnification is used. Abbé devised a number of interesting experiments to illustrate the theory. By means of suitably perforated screens placed within or above the objective of a microscope, one or more of the diffraction spectra produced by a glass grating (the object viewed through the micro-

scope) were cut off, and the appearance of the grating shown to be modified in a most remarkable manner. If all of the spectra were screened off, all trace of the lines vanished. It is sometimes assumed that there is an essential difference between microscopic and ordinary vision, and that the phenomena of diffraction play no part in the latter. Whatever difference there may be between the two cases, arises, however, from the relative size of the objects involved, and from the special methods of illumination employed with the microscope, as has been shown by A. B. Porter, who devised the following ingenious experiment for demonstrating that the images of periodic structures formed by the naked eye itself are due to diffracted light. Light from an arc lamp or the sun passes through a pin-hole in a screen and is focussed by means of a photographic lens on a cardboard screen, about 30 cms. from the lens — immediately in front of which a piece of wire gauze having about thirty wires to the centimetre is placed. The diffraction pattern produced on the screen by the wire gauze consists of a central image with a large number of radiating spectra surrounding it. There are two sets of spectra at right angles to each other, formed by the two sets of wires, with two intermediate sets also at right angles, but rotated through forty-five degrees with respect to the others. These latter may be regarded as the spectra of spectra, and are always seen when a source of light is viewed through a pair of crossed diffraction gratings. (See Fig. 169.) By cutting small holes in the screen we may transmit any portion of the diffraction pattern and allow it to enter the eye. If the screen is pierced by a hole only large enough to transmit the central image, the wire gauze is quite invisible. If a narrow slit is used which transmits only the central image and the horizontal line of spectra, the vertical wires alone are seen; if the slit is turned vertically so as to transmit the vertical line of spectra, the horizontal wires alone are visible. If the slit is turned at an angle of forty-five degrees so that the diagonal set of spectra are transmitted, neither the vertical nor horizontal wires are seen, but a very real-looking set of wires appears running diagonally in a direction perpendicular to the slit. Such a set of wires would, if acting alone, give rise to the spectra transmitted. If the card is pierced with three pin-holes which transmit the central image and two second order spectra, a set of vertical wires is seen, twice as close together as the actual wires. By cutting two slits in the screen at right angles to each other, and arranging them so as to transmit the diagonal spectra, we see the gauze as if rotated through forty-five degrees.

In these experiments the object of placing a lens between the gauze screen and the eye is to enable us to remove any desired group of parallel diffracted rays. This can only be accomplished by bringing all of the parallel sets each one to its own focus, and intercepting them at this point. A white card should be mounted at the point where the spectra appear sharpest, and the size and shape of the aperture desired for the transmission of any portion of the diffraction pattern traced upon it with a sharply pointed pencil.

These experiments are similar to the ones devised by Abbé, and furnish a very easy and convenient means of illustrating his theory.

The subject of the distribution of the light in the spectra in its connection with the formation of images of periodic structures by the microscope has recently been investigated by A. B. Porter, and some extremely interesting effects found which were verified by experiment (*Phil. Mag.*, 1905).

The case of a grating formed by opaque bars and transparent intervals was first examined by Fourier's theorem. The amplitude of the transmitted light is in this case represented by a square-topped curve, which, by Fourier's theorem, may be represented by an infinite series of cosine terms. If the edges of the opaque bars are not sharply defined, *i.e.* if they are shaded slightly, the analysis shows that the higher harmonic terms in the series are absent, and since each one of these gives a spectrum, the higher orders of spectra are absent. This was verified experimentally by making a contact print on a photographic plate of a grating with 400 very sharply ruled black lines to the inch; one edge of the grating was in contact with the plate, while the other was separated from it by means of a piece of paper. The blurring thus increased progressively across the plate. The original grating showed the first 35 orders of spectra of a sodium flame, while the print showed but three, when the flame was viewed through the edge which was in contact with the original. As the eye was moved along the print, the 2d and 3d order spectra rapidly faded away, the first order only being yielded by the end which was separated from the original. Applying this principle to Abbé's theory, we see at once that the sharpness of the edges of the images of a series of black lines depends upon the transmission of the spectra of high orders. If only the first order spectra are passed, the lines appear greatly blurred. If four or five orders were passed, the images became sharper and less blurred, but a fine dark line appeared down the centre of each. This was predicted by Porter from curves drawn representing the summation of the first five terms of the Fourier series, and subsequently verified by experiment. The result is rather remarkable in showing that a falsification of the image may result from an improvement of the lens. When 8 spectra were transmitted, two dark lines appeared running down the centre of each bright band. The results were verified by examining a grating under a microscope, the objective of which was furnished with an iris diaphragm, by which any desired number of spectra could be cut off. Monochromatic light was used, and the experiments show that some caution must be used in interpreting minute details in micro-photographs obtained with monochromatic light.

Porter has also examined the effects of the spectra produced by structure underlying the structure upon which the microscope is focussed. It was found that when monochromatic light was employed, the presence of a second grating, a short distance below the one under observation, in certain cases caused a complete obliteration of the lines over the greater part of the field. It is well known that when two gratings are superposed, with their lines parallel and separated by a short distance, the spectra formed by the double grating are intersected by transverse shadow bands. If under certain specified conditions these shadow bands lie in the

yellow of the first order spectra, and the iris diaphragm of the objective is contracted until only the first order spectra are passed, we shall have no spectra at all when the microscope is illuminated with yellow light, and the first order spectra when it is illuminated with light of some other color. In Porter's experiment, two gratings of 3000 lines to the inch, separated by a space 1 mm. thick, were placed upon the stage of the microscope, which was focussed upon the upper grating. The instrument was illuminated with monochromatic light furnished by a spectroscope, and it was found that the lines completely disappeared when the illumination was with yellow light, though they were distinctly visible with all other colors.

These experiments are very interesting, as they were all made with the microscope under what would be considered normal working conditions, with central illumination and circular diaphragms centred on the optic axis. "Nevertheless," as Porter says, "when certain relations existed between the aperture of the lens and the coarseness of structure of the object, images were formed which were utterly false in their smaller details, and other images were profoundly modified by the presence of structure lying entirely beyond the focal plane. It therefore seems that a working knowledge of the phenomena and laws of diffraction might well form a part of the equipment of every one who uses the microscope and attempts to interpret its indications."

Abbé's theory has sometimes been lightly treated, as most of his anomalous appearances of images were produced by diaphragms of peculiar form, slits, crosses, triangles, etc., which are never met with in practice. Porter's experiments were, however, all made with circular diaphragms under normal conditions, and the effects observed may frequently occur in practice, and be falsely interpreted. The reader should refer to the original paper for other interesting cases, and a more complete treatment than the scope of this book permits.

Concentration of Light into a Single Spectrum. — It is clear that the efficiency of a grating would be enormously increased if the light could all be concentrated into a single spectrum. By making the laminae of prismatic form it is possible to throw all of the light into a single spectrum, as has been shown by Lord Rayleigh. The angles of the elementary prisms must be such as would cause refraction of the incident rays in the direction of the diffracted rays of the spectrum into which the light is to be thrown. The possibility of its experimental realization was demonstrated by Mr. Thorp of Manchester, who punched a series of parallel saw-tooth grooves on a soft metal surface by means of a die. The surface was then flowed with a solution of celluloid in amyl acetate, which upon solidification was stripped off in the form of a film, bearing an accurate cast of the metal surface. Fully ninety per cent of all the light was concentrated into the first order spectrum on one side. These gratings were very small and of no use for optical purposes. The echelette gratings recently made by the author are of this type. They will be described later on. The same thing has been

accomplished in Michelson's remarkable echelon grating, which will be treated in another chapter.

It is clear that the same thing will be accomplished in the case of a reflecting grating if the grooves have flat sides which slope in such a direction as to reflect the energy in the direction of a given spectrum. The author has recently made gratings for use with long heat waves which act in this way. An investigation of their properties in collaboration with A. Trowbridge has shown that they form the most efficient spectroscopic apparatus of high resolving power for infra-red work at present at our disposal. Inasmuch as they constitute a link between the ordinary grating and Michelson's echelon, with its large steps (see next chapter) they have been termed echelette gratings. Their behavior with short waves (visible light) is most interesting, and teaches us a great deal about the action of a grating, and the way in which the form of the groove influences the distribution of intensity.

The Echelette Grating, for the Infra-Red. — One of the most important problems in Optics is the question of the distribution of intensity among the spectra of different orders produced by a diffraction grating. Practically no rigorous experimental investigation has been made, owing to the impossibility of determining the actual form of the groove ruled by a diamond point on a glass or metal surface. It is very difficult to learn anything from a microscopical examination, and it is by no means certain that the form of the groove will conform to what we believe to be the shape of the ruling point. A promising method of attack would be to manufacture gratings with grooves of such large size as to make the determination of their exact form, width, etc., a matter of certainty and then investigate the energy distribution by means of the long heat-waves discovered by Rubens and his collaborators. By employing the residual rays from quartz (see Chapter on Theory of Dispersion) and a grating with 1000 lines to the inch, we should have about the same ratio of wave-length to grating space as obtains in the case of a Rowland grating with 14,000 lines to the inch, and red light. Gratings with constants varying from 0.1 mm. to .01 mm. could be studied by means of residual rays, or narrow regions of the infra-red spectrum, isolated by a salt-prism spectrometer, and the relation between the intensity distribution and the form of the groove determined. Methods were worked out by the author by which a groove of any desired form could be ruled, with optically flat sides (a very important point), the angular shape of each side of the groove measured, and the exact nature of the ruling determined, *i.e.* whether the metal had been forced up between the grooves, or whether the angle between the opposed faces was equal to the angle between edges of the ruling knife. This by no means follows, as the ruling of groove No. 2 may force the metal to one side and increase the angle of slope of the adjacent side of groove No. 1. Gratings were finally obtained, which have proved so efficient in the investigation of infra-red spectra, that it seems worth while to designate them by a name of their own. They throw practically all of the energy into one or two spectra

to the left of the central image, which is completely absent, with visible light, and show no trace of any energy to the right of it. With visible light they send the greater part into a group of spectra, say, from the 12th to the 16th, or from the 24th to the 30th order. They may thus be regarded as reflecting echelons, of comparatively small retardation, and may be termed "echelette," to distinguish them from the ordinary grating and the Michelson echelon.

After considerable experimenting with various metals and ruling points, the method finally adopted was the following: A sheet of polished copper plate, such as is used by photo-engravers for the half-tone process, was gold plated and polished. The plates were found sufficiently flat for the purpose and had a fair optical surface.

A carborundum crystal was used for the ruling point, selected by breaking up a mass of the substance as it comes from the furnace. Specimens of these iridescent crystalline masses are to be found in most chemical or mineralogical museums. The crystals have a hexagonal form, and are mounted so as to rule with the 120° angle. The natural edges are so straight that they rule a groove with optically perfect sides. Everything depends upon the nature of the edge and the angle at which it is set with respect to *the direction of the line, i.e.* the tilt forwards or back. Some edges will not rule properly at any angle, "chattering" over the surface and tearing off a thread of metal. *No metal is removed* when the ruling is going on properly, the groove being formed by compression of the metal. If the edge is properly chosen, mounted at the proper angle and correctly weighted, a beautiful groove is made with a very little elevation of the metal above the original surface at the edges. The first gratings were ruled on copper and subsequently gold-plated to prevent tarnish, but it was found that even the lightest polishing on the buffing wheel destroyed the sharpness of the edges and caused the development of strong central images.

The angle of the ruling edges of the carborundum hexagonal plates is 120°, consequently the sides of the groove make approximately this angle. By placing the crystal in various positions we obtain grooves of various shapes, one side, for example, sloping at an angle of 12° with the original surface, the other at 48°. These angles are subsequently determined with a small spectrometer or by simply mounting the gratings on a graduated circle, and observing the reflection of a lamp flame in them. In the best ones no trace of the central image can be seen, which is what we should expect if the edges of the grooves were sharp and none of the original plane surface remained.

In the majority of cases the crystal was mounted so as to rule a groove one edge of which made an angle of 20° or less with the original surface. With normal incidence this gives us a concentration of energy at an angle of 40°, with practically no energy thrown off from the other edges of the groove, in the case of visible light, owing to the steepness of the angle. The best gratings show no reflection in the normal direction, *i.e.* they give no central image. They give, however, a very good reflected image of one's face, when held at an angle of 20°, the image being uncolored, but slightly

diffused by diffraction in a direction perpendicular to the grooves. The image is so sharp, however, that the pupil of the eye can be seen without difficulty. The gratings behave, with infra-red radiation of wave-length, above, say, 3 μ, as almost ideally perfect gratings, that is they give spectra similar to what we should have with an ordinary grating which threw practically all of the light into one or two orders on one side of the central image.

With visible light their behavior is most curious and interesting. The central image is absent, and we get a blaze of light when the grating is turned at the proper angle. With a symmetrical groove the blaze is seen on both sides, at angles of 45° for normal incidence.

If the source of light is white, a lamp flame, for example, the appearance is as shown in Fig. 1, Plate VII, which is for a grating showing a faint central image in the position indicated by the arrow. This is bordered on each side by the ordinary grating spectra, which are close together on account of the coarseness of the ruling. They are much fainter than I have indicated on the plate. Well to one side, at an angle of about 40° with the normal, we see a very bright and greatly broadened white image of the flame, accompanied by lateral spectra, colored as shown. These are not grating spectra, but the first class spectra (as Fraunhofer termed them) due to a single slit, or in this case to the superposed images formed by the reflecting edges of the grooves. With a sodium flame the appearance is as shown in Fig. 2, Plate VII. We have in this case four orders of spectra in the region occupied by the central maximum of the spectra of the first class. Their order is indicated below. Two or three orders to the right and left of this group are absent, since they fall in the region of the minima due to a single slit. They are the "absent spectra" of grating theory. Other groups of orders appear in the regions occupied by the first class colored spectra, their intensity being much less, however, than the intensity of the ones falling within the region of the central maximum. The existence of these images of the soda flame shows us that the perfection of ruling is such that interference, with a path difference of about 30 wave-lengths, is still taking place. In other words, our grating is acting as a reflecting echelon with steps 15 wave-lengths in height.

The width of the region in which we have these maxima and mimima of the first class becomes less as we increase the width of the reflecting steps of the grating. In the case of a grating of such a small constant as .0123 mm., this region of diffraction of the energy from each individual element covers a range of fully 10°, embracing as many as 12 or more orders of second class spectra. In the case of the Michelson echelon, the width of the step is from 0.5 to 1 mm., and the range of diffraction is so small that but one or two orders of spectra are included within it. By the study of these echelette gratings we can pass gradually from the case of the ordinary grating to that of the echelon.

The results appear to indicate that with a simple groove, such as we have here, we cannot secure a concentration of light in a region narrower than the diffraction range from a single reflecting

element. In one case the central maximum of the spectra of the first class, instead of being white, was distinctly blue, while the maxima immediately to the right and left of it contained only red, orange, and yellow light, as shown in Fig. 3, Plate VII. This curious distribution of color in the maxima of the first class was observed only in the case of one grating. To explain it we must devise some type of single reflecting element which will give, in the case of red light, zero illumination at the center of symmetry with strong lateral maxima, and with blue light a strong maximum at the center, bordered by minima, which occupy the positions of the red maxima, and maxima in the positions of the red minima. This can apparently be brought about only by an element consisting of two parts, in other words a double reflecting strip, with a half wave retardation for red light, as in the case of the laminary grating. The central maximum of the first class will vanish in this case for red light, as can easily be seen by constructing the diffracted wave fronts.

If blue light is used, the retardation becomes very nearly a whole wave and we have the center of the system bright. An examination of the grating with a microscope showed that there were, in fact, two narrow reflecting strips in contact, the grating being built up of paired reflecting elements, separated by inoperative strips of about the same width. A fuller account will be found in *Philosophical Magazine.*

Some of the gratings, with a constant of .0123 mm., gave strongly colored central images, and lateral spectra of low order in which a certain color or colors were wholly absent. The first order spectrum on one side, for example, may contain no yellow-green light, which appears in excess in the central image. The higher orders are also wanting in certain colors, a color wholly absent in one spectrum constituting all that there is of a spectrum of higher order. The spectra formed by one of these gratings showing a most remarkable distribution of color is represented in Fig. 4, Plate VII. Replicas made from it with celluloid show similar anomalies.

Spectral Intensity and Form of Grooves. — An investigation has been made by A. Trowbridge in collaboration with the author of the distribution of the energy amongst the spectra of different orders, as influenced by the form of the groove (*Phil. Mag.*, Oct. 1910). The echelette gratings, with grooves of known form, were used in connection with a remarkably perfect vacuum spectrometer and bolometer, both designed by Trowbridge. The slit of the instrument was illuminated with either the residual rays from quartz (see Chapter on Theory of Dispersion), the wave-length of which (8.5 μ) was about fifteen times that of yellow light, or with the radiations from the flame of a Bunsen burner, which emits a very strong band of wave-length. 4.3 μ. The curves obtained with the bolometer indicated how the energy was partitioned between the central image and spectra of various orders. Extremely interesting results were obtained. For example, a grating with a constant of .0123 mm. and a groove with

one side sloping at an angle of 8° with the original surface, and the other side at an angle of more than 45°, gave with quartz rays 66 per cent of the energy in the central image and 34 per cent in the first order spectrum. The reflecting edges of 8° angle reflected the energy at an angle somewhat less than that subtended by a point midway between the central image and the first order spectrum, for $\lambda = 8.5\,\mu$, consequently the energy was divided between them, with an excess in the central image. For the $4.3\,\mu$ waves from the Bunsen burner, the first order spectrum was very nearly in the direction of reflection, and contained 70 per cent of the total energy, 22 per cent being in the central image, and 8 per cent in the first order spectrum on the other side. A grating of the same spacing with angles of slope of 29° and 18° gave for quartz rays 17 per cent in the central image, 57 per cent in the first order spectrum to the left, and 26 per cent in the first order to the right. The direction of reflection from the 29° edges was about 7° greater than the angle of the first order spectrum to the left, consequently more than half of the energy was found here. For the 4.3 radiations, the first order to the left was wholly absent, the second order contained 37 per cent, the third 39 per cent, the central image 17 per cent, and the first order to the right 7 per cent. The absence of the first order spectrum to the left is of especial interest. It can result only from the destructive interference between the disturbances coming from the 18° edges of the grooves, with those coming from the 29° edges. We may thus regard the grating constant as halved, calling each edge of the groove an element of the grating. The first order spectrum referred to the original constant is therefore absent. A large number of gratings have been examined, and the work is not yet completed. For a complete solution of each case it is necessary to know whether any of the original flat surface has been left between the grooves. This is often the case with the coarser rulings, and results in the formation of strong central images when the gratings are examined with visible light. An elementary method of handling the energy distribution in these cases has not been found, for there are three elements of surface operating. It has been shown, however, that even with flat reflecting surfaces at just the right angle it is impossible to get all of the energy into a single spectrum, on account of the effects from the other sides of the grooves, and the diffraction of the energy over a rather wide angular range, resulting from the narrowness of the reflecting elements.

Concave Gratings. — To obtain sharply focussed spectra by means of the plane diffraction grating, two lenses are required: one to render the light parallel before its incidence upon the grating, the other to unite the parallel diffracted rays in a focus. The brilliant discovery was made by Rowland that gratings ruled on concave spherical surfaces would of themselves furnish focussed spectra, excelling in sharpness those obtained by means of lenses. This discovery marked an epoch in the history of spectroscopy, for by dispensing with the lenses, and the absorbing action which they exerted on the ultra-violet, the region of short waves could be explored with an accuracy never before attainable, and the gratings ruled upon

Professor Rowland's machine have become the standard instruments for spectroscopic work throughout the world. They combine the image-forming power of concave mirrors with the spectrum-producing power of gratings. A marked advantage of the concave grating lies in the fact that the superposed spectra of different orders are all in focus, which is not the case with plane gratings and lenses, owing to the fact that complete achromatization can never be obtained. It is thus possible to measure the relative wave-lengths with great accuracy. An ultra-violet line of wave-length 2950 of the second order spectrum will be photographed nearly in coincidence with the *D* lines of sodium, and its wave-length can be very accurately measured relatively to these lines. This method of coincidences was originated by Rowland. But the greatest advantage of all is the fact that, when properly mounted, the concave grating yields spectra which are truly normal, *i.e.* spectra in which the distances between the lines are proportional to their wave-lengths.

Various methods of mounting the concave grating have been devised. That due to Rowland is the following: The theory of the grating, which we shall take up presently, shows that if the grating and the illuminated slit are both situated on a circle, the diameter of which is equal to the radius of curvature of the grating, the spectra of different orders will all be in focus upon the same circle. The spectra are normal along that portion of the circle diametrically opposite the grating, consequently if a photographic plate is placed at this point and bent to the radius of curvature of the circle, the photographic image will be everywhere in focus, and the spectrum will be normal. To pass from one part of the spectrum to another we have only to move the slit around on the circle, a method sometimes employed. With fixed sources of light, such as the image of the sun formed by a lens in combination with a heliostat, this is impossible, and Rowland devised the following extremely ingenious mechanical device, by which the camera and grating could be moved, with reference to a fixed slit, so as to comply with the required conditions.

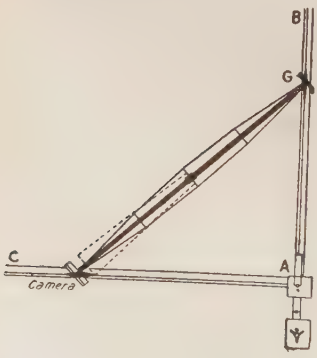


Fig. 171.

Two tracks *AB*, *AC* (Fig. 171) are rigidly mounted on fixed beams, so as to meet accurately at a right angle. On these tracks roll a pair of carriages which support a trussed tube of iron, the length of which is equal to the diameter of the large circle, *i.e.* the radius of curvature of the grating. One of the carriages carries the camera, the other the grating *G*, while the slit is permanently mounted above the point where the rails meet. As the camera is moved away from the slit, the grating is drawn towards it, the three always remaining on the circumference of the circle, with the grating and

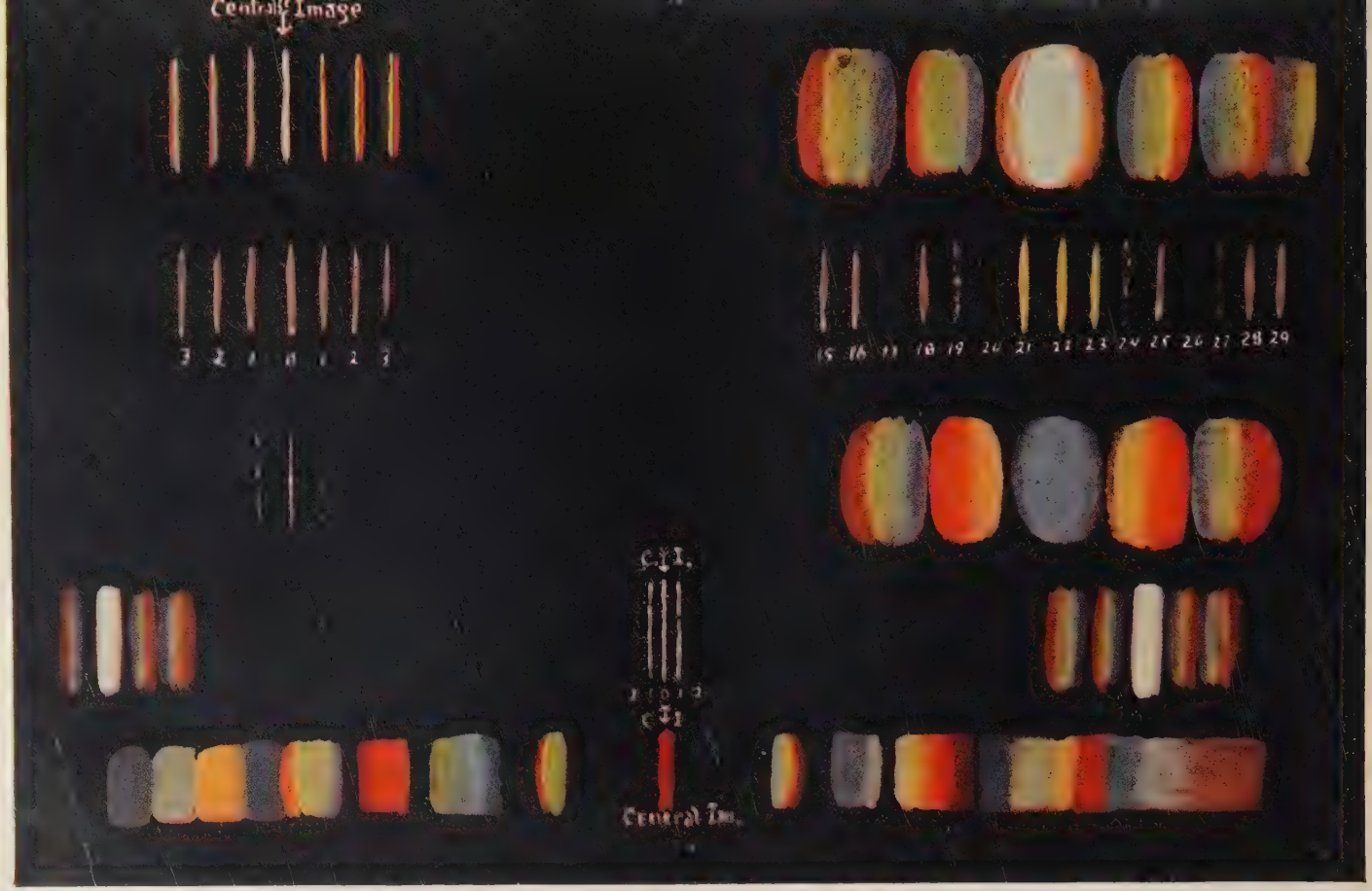

Central Image
3 2 1 0 1 2 3
15 16 17 18 19 20 21 22 23 24 25 26 27 28 29
C. I.
Central Im.

Plate V.

camera always at opposite ends of a diameter. The grating is turned into such a position that its centre of curvature coincides with the centre of the photographic plate.

Full and explicit directions for mounting and adjusting the grating will be found in Kayser's *Handbuch der Spectroscopie*, vol. i.

Theory of the Concave Grating. — The theory of the concave grating has been treated by Rowland, Mascart, and others. The following is due to Runge, who specifies definite conditions regarding the position of the grating, slit, and spectrum, and then investigates the nature of the ruling necessary to produce sharply focussed spectra; for example, should the lines be equally spaced along the *arc* of the grating or along the chord? The manner of ruling adopted by Rowland gives equal spacing along the chord, for, as the diamond point moves back and forth along a fixed line, the concave surface, which lies flat upon the moving carriage of the dividing engine, is advanced through equal distances by means of the screw.

Suppose that we have a small source of monochromatic light at the point A (Fig. 172), and wish to determine the resultant at A' of the disturbances coming from the different elements of the concave surface GP.

Let P be any point on the concave surface. An image of A will be formed at A' whenever the disturbances arising at every point P reach A' in the same phase. This condition is fulfilled if $AP+PA' = \text{const.}$, or if the curved surface is a portion of an ellipsoid of revolution having A and A' as foci. Construct now a series of confocal ellipsoids, under the condition that the constant distance $AP+PA'$ for each increases by $\frac{\lambda}{2}$ for each successive surface. These ellipsoids will cut the spherical surface G up into zones, in such a manner that the disturbances from any two adjacent ones will reach A' in opposite phase. If AP, $A'P$ and the radius of curvature of G are large, the zones will have practically the same width, and the resultant effect of all at A' will be zero. If every other zone is blotted out, or if a line is ruled on every other one, so as to get rid of this destructive interference, we shall have illumination at A'. Runge then shows that with any other wave-length differing even but slightly from the one considered, we shall have zero illumination at A'. This is merely an explanation of the formation of the spectrum and can be omitted.

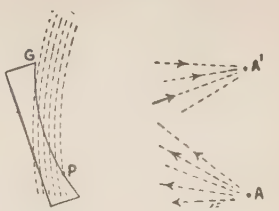


FIG. 172.

Consider the spherical surface of radius ρ as fixed with its vertex at the origin of coordinates x, y, z, and tangent to the yz plane. Its equation in this position is

$$x^2+y^2+z^2-2\rho x=0.$$

Let the points A and A' lie in the xy plane, and let their coordinates be a, b, and a', b' (Fig. 173). We require expressions for AP

and $A'P$, for it is the sum of these quantities which enters into our expression for the illumination at A'.

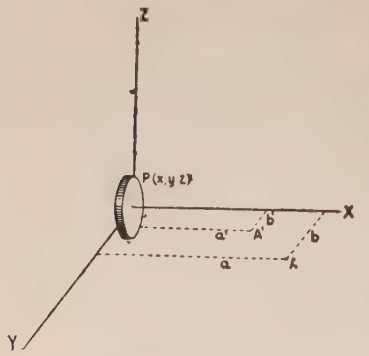


Fig. 173.

The coordinates of P, a point on the spherical surface, are x, y, and z.

Then

$$\overline{AP}^2=(x-a)^2+(y-b)^2+z^2,$$

or writing r^2 for (a^2+b^2),

$$\overline{AP}^2=r^2-2ax-2by+x^2+y^2+z^2.$$

From the equation of the spherical surface we have

$$2x=\frac{x^2+y^2+z^2}{\rho},$$

and substituting in the last equation for $2ax$, the value given above, *i.e.* $\frac{a}{\rho}(x^2+y^2+z^2)$,

$$\overline{AP}^2=r^2-2by+\left(1-\frac{a}{\rho}\right)y^2+\left(1-\frac{a}{\rho}\right)z^2+\left(1-\frac{a}{\rho}\right)x^2.$$

From the equation of the sphere, x is of the second order with respect to y and z; the above equation reduces to (neglecting third order terms)

$$AP=r-\frac{b}{r}y+\frac{a}{2r}\left(\frac{a}{r^2}-\frac{1}{\rho}\right)y^2+\frac{1}{2r}\left(1-\frac{a}{\rho}\right)z^2.$$

A similar equation is obtained for $A'P$, and the sum may be written

$$AP+PA'=r+r'-\left(\frac{b}{r}+\frac{b'}{r'}\right)y+\left[\frac{a}{2r}\left(\frac{a}{r^2}-\frac{1}{\rho}\right)+\frac{a'}{2r'}\left(\frac{a'}{r'^2}-\frac{1}{\rho}\right)\right]y^2$$
$$+\left[\frac{1}{2r}\left(1-\frac{a}{\rho}\right)+\frac{1}{2r'}\left(1-\frac{a'}{\rho}\right)\right]z^2.$$

We can simplify this equation by imposing certain conditions. If we limit the vertical aperture of the mirror sufficiently, the terms in z^2 may be neglected. This condition is fulfilled if the ruled lines are short, and in practice the ratio of the length of line to the radius of curvature is never made greater than a certain amount, say $\frac{1}{50}$.

We can get rid of the terms in y^2 by proper disposition of A and A'. The condition to be fulfilled is that

$$\frac{a}{2r}\left(\frac{a}{r^2}-\frac{1}{\rho}\right)+\frac{a'}{2r'}\left(\frac{a'}{r'^2}-\frac{1}{\rho}\right)=0,$$

which will occur if $r^2=a\rho$ and $r'^2=a'\rho$, that is if A and A' are situated on a circle, the centre of which is on the x-axis at a distance $\rho/2$ from the origin.

This is the condition already specified in the treatment of the mounting of the grating.

The equation now reduces to

$$AP+A'P=r+r'-\left(\frac{b}{r}+\frac{b'}{r'}\right)y.$$

Under the conditions specified r and r' are independent of the position of the point P on the surface of the grating; we need only consider the term $\left(\frac{b}{r}+\frac{b'}{r'}\right)y$ in determining the illumination at A'.

Call e the distance between the y coordinates of the lines n and $(n+1)$, or the distance between their adjacent zones; there will be illumination at A' when the path-difference between the streams of light from the two zones amounts to a whole number of wave-lengths. This condition is obviously represented by

$$\left(\frac{b}{r}+\frac{b'}{r'}\right)(y+e)-\left(\frac{b}{r}+\frac{b'}{r'}\right)y=m\lambda,$$

or

$$e\left(\frac{b}{r}+\frac{b'}{r'}\right)=m\lambda,$$

m being an entire number.

The consecutive values of the y coordinates of the lines must therefore differ by a constant amount; in other words, the spacing of the grating must be equal when measured along a chord of the arc, and not along the arc itself. The manner of ruling the gratings insures this, as has been pointed out.

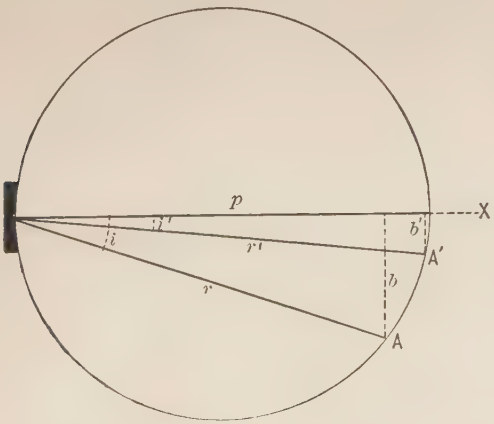


Fig. 174.

In Fig. 174 let A be the slit and A' the spectrum line,

$$e\left(\frac{b}{r}+\frac{b'}{r'}\right)=e(\sin i+\sin i')=m\lambda;$$

therefore keeping i constant,

$$e \cos i' di' = m d\lambda.$$

Now

$$\rho di' = ds,$$

therefore

$$e \frac{\cos i'}{\rho} ds = m d\lambda,$$

or

$$\frac{ds}{d\lambda} = \frac{m\rho}{e \cos i'}.$$

$\frac{ds}{d\lambda}$ is the scale of the spectrum, that is to say, the distance measured in a sufficiently small unit of length between two lines whose wave-lengths differ by one unit. The scale is a minimum when $i' = 0$, that is to say, when A' lies on the normal of the grating. For A' in the neighborhood of the normal the scale varies slowly; that is, the spectrum is approximately normal.

Diffraction by a Circular Aperture. — This case is of especial interest in connection with the theory of optical instruments.

Let R be the radius of the aperture, and θ the angle of diffraction of parallel rays which meet at M, the focus of the lens. AB is a diameter of the aperture and ON the normal at the centre.

Let the displacement at M due to an area $\rho d\Phi d\rho$ at A be expressed by $\sin 2\pi \frac{t}{T} \rho d\Phi d\rho$ and let $OP = \rho$ and $\angle AOP = \Phi$ be the coordinates of a point P of the aperture. The path-difference between the rays leaving A and H is $AH \sin\theta$.

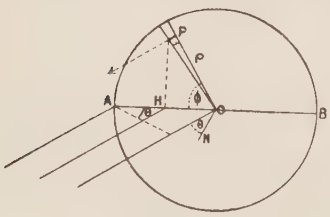


Fig. 175.

A ray leaving P, parallel to the other three rays, will have the same path-difference with respect to A as has the ray from H, the foot of the perpendicular, let fall from P upon AO. The displacement due to the disturbance from P is therefore expressed by

$$\rho \sin 2\pi \left(\frac{t}{T} - \frac{AH \sin\theta}{\lambda} \right) d\Phi d\rho,$$

$\rho d\Phi d\rho$ being the area of the surface element at P, and since

$$AH = R - \rho \cos\Phi,$$

we can write the above expression

$$\rho \sin 2\pi \left(\frac{t}{T} - \frac{R \sin\theta}{\lambda} + \frac{\rho \cos\Phi \sin\theta}{\lambda} \right) d\Phi d\rho,$$

which being the sine of the sum of two quantities can be treated in the same way as the expression of similar form which we developed in determining the effect of the concave spherical wave.

The resultant obtained by integrating over the whole aperture is then

$$\left(\int_0^{2\pi}\int_0^R \rho\cos 2\pi\frac{\rho\cos\Phi\sin\theta}{\lambda}d\Phi d\rho\right)^2 + \left(\int_0^{2\pi}\int_0^R \rho\sin 2\pi\frac{\rho\cos\Phi\sin\theta}{\lambda}d\Phi d\rho\right)^2.$$

The second integral is zero, for the elements of it, arising from any two points situated at equal distances on opposite sides of PO, are equal and of opposite sign.

The intensity is therefore

$$I=\left(\int_0^{2\pi}\int_0^R \rho\cos 2\pi\frac{\rho\cos\Phi\sin\theta}{\lambda}d\Phi d\rho\right)^2.$$

This expression is integrated with respect to r by parts, and with respect to Φ in series, the final result being

$$I=(\pi R^2)^2\left\{1-\frac{1}{2}\left(\frac{m}{1}\right)^2+\frac{1}{3}\left(\frac{m^2}{2}\right)^2-\frac{1}{4}\left(\frac{m^3}{2\times 3}\right)^2+\frac{1}{5}\left(\frac{m^4}{2\cdot 3\cdot 4}\right)^2\right\}^2,$$

in which m is defined by $2m=\frac{2\pi R}{\lambda}\sin\theta$.

This result was obtained by Airy (*Camb. Phil. Trans.*, p. 283, 1834). The series is convergent for all values of m, and becomes alternately positive and negative as m increases. The intensity is therefore zero for certain values of m, *i.e.* for certain values of θ. We have in consequence a series of concentric bright and dark rings. The angle θ corresponding to any bright or dark ring is found by ascertaining the corresponding value of m in the series, and equating it to

$$\pi R\frac{\sin\theta}{\lambda} \text{ or } \sin\theta=\frac{m\lambda}{\pi R},$$

an equation which shows that the deviation θ for any ring is proportional to λ, and inversely as the radius of the aperture.

The diameters of the rings and central spot consequently become less as the aperture is increased in size. It is on this account that the images of the fixed stars appear smaller in telescopes of large aperture than in smaller instruments.

The following table gives the values of $\frac{m}{\pi}$ for the first few maxima and minima:

	$\frac{m}{\pi}$.	Intensity.
1st Max.,	0	1
1st Min.,	0.61	0
2d Max.,	0.81	.0174
2d Min.,	0.116	0
3d Max.,	1.333	.0041
3d Min.,	1.619	0

Resolving Power of Telescope. — The images of two stars can be seen separated if the central spot of the diffraction pattern of one falls at or beyond the first minimum (*i.e.* dark ring) of the image of the other. Let R be the radius of the telescope's aperture. The diffraction angle θ for the first minimum is given by

$$\sin\ \theta = 0.61\frac{\lambda}{R}.$$

The angular distance between two stars must therefore be greater than θ, as defined above, if they are to be seen separated, *i.e.* we must have the angular separation $\Phi > .61\frac{\lambda}{R}$ (writing Φ for $\sin \Phi$).

Calling $R = .00056$ mm. and expressing Φ in minutes, we have $\Phi > \frac{1.17'}{R}$. A telescope of 200 mms. aperture will therefore resolve a double star with an angular separation of $.0117' = .7''$. The equation shows us that the angular separation of two stars which can be separated by a given lens is roughly equal to the angle subtended by the wave-length of light at a distance equal to the diameter of the lens.

In some cases it becomes necessary to reduce the aperture of the telescope, as when viewing a very brilliant object such as the sun or the brighter planets. This is usually accomplished by covering the lens with a screen perforated with a circular hole. The resolving power is immediately reduced, and if the reduction is carried too far all definition disappears.

Lord Rayleigh has pointed out that a better method would be to cover the central portion of the lens with a circular opaque screen. This would permit of the interference of the rays from the outer zones of the lens and the resolving power would not be reduced. In fact calculations showed that an actual improvement resulted, the diameter of the first dark ring around the central bright spot being less than with the full aperture.

Babinet's Principle. — This principle is one which is applied to complementary diffraction screens, by which we mean a pair of screens in which the transparent portions of one are replaced by opaque portions in the other, and *vice versa*. An example would be a metal plate with a number of small circular apertures and a glass plate with metal disks of similar size and distribution. Babinet's principle states that the diffraction patterns are the same in each case. This we can see from the following considerations:

In the case above the illumination at a point M on the screen, where the parallel diffracted rays of diffraction angle δ from the collection of circular apertures come together, is represented by the sum of the squares of two integrals taken over the areas of the apertures. This we will call $A_1^2 + B_1^2$. In the same way the illumination at the same point due to the collection of disks is $A_2^2 + B_2^2$. If the two sets of disturbances act simultaneously, *i.e.* if the wave is disturbed by no screen, the illumination is zero, provided the point M is situated at some point not coincident with the point at which

the wave comes to a focus; in other words, no diffraction effects are produced. This means that the resultant of one set of disturbances is able to exactly destroy the resultant of the other set, or

$$(A_1+A_2)^2+(B_1+B_2)^2=0, \text{ or that } A_1=-A_2 \text{ and } B_1=-B_2.$$

The illumination is, therefore, the same in the two cases, and the only result of changing the screens is to alter the resultant phase by 180°.

The principle of Babinet cannot be applied universally to all diffraction problems, for example the circular aperture and disk in the Fresnel class, one of which gives maxima and minima along the axis, the other only a maximum. Its application is restricted to points lying outside of the projection of the aperture, where the illumination due to the whole aperture is zero. As an illustration of a case in which it can be applied, suppose we have a large aperture AB (Fig. 176) filled with small circular disks. The illumination at points in the regions CD and EF remains the same, when circular apertures are substituted for the disks, but alters in the region DE, the projection of the large aperture. Strictly speaking we cannot apply the principle quite up to the points D and E, for if we get very near them we are in a region where the illumination due to the whole aperture is not zero, owing to diffraction by its edges. The case above figured belongs to the Fresnel class. If we place a lens behind the aperture, we can apply the principle to all points lying outside of the system of small diffraction rings formed by the open aperture and the lens. If the aperture is fairly large and the lens of short focus the ring system is exceedingly small, and the principle applies everywhere except at the image of the source of light thrown by the lens, which is sensibly a point.

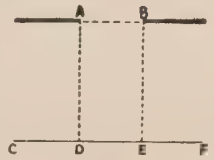


FIG. 176.

Diffraction by Two Small Apertures. — If we have two small circular apertures of the same size and close together we shall have interference between the disturbances exactly as in the case of the two parallel slits. The intensity due to a single aperture may be represented by

$$i=[f(\delta,\ \delta')]^2,$$

in which δ and δ' are the two angular coordinates which determine the direction of the diffracted ray.

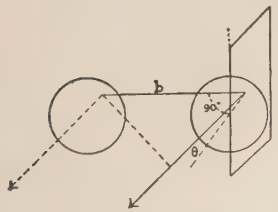


FIG. 177.

If b is the distance between the centres of the apertures (Fig. 177), and consequently the distance between any two homologous points, and Φ is the angle between the diffracted ray and a plane perpendicular to the line joining the points, the path-difference between the rays will be $b \sin \Phi$, and the intensity

$$I=2[f(\delta,\ \delta')]^2\cdot\left(1+\cos 2\pi\frac{b\sin\Phi}{\lambda}\right).$$

There will thus be a system of circular maxima and minima, crossed by a system of parallel dark strips perpendicular to the line joining the points, the position of which is given by

$$2\pi\frac{b\sin\Phi}{\lambda}=(2m+1)\pi \text{ or } \sin\Phi=(m+\tfrac{1}{2})\frac{\lambda}{b}.$$

By Babinet's principle we may substitute for the apertures two small circular disks, without changing the distribution of intensity in the diffraction pattern. In this case, however, the diffused light and the intensity of the illumination at the centre make it difficult or impossible to see the rings and fringes.

Diffraction by a Large Number of Irregularly Arranged Circular Apertures or Disks. — In this case the phases between the parallel disturbances from homologous points vary in an irregular manner, and we have on the whole as much reënforcement as destructive interference in any given direction, the case being similar to the parallel but not equidistant slits. The illumination at any point is the same as that produced by a single aperture multiplied by the number of apertures.

Halos. — The halos which are sometimes seen surrounding the sun or moon are due to diffraction by small drops of water, which by Babinet's principle will produce the same effects as small circular apertures of the same size. The smaller the drops the larger the halos, but we distinguish between the diffraction halos, which are always close to the sun, and the large rings due to ice spicules floating in the air. These halos can be imitated by viewing a candle flame or other source of light, through a glass plate, on the surface of which lycopodium has been dusted, or better, by viewing the light through a large glass flask, wet on the inside and connected with an air pump. On partially exhausting the flask with one or two strokes of the pump a cloud forms in the flask, and the light is seen to be surrounded by brilliantly colored rings.

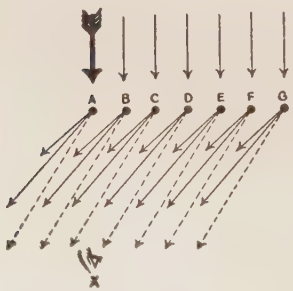


Fig. 178.

When the halo is produced at a great distance, as is the case in the atmospheric phenomenon, instead of by particles immediately in front of the lens of the eye or telescope, the complete ring system as seen is of course not produced by each individual particle.

The production of the colored ring is illustrated in Fig. 178. The broad arrow indicates the direction in which the sun is seen by an eye at X. A, B, C, D, etc., are small globules of water. The dotted arrows represent the directions of the diffracted rays, giving the first maximum to the left of the central maximum for the blue rays, the long solid arrows the directions of the diffracted rays for the green, and the short arrows for the red. It is obvious from the diagram that the particle D will send blue light to the eye, the

particle E green light, and the particle F red light. The phenomenon in space will be represented by rotating the diagram on AX as an axis. Each particle of water thus forms an infinitesimal element of the halo. If the particles vary in size in different parts of the sky, the angles of diffraction will vary also, and we may thus have a halo which is not a perfect circle.

In the same way a cobweb in the sunshine sends approximately monochromatic light to the eye, the color depending on its angular position, and a plane diffraction grating at a distance of eight or ten feet from the eye appears illuminated in light of a uniform color.

Young's Eriometer. — The dependence of the diameter of the halo on the size of the diffracting particles was utilized in an ingenious piece of apparatus devised by Young for measuring the diameters of fibres, or small particles of any sort. It consists of a metal plate with a small hole 5 mms. in diameter, surrounded by a circle of smaller holes about 1 cm. in radius. The plate is placed in front of a lamp flame, and viewed through the particles or fibres to be measured, which are best spread out on a glass plate. The halo surrounding the central aperture can be brought into coincidence with the circle of small holes by varying the distance of the screen, which can be done by sliding the plate carrying the particles along a graduated rod, on the end of which the diffracting screen is mounted, the distance varying inversely as the diameter of the halo, which in turn varies inversely as the diameter of the particles. The constant of the instrument is determined by making an observation with particles of known size. If d is the distance between the screen and the particles of known radius r, when the halo is in coincidence with the ring, and d' is the distance for particles of unknown radius r', we have

$$\frac{r'}{d'} = \frac{r}{d} \text{ or } r' = r\frac{d'}{d}.$$

Effect of Moving one of the Two Apertures in the Direction of the Source. — This case is of especial interest in connection with the study of the so-called diffusion rings, which we shall take up next. Suppose the screen with the two circular apertures to be divided in two along a line perpendicular to the line uniting the centres of the apertures at its middle point, and the two halves displaced in the direction of the incident rays by a distance a. In Fig. 179 call 2ϵ the distance between two homologous points A, B of the apertures, Φ the angle between the incident ray and AB, and χ the angle between the diffracted ray and AB. Let D be the position of B before it was moved forward through the distance a. We require the path-difference between the parallel diffracted rays AE and BF. When the incident wave-front reaches AD, the secondary disturbance leaves A, travelling along AE and reaching a point E (so situated that $AE = a$) at the moment when the incident wave reaches B. Let fall a

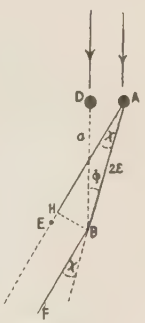


Fig. 179.

perpendicular from B upon AE meeting it at H, which may be above or below E according as χ is greater or less than Φ. If $\Phi=\chi$ the two points will coincide, since then the rt. triangles DAB and AHB will be equal. The path-difference between the parallel diffracted rays is evidently HE, or

$$AE-AH=a-AH=2\epsilon(\cos\Phi-\cos\chi).$$

If Φ and χ are small this is approximately equal to

$$\epsilon(\chi^2-\Phi^2).$$

Substituting this in the expression obtained in the last article we get

$$I=2[f(\delta,\delta')]^2\left(1+\cos 2\pi\frac{\epsilon(\chi^2-\Phi^2)}{\lambda}\right).$$

The minima will be given by $2\pi\frac{\epsilon(\chi^2-\Phi^2)}{\lambda}=(2m+1)\pi$

or

$$\chi^2=\Phi^2+\frac{(2m+1)\lambda}{2\epsilon}.$$

This equation represents concentric circles surrounding a point on the prolongation of AB. These minima of course correspond to the minima described in the last article. When the two apertures are side by side the minima are practically vertical straight lines. Shifting one of the sources in the direction of the incident light-rays causes the minima to become arcs of circles, the centre of the concentric system coming nearer to the image of the source as the apertures are further displaced. If the apertures were in line, *i.e.* if $\Phi=0$, the image of the source would be at the centre of the system.

This will be the case treated in the next article, in which, however, the apertures are replaced by small opaque particles. The linear and circular minima, which we have discussed in this and the preceding article, should be compared with the minima produced by two similar sources in directions at right angles to, and parallel with, the line joining them.

Diffraction by Small Particles on the Surface of a Mirror. — The so-called diffusion rings observed when a small source of light is viewed in a silvered glass mirror, the front surface of which is slightly dimmed with a deposit of dust, such as lycopodium, are in reality diffraction phenomena. They are sometimes erroneously attributed to the interference of diffused light, and Stokes was the first to treat them as diffraction effects.

We have here a case of the interference of the secondary disturbances from a particle interfering with the reflected secondary disturbances from the same particle, the path-difference depending on the thickness of the glass plate and its refractive index. By a suitable arrangement of the apparatus employed for viewing them, they may be brought under the Fraunhofer class of diffraction phenomena. The method is due to Lommel. We require the normal incidence of parallel rays upon the surface of the mirror, and a lens

or telescope for the purpose of rendering concave the reflected wave-front.

Sunlight is concentrated upon the slit of a spectrometer, which should be wide open. The parallel rays emerging from the collimator are reflected from a piece of plane-parallel glass, placed on the table of the instrument at an angle of 45° (Fig. 180), against the dusted mirror, and by this back through the inclined reflector into the telescope, in which the image of the source is seen surrounded by brilliant colored rings.

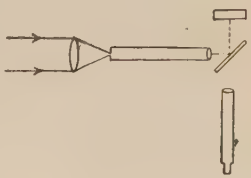


FIG. 180.

The investigation can be simplified by considering the glass plate absent, *i.e.* by reducing the diffracting system to a reflecting surface with a large number of small particles lying in a plane parallel to and in front of it. We may further simplify the case by considering the reflecting surface absent, and a second layer of particles, absolutely identical with the first, occupying the position of the reflected image of the first layer. The source of light we consider the vertical image of the actual source, seen behind the reflecting surface. The case as it now stands is a source of light at an infinite distance, two parallel equidistant layers of small particles identical with each other, and a lens for bringing the parallel diffracted rays to a focus where they interfere. If the distance between the layer of dust and the reflecting surface is ϵ, the distance between the two layers in the modified case is obviously 2ϵ. Every particle in one layer has a similar neighbor in the direction from which the light comes, and we will consider the particles so small, in comparison to the angular diameter of the light source, that they do not shade their neighbors to any sensible degree. Let A and B be two particles at distance 2ϵ (Fig. 181). We are to investigate the mutual interference of the diffracted disturbances in the direction of the dotted lines, making an angle χ with the incident rays. The path-difference will be the same as in the case treated in the last article, $2\epsilon(\cos\Phi - \cos\chi)$. For normal incidence $\Phi = 0$. If I_1 is the intensity of the field in the direction χ, due to a single layer of particles, the intensity when both layers are present will be

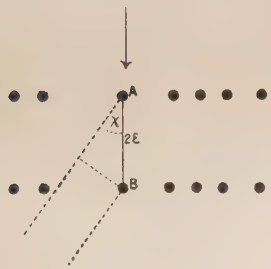


FIG. 181.

$$I = 2I_1\left(1+\cos 2\pi\frac{2\epsilon(\cos\Phi-\cos\chi)}{\lambda}\right).$$

The position of the maxima and minima, due to the second variable factor, are given by

$$2\epsilon(\cos\Phi-\cos\chi) = m\frac{\lambda}{2},$$

odd values of m giving minima, even maxima

$$\cos \chi = \cos \Phi - \frac{m\lambda}{4\epsilon}.$$

The maxima and minima are concentric circles, the common centre of which is in the direction of the normal to the reflecting surface. The white central maximum for which $m=0$ is given by $\chi=\Phi$. This means that if we incline the mirror, the centre will move to one side and eventually disappear, the fringes becoming approximately straight.

Fresnel Diffraction Phenomena. — In the preceding section we have discussed various diffraction problems under the simplified conditions of light source and screen at infinity. We will now proceed with the more general treatment of cases in which the source and screen are both at finite distances from the diffracting aperture, and no lenses are used for rendering the rays parallel or convergent. We cannot now treat the phase as the same at all points in the plane of the diffracting aperture, nor can we solve the problems by determining the resultant of parallel disturbances as in the Fraunhofer class.

Fresnel discussed only the diffraction patterns produced by screens bounded by straight lines of infinite length, such as straight edges, wires, and slits. He first showed that the relative intensities at different points on the projection screen, along a line perpendicular to the diffracting edge, could be determined by considering only the secondary disturbances coming from a circular section of the wave-front, the problem reducing itself to the discussion of the resultant of an infinite number of disturbances from a limited portion of a linear circular wave.

In Fig. 182 AB and CD are the sections of the projection screen and spherical wave-front respectively, F is the section of the diffracting screen, and O the source of the light. The *relative* illumination along AB will be the same whether we take the resultant of the disturbances from the circular *section* of the wave which is not intercepted by the screen F, or the disturbances from that portion of the *complete* spherical wave which is not screened off.

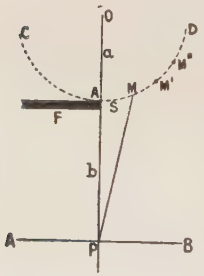


FIG. 182.

We have then to determine the effects at a point P of disturbances coming from points M, M', M'', etc. Let the distance from O to the edge of F be a, and from F to the screen b, and let ds be a small element of the wave at A. If the displacement at A be proportional to $\sin 2\pi \frac{t}{T}$, that at P, contributed by ds, will be $\sin 2\pi\left(\frac{t}{T}-\frac{b}{\lambda}\right)ds$, while an element at M will contribute a displacement represented by $\sin 2\pi\left(\frac{t}{T}-\frac{b+\delta}{\lambda}\right)ds$, in which $b+\delta=MP$.

The displacement at P due to the simultaneous action of all the elements ds of the circular arc will be

$$\int \sin 2\left(\frac{t}{T}-\frac{b+\delta}{\lambda}\right)ds,$$

and the intensity (compare article on Diffraction by Parallel Slits)

$$I=\left(\int \cos 2\pi\frac{\delta}{\lambda}ds\right)^2+\left(\int \sin 2\pi\frac{\delta}{\lambda}ds\right)^2,$$

in which we have resolved each disturbance into two rectangular components, which are separately added.

If we can confine our attention to points not far removed from A we can write $\delta=\frac{s^2(a+b)}{2ab}$, as can be easily shown by considering a and b as the longer sides of two right triangles similar to the small triangles which have the side s in common; δ is then equal to the sum of the short sides of the small triangles.

This gives us for the intensity

$$I=\left[\int \cos\pi\frac{(a+b)s^2}{ab\lambda}ds\right]^2+\left[\int \sin\pi\frac{(a+b)s^2}{ab\lambda}ds\right]^2.$$

Writing $\frac{\pi(a+b)s^2}{ab\lambda}=\frac{\pi}{2}v^2$, which gives us

$$s=v\sqrt{\frac{ab\lambda}{2(a+b)}} \text{ and } ds=\sqrt{\frac{ab\lambda}{2(a+b)}}dv,$$

and the expression for the intensity reduces to

$$I=\frac{ab\lambda}{2(a+b)}\left[\left(\int \cos\frac{\pi}{2}v^2dv\right)^2+\left(\int \sin\frac{\pi}{2}v^2dv\right)^2\right].$$

The two integrals occurring in this expression are known as the Fresnel integrals. Integrating them between certain values of v gives us the resultant of the secondary disturbances from a corresponding portion of the wave-front, v varying with s the distance of the wave-front elements from the pole of the wave, the latter taken with reference to the point at which we are determining the illumination. The values of these integrals between 0 and upper limits of various values have been evaluated by different methods by Fresnel, Knochenhauer, Cauchy, and Gilbert, and the results given in tables. As we gradually increase the upper limit, the values of the integrals pass through maxima and minima, approaching $\frac{1}{2}$ as a limit, as we see from substitution in the formula

$$\int_0^\infty \sin mx^2dx=\int_0^\infty \cos mx^2dx=\sqrt{\frac{\pi}{8m}},$$

which gives $\int_0^\infty \cos\frac{\pi}{2}v^2dv=\int_0^\infty \sin\frac{\pi}{2}v^2dv=\sqrt{\frac{\pi}{4\pi}}=\frac{1}{2}$.

Fresnel's method of integration was as follows:

Since the absolute value of the integral remains the same when the upper limit changes sign, it was sufficient to integrate between 0 and $+v$. Assuming the value of the integral to be known between the limits 0 and i, we deduce the expression for the value between i and $i+t$, where t is a small fraction of the unit, for example 0.1. Writing

$$v = i + \frac{2}{t} + u,$$

where u is a variable which increases from $-\frac{t}{2}$ to $+\frac{t}{2}$, we have

$$\int_i^{i+t} \cos \frac{\pi}{2} v^2 dv = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \cos \frac{\pi}{2}\left(i + \frac{t}{2} + u\right)^2 \cdot dv.$$

Fresnel found for this the value

$$\int_{-\frac{t}{2}}^{+\frac{t}{2}} \cos \frac{\pi}{2}\left[i^2 + it + \frac{t^2}{4} + 2\left(i + \frac{t}{2}\right)u \text{ (in which } u^2 \text{ has been neglected as small).}\right.$$

$$= \frac{1}{\pi\left(i + \frac{t}{2}\right)}\left[\sin \frac{\pi}{2}\left(i + \frac{t}{2}\right)\left(1 + \frac{3t}{2}\right) - \sin \frac{\pi}{2}\left(i + \frac{t}{2}\right)\left(i - \frac{t}{2}\right)\right].$$

An expression was developed in the same way for the other integral, and with these formulae Fresnel calculated his table for $t = .1$, and i (in succession) $= 0, .1, .2, .3$, etc., getting values for $\int_0^{.1}$, $\int_{.1}^{.2}$, $\int_{.2}^{.3}$, etc., which by addition give $\int_0^{.1}$, $\int_0^{.2}$, $\int_0^{.3}$.

The values of the integrals are usually given in the form of a table, thus:

TABLE OF FRESNEL'S INTEGRALS (Gilbert)

v	A $\int_0^v \cos \frac{1}{2}\pi v^2 dv$	B $\int_0^v \sin \frac{1}{2}\pi v^2 dv$	v	A $\int_0^v \cos \frac{1}{2}\pi v^2 dv$	B $\int_0^v \sin \frac{1}{2}\pi v^2 dv$
0.0	0.0000	0.0000	2.6	0.3389	0.5500
0.1	0.0999	0.0005	2.7	0.3926	0.4529
0.2	0.1999	0.0045	2.8	0.4675	0.3915
0.3	0.2994	0.0141	2.9	0.5624	0.4102
0.4	0.3975	0.0334	3.0	0.6057	0.4963
0.5	0.4923	0.0647	3.1	0.5616	0.5818
0.6	0.5811	0.1105	3.2	0.4663	0.5933
0.7	0.6597	0.1721	3.3	0.4057	0.5193
0.8	0.7230	0.2493	3.4	0.4385	0.4297
0.9	0.7648	0.3398	3.5	0.5326	0.4153
1.0	0.7799	0.4383	3.6	0.5880	0.4923
1.1	0.7638	0.5365	3.7	0.5419	0.5750
1.2	0.7154	0.6234	3.8	0.4481	0.5656
1.3	0.6386	0.6863	3.9	0.4223	0.4752
1.4	0.5431	0.7135	4.0	0.4984	0.4205
1.5	0.4453	0.6975	4.1	0.5737	0.4758
1.6	0.3655	0.6383	4.2	0.5417	0.5632
1.7	0.3238	0.5492	4.3	0.4494	0.5540
1.8	0.3363	0.4509	4.4	0.4383	0.4623
1.9	0.3945	0.3734	4.5	0.5258	0.4342
2.0	0.4883	0.3434	4.6	0.5672	0.5162
2.1	0.5814	0.3743	4.7	0.4914	0.5669
2.2	0.6362	0.4556	4.8	0.4338	0.4968
2.3	0.6268	0.5525	4.9	0.5002	0.4351
2.4	0.5550	0.6197	5.0	0.5636	0.4992
2.5	0.4574	0.6192	∞	0.5000	0.5000

Knochenhauer developed the integrals in series by partial integration, thus:

$$\int_0^v \cos \frac{\pi}{2}v^2 dv = v \cos \frac{\pi}{2}v^2 + \pi \int_0^v v^2 \sin \frac{\pi}{2}v^2 dv,$$

$$\int_0^v v^2 \sin \frac{\pi}{2}v^2 dv = \frac{v^3}{3} \sin \frac{\pi}{2}v^2 - \frac{\pi}{3}\int_0^v v^4 \cos \frac{\pi}{2}v^2 dv,$$

$$\int_0^v v^4 \cos \frac{\pi}{2}v^2 dv = \frac{v^5}{5} \cos \frac{\pi}{2}v^2 + \frac{\pi}{5}\int_0^v v^6 \sin \frac{\pi}{2}v^2 dv,$$

. . . . . .

his final expression giving the integral in the form of the sum of two convergent series:

$$\int_0^v \cos \frac{\pi}{2}v^2 dv = \cos \frac{\pi}{2}v^2\left(v - \frac{\pi^2 v^5}{1\cdot 3\cdot 5} + \frac{\pi^4 v^9}{1\cdot 3\cdot 5\cdot 7\cdot 9\ldots} - \cdots\right)$$

$$+ \sin \frac{\pi}{2}v^2\left(\frac{\pi v^3}{1\cdot 3} - \frac{\pi^3 v^7}{1\cdot 3\cdot 5\cdot 7} + \cdots\right).$$

The convergence becomes less as v increases, consequently the expression can only be used for small values of the upper limit.

Cauchy in a somewhat similar way developed the integral in series which were convergent for large values of v.

Turning back now to the expression for the illumination, we see that it consists of the sum of the squares of two integrals. The two integrals, therefore, represent the components along two rectangular axes of the resultant amplitude. The illumination is thus represented by the square of a line joining the origin with a point, the coordinates of which are the two integrals. Taking ξ and η as the coordinates of the point for different values of v, we will investigate the curve along which the point moves as v varies. This geometrical discussion of the equation is due to Cornu, and the curve is known as Cornu's Spiral. By its aid the classical problems of diffraction can be solved in a geometrical manner, the intensity curve of the diffraction pattern being plotted from measurements made on the spiral.

Cornu's Spiral. — Let

$$\xi = \int_0^v \cos \frac{\pi v^2}{2} dv, \quad \eta = \int_0^v \sin \frac{\pi v^2}{2} dv.$$

The curve passes through the origin, since for $v=0$, ξ and η also equal zero. Changing the sign of v does not change the values of ξ and η, but only their sign; the curve is therefore symmetrical about the origin.

The tangent to the curve makes an angle τ with the ξ axis given by

$$\tan \tau = \frac{d\eta}{d\xi} = \tan \cdot \frac{\pi v^2}{2} \text{ or } \tau = \frac{\pi}{2} v^2.$$

At the origin, where $v=0$ the curve is parallel to the ξ axis. For $v=1$ or $s=1$ it is parallel to the η axis, for $s^2=2$ again parallel to the ξ axis, and for $s^2=3$ parallel to the η axis.

The radius of curvature is given by

$$\rho = \frac{ds}{d\tau} = \frac{1}{\pi v} = \frac{1}{\pi s}.$$

For $v=0$ the radius is infinite, and the curve has a point of inflection at this point: as v increases the radius decreases, the curve having the form of a double spiral, which winds about the asymptotic points P and P', which correspond to the values of the integrals when the upper limits are

$$+\infty \text{ and } -\infty.$$

We can construct the curve by employing the equations for the radius of curvature, and the angle which the tangent makes with the ξ axis.

A small element of the curve at distance ξ from the origin corresponding to $s=0.1$ has a radius of curvature $\rho = \frac{1}{\pi s} = \frac{10}{\pi}$, the centre of curvature lying in a direction from the point such that it

makes the same angle with the η axis as the tangent makes with the ξ axis, viz.,

$$\tau = \frac{\pi s^2}{2} = .01\frac{\pi}{2}.$$

On this circle we lay off the arc $s = .1$. We proceed in the same way for points corresponding to $s = 0.2$, $s = 0.3$, etc., and thus build up the entire curve. This somewhat laborious geometrical method is not necessary, for in Fresnel's table of integrals we have successive values of ξ and η and can plot the curve at once by taking them as ordinates and abscissae. In this way the curve shown on Plate V. at the end of the book was constructed. This curve we have already used in the solution of certain diffraction problems. We will now examine the method employed by Fresnel.

Diffraction by a Straight Edge. — In the elementary treatment we have seen that the illumination within the geometrical shadow falls off gradually without showing maxima and minima, while outside of the edge of the shadow we have maxima and minima, which decrease in distance, and become more nearly of the same intensity as we recede from the edge, until finally we have uniform illumination. We will now apply Fresnel's expression to a point P within the geometrical edge of the shadow. The integration must be taken from the edge of the screen to infinity. The pole of the wave with reference to P is cut off by the screen, and if S represents the arc from the pole to the edge, the expression for the intensity is obviously

$$I = \left[\int_s^\infty \cos\frac{\pi(a+b)s^2}{ab\lambda}ds\right]^2 + \left[\int_s^\infty \sin\frac{\pi(a+b)}{ab\lambda}ds\right]^2,$$

or introducing the quantity v as before,

$$I = \frac{ab\lambda}{2(a+b)}\left[\left(\int_v^\infty \cos\frac{\pi}{2}v^2dv\right)^2 + \left(\int_v^\infty \sin\frac{\pi}{2}v^2dv\right)^2\right],$$

in which

$$V = \sqrt{\frac{2(a+b)}{ab\lambda}}\,S.$$

Let x equal the distance of the point P from the edge of the geometrical shadow (Fig. 183),

$$x = \frac{a+b}{a}S = \sqrt{\frac{(a+b)b\lambda}{2a}}\cdot V.$$

Write $\int_0^v \cos\frac{\pi}{2}v^2dv = C_v$, $\int_0^v \sin\frac{\pi}{2}v^2dv = S_v$.

We have shown that the integrals taken between 0 and ∞ are both equal to $\frac{1}{2}$, therefore

$$I = \frac{ab\lambda}{2(a+b)}[(\tfrac{1}{2}-C_v)^2 + (\tfrac{1}{2}-S_v)^2].$$

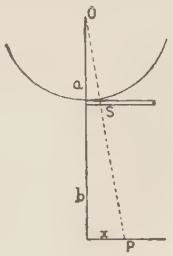


FIG. 183.

The value of the quantity within the brackets was determined by Fresnel for different values of V (which corresponded to certain

values of S and x) by the method which we have already seen. It was found that the illumination fell off rapidly without passing through maxima and minima, becoming 0 as soon as V reached any considerable value. In the same way the illumination outside of the edge of the shadow was found to depend on the quantity

$$(\tfrac{1}{2}+C\ )^2+(\tfrac{1}{2}+S_r)^2.$$

The values calculated from the Fresnel tables for this sum showed that it passed through maxima and minima, the increment of x necessary for the change becoming less as the distance from the edge of the shadow increased.

The geometrical solution of this and kindred problems by the aid of the Cornu spiral which we have already considered is much more convenient and direct however.

Diffraction by Thin Laminae. — Laminary diffraction phenomena of the Fraunhofer class have already been briefly discussed in connection with the theory of the diffraction grating. We will now examine the case of diffraction by a straight edge, when the screen, instead of being opaque, consists of a thin transparent lamina of thickness ϵ and refractive index n. The path-difference between two rays, one passing *by* the edge and the other *through* the edge of the lamina, is $(n-1)\epsilon$. The phase-difference Δ will be given by

$$\frac{\Delta}{2\pi}=\frac{(n-1)\epsilon}{\lambda},\quad \Delta=2\pi\frac{(n-1)\epsilon}{\lambda}.$$

If this is an odd multiple of π, the edge of the geometrical shadow of the lamina will be in total darkness, for the disturbances from homologous points of the lamina and the clear space reach points situated on the edge of the shadow with a path-difference of half a wave-length, and mutually destroy one another. There will in addition be interference fringes both within and without the edge of the shadow, which we shall investigate presently.

If Δ equals an even multiple of π, not only will there be no minimum at the edge of the shadow, but the fringes will disappear, the illumination being the same as if the lamina were absent. Owing to the dispersion in the lamina, the former condition may hold for one color, the latter for another, and we may have the fringes very distinct with blue light, and scarcely visible with red.

Assuming that we are working with monochromatic light and a lamina giving a half-wave retardation, we can easily construct the intensity-curve with the help of the Cornu spiral.

The whole wave is utilized in this case, the portion of the spiral representing the part passing through the lamina being rotated through 180°. For $x=0$ the upper half of the spiral is rotated around the origin through angle π and comes into coincidence with the lower half. The vector sum of the lines from the asymptotic points to the origin is zero, since they are equal and oppositely directed, and the illumination is therefore 0 at the center of the system.

As we increase x we have our moving point tracing out maxima and minima as in the case of the opaque straight edge, but we

shall find that the effect of adding the vector of the portion of the wave which was before left out, and is now utilized after a rotation of 180°, is to increase the intensity at the maxima and decrease it at the minima. For example, with an opaque screen the amplitude would be JM for the value of x giving the first maximum (Fig. 184). With the lamina screen the portion of the spiral MJ' is used. The natural direction of the vector is towards J', but the half-wave retardation directs it away from J', the resultant being JM', which is larger than JM. If our point has moved around to N, the first minimum, the direction of the smaller vector after rotation is $J'N$, which, being opposed to JN, means a subtraction, and a smaller value of the amplitude.

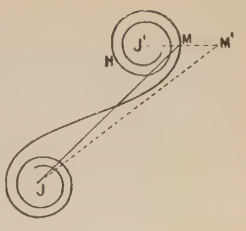


Fig. 184.

It is obvious that the fringes are similar on both sides of the central minimum, for the above construction applies in whichever direction we take x.

The fringes are best shown by means of a lamina of variable retardation, which can be made by etching a strip of plate glass with dilute hydrofluoric acid.

Fig. 185.

One-half of the strip is protected by means of a coating of paraffine, running parallel to the long edge. The paraffine can be applied with a brush, or better by heating another strip of glass and pressing the melted paraffine between the two strips. With a little practice it is possible to obtain in this way a uniform coating bounded by a straight edge. The acid, diluted with about five parts of water, is placed in a beaker glass, previously coated with paraffine, and the strip of glass lowered very gradually into it, a millimetre or two per second, until about three-quarters is immersed. It is then quickly withdrawn and transferred to a dish of water to stop the action of the acid.

After removing the paraffine a fine line will be seen running down the middle of the plate, marking the division between the regions of different thickness. Sun or arc light is concentrated upon a pin-hole in a thin metal screen, or upon a slit, and the strip mounted vertically at a distance of two metres. The fringes can be observed with an eye-piece at a distance of a metre behind the plate. The unetched portion of the plate represents the retarding lamina, and we have an increasing retardation varying from 0 to several wavelengths as we pass down the plate. Where the retardation is considerable, we find the centre of the system brilliantly colored.

A photograph of the fringes obtained in this way is reproduced in Fig. 185, the retardation at various points being indicated. Where it amounts to one or more whole wave-lengths the centre of the system is bright. A still easier way to make the lamina is to flow a very dilute solution of celluloid in amyl acetate over a strip of glass, standing it on edge to dry. Celluloid solution can be easily obtained, as it is sold with most brands of gilt paint; it must be diluted, however, with the amyl acetate. After the film is dry make a straight cut down the centre with a sharp knife, and remove one-half of the film, which is best done by placing a few drops of water at the edge and lifting it up with the point of a knife.

The Colors of Mixed Plates. — Interference colors of this type were discovered by Young, and described in the *Philosophical Transactions* for 1802, and were subsequently studied by Sir David Brewster. Verdet and other writers on Optics have classified them with Newton's thin-film colors, and have given treatments which are not very rigorous, and fail to show where the energy goes to.

The colors are very easily obtained by pressing a little white of egg between two pieces of plate glass, separating the plates and squeezing them together a number of times so as to form a froth. The plates are to be pressed firmly together with a rotary sliding motion just before the froth becomes sticky, enclosing a film made up of air and albumen in the form of a mosaic. The colors are best seen by holding the plate towards a distant window or other bright source of light on a dark field. Certain wave-lengths will be found to be absent in the directly transmitted light. Young's explanation was that the path-difference between a ray passing through an air-space and one passing through the albumen was an odd number of half wave-lengths for such colors as failed to appear in the transmitted light. Neither Young nor subsequent writers, so far as I have been able to find, show what becomes of these absent colors, though both Young and Brewster observed the colored fringes which appeared against the dark background to one side of the source of light. Brewster published a paper in the *Philosophical Transactions* for 1837 in which he referred the colors to diffraction, though his treatment was not very complete, and concerned chiefly the case of diffraction by a transparent lamina bounded by a straight edge. Verdet objected to this explanation on the ground that the colors are independent of the size and arrangement of the air-bubbles, depending only on the thickness of the albumen-film and the angle of incidence. The interference phenomena of mixed plates are easily explained by the elementary theory of diffraction, and they should be classed with laminary diffraction effects, and not with thin film interferences, as is usually the case.

In Fig. 186 let AA represent the glass plates with the albumen and air elements between them. We will assume the thickness of the albumen such that green light suffers a retardation of $\frac{\lambda}{2}$ in traversing it. If B is the lens of the eye, and parallel rays traverse the plate, the secondary disturbances represented by the dotted lines (normally diffracted rays) will be brought to a focus at E; that is,

the reduced paths of all these rays are equal and the disturbances arrive at E in the same phase, if there be no retardation. The disturbances coming from the albumen elements are retarded, however, and reach E half a wave-length behind the disturbances coming from the air elements. The two sets destroy each other at this point, and green light will not be represented here. In general, light will be absent at this point if the retardation of a ray passing through albumen with respect to one passing through the adjacent air-space is $(2n+1)\frac{\lambda}{2}$. If the film is fairly thick, this condition may hold for a number of colors in the spectrum, which will consequently be absent in the image of any source of light seen through the plate. The question now is: What becomes of this energy? In the case of thin film interferences, the wave-lengths absent in the transmitted light appear in the reflected. This is not the case with mixed plates, which show little or no color by reflection. If we refer to Fig. 186 it is clear that if we take parallel rays diffracted in an oblique direction, the phase-difference introduced by the retardations in the mosaic may be compensated by the obliquity, the agreement of phase being more or less complete for green light in the point F. The case is analogous to a laminary grating, which yields colored central images, the absent wave-lengths appearing in the spectra. Mixed plates throw the light absent in the direct image into a halo or ring, which is seen to surround the source of light.

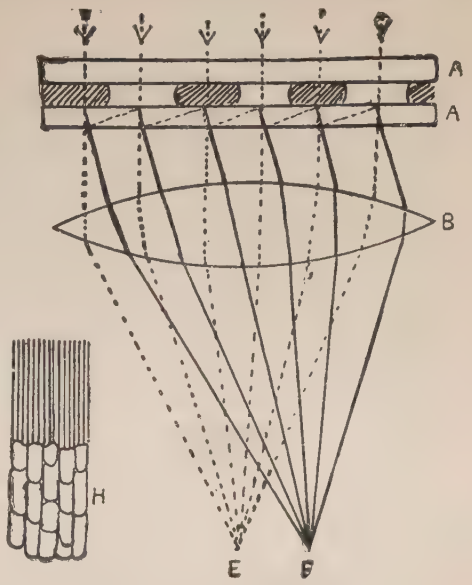


FIG. 186.

Laminary diffraction phenomena, which we have just discussed, and mixed plates belong to the same class, the case being best defined as laminary diffraction by a great number of irregularly distributed transparent disks. If the patches of the mosaic were of uniform size, the halo would be sharply defined and separated from the direct image by a dark space, which would become wider as the size of the elements of the mosaic decreased. Though it is easy to obtain very perfect halos in some cases, separated by a dark area of considerable size, the variation in the size of the elements usually causes the halo to take the form of a disk, the centre of which is occupied by the direct image.

If the plates are held close to the eye and a distant lamp-flame

viewed through them, the flame will, for example, appear purple and the surrounding halo green. If a small sodium flame is employed, parts of the mosaic will show it much blurred, and surrounded by a halo, while other parts, where the retardation is a whole number of half-waves, show it perfectly sharp and distinct. The distribution of the light in the halo depends on the form of the elements of the mosaic. By pressing the plates firmly together and sliding one over the other, the circular air-bubbles can be deformed into ellipses. The light in the ring will be more or less concentrated on opposite sides of the halo. If the ellipses were drawn out indefinitely, we should pass over to the grating, and the points of concentration would become first order spectra, the rest of the halo disappearing.

A very curious and interesting example of this concentration of light in a halo was observed by the author when copying some diffraction gratings on bichromatized albumen. The original grating was ruled on glass, 14,400 lines to the inch, a spacing so fine that copies were only obtained with considerable difficulty.

Some of the films were found to have frilled in the process of washing, the buckling of the film following the grooves of the grating to a certain extent. The albumen surface was seen by the microscope to have frilled into oval patches of varying length, but of fairly constant width, the width being equal to three lines of the original grating. In Fig. 186, *H*, we have a diagram illustrating this condition. This plate when held before the eye showed a ring of wide aperture surrounding a brilliant source of light, with four distinct concentrations, two very bright, and two quite faint. The appearance reminded one most forcibly of a solar halo, with parhelia or mock suns. A photograph of this curious diffraction pattern was made by directing a camera towards a brilliant point source of light, and placing one of the frilled plates before the lens. This photograph is reproduced in Fig. 187.

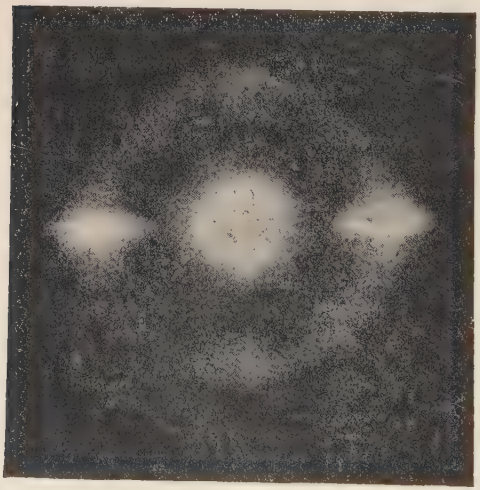


Fig. 187.

The arrangement of the colors in the "mock suns" produced in this way is, however, exactly the opposite of the arrangement in the real ones, since they are produced by diffraction instead of refraction. The parallel oval patches correspond to the oriented ice crystals which produce the parhelia, as we shall see in the Chapter on Meteorological Optics.

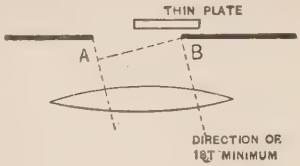
Talbot's Bands. — A curious type of interference bands was de-

scribed by Talbot[1] in 1837. If, when viewing a continuous spectrum in a spectroscope, we place a thin piece of glass or mica in front of one-half of the pupil of the eye, with its edge towards the red end of the spectrum, a set of extremely black bands appears, crossing the spectrum parallel to the Fraunhofer lines. If, however, the thin plate be turned with its edge pointing towards the violet, no trace of the bands is to be seen. A thin cover-glass, such as is used for microscope slides, answers the purpose. If the glass is placed in front of the telescope lens it must be introduced from the side towards the blue end of the spectrum.

Talbot gave an imperfect explanation of the bands on the elementary principles of interference. The thin plate retards certain colors an odd number of half wave-lengths. These waves arrive at the retina in a condition to interfere destructively with the waves which enter the uncovered portion of the pupil, the lens of the eye bringing them to the same point, consequently these colors are absent in the spectrum. The distinctness of the bands depends on the thickness of the plate, and Talbot's explanation neither accounts for this nor for the apparently remarkable circumstance that the bands do not appear at all when the plate is turned with its edge the other way, covering the other half of the pupil, a change which should produce no effect on the interference theory; neither is it quite clear what becomes of the light. The correct explanation was first given by Airy.[2] His treatment involves so much mathematics that it is difficult to follow the physical significance of the different steps.

The following elementary treatment will make clear just how the bands are formed, and why they appear only when the plate is introduced on one side. Compare this case with the laminary grating just treated.

Incidentally we shall find that the echelon grating, which will be described in the next chapter, is a special case of Talbot's plate, the aperture partly covered by the thin plate forming an echelon grating of two elements. To see the bands at their best, the object-glass of the spectroscope should be covered with a screen perforated with an aperture about 1 cm. in height by 5 mms. in width. The glass or mica plate can be mounted so as to cover the right or left hand half of this aperture, or simply held in front of the eye.



Let AB, Fig. 188, be the aperture, one-half of which may be covered with the retarding plate.

In the absence of the plate, the lens will form an image of the slit (which for the present we will consider illumi-

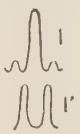


FIG. 188.

[1] Gilbert, *Ann.*, 1837.
[2] Airy, *Pogg. Ann.*, liii.-lviii.; *Philos. Trans.*, 1840, vol. ii.; 1842, vol. i.

nated with monochromatic light), bordered by faint maxima and minima. For the present we consider the prism removed, and the telescope directed in line with the collimator. The intensity curve, which represents the image of the slit diagrammatically, is shown at I.

The first diffraction minima to the right and left of the central maximum will lie in directions such that the path-difference between the disturbances coming from the edges of the aperture is a whole wave-length. If the right-hand half of the aperture is covered with a transparent plate which retards the wave-train one-half wave-length, we shall have zero illumination at the centre, and two bright maxima of equal intensity in the position previously occupied by the first minima, as shown at I'. It is clear that the path-difference for the left-hand maximum is one wave-length, and for the right-hand one zero. We may therefore call the former the spectrum of the first order, and the latter the spectrum of zero order, considering the aperture as a grating of two elements. If we consider the change as taking place progressively, starting with a transparent lamina of infinite thinness which gradually thickens, it is clear that the central maximum moves to the right, decreasing in intensity, while its neighboring maximum to the left increases in brightness moving in the same direction. When the thickness corresponds to a half-wave retardation these two maxima are of equal intensity, and occupy the positions previously occupied by the minima to the right and left of the central maximum. If we increase further the thickness of the plate the maxima drift further to the right, the left-hand one increasing in brilliancy, the right-hand one decreasing. When the retardation becomes one whole wave-length, we again have a single bright maximum at the centre, as with the uncovered aperture. In this case, however, it is the spectrum of the first order which is at the centre. As we go on increasing the thickness of the plate the march of the fringes continues, new ones coming into view on the left and disappearing on the right, and the order of spectrum increasing. The condition when we have a pair of bright maxima will be recognized as the condition which obtains when an echelon grating is set for the position of double order, the single order position corresponding to the case when we have a single bright maximum at the centre.

Suppose now that we have a thick retarding plate and gradually decrease the wave-length of the light which illuminates the slit from which the plane waves are coming. We shall have the same march of the fringes as before, the gradual increase in retardation resulting from the decrease in wave-length. If our source emits two wave-trains of different wave-length, and the retardation for one of the trains is an odd, and for the other an even number of half wave-lengths, it is clear that the single and double maxima exist simultaneously, and all trace of the maxima and minima will disappear owing to the form of the intensity curve; in other words, the lines are not resolved.

If, however, we increase the number of retarding plates, placing them so as to divide the aperture into a number of vertical strips

of equal width, the width of the maxima decreases in proportion to the distance between them, as in the case of the ordinary diffraction-grating when we increase the number of lines, and we have resolution of the lines. Our aperture is now acting as an echelon grating, set in position of single order for one train of waves and double order for the other. With very thick plates, as in the case of echelons such as are actually used, it is evident that a very minute change in wave-length will be sufficient to change the retardation by the amount necessary for resolution. We are now in a position to discuss the effect of the thickness of the plates, their number, and the width of the steps. Decreasing the width of the steps increases the width of the space between the spectra, in other words, forms the system of maxima and minima on a larger scale. Increasing the thickness decreases the change in wave-length necessary in passing from the condition of single order to that of double order, in other words the greater the thickness the greater will be the shift of a maximum for a very minute change in λ, and in consequence the greater the resolving power. An increase in the number of plates merely reduces the width of the maxima, without affecting the distance between them, or the amount of shift produced by a given change of λ. This then is the whole theory of the echelon in a nut-shell (a coco-nut perhaps).

We are now ready for the Talbot fringes.

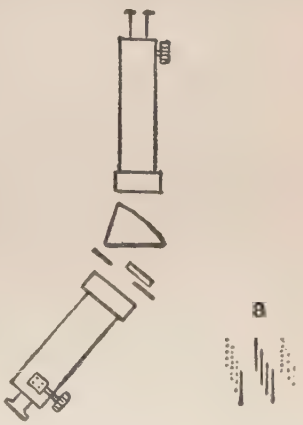


FIG. 189.

Consider the slit of a spectrometer illuminated with monochromatic light, the prism, aperture, and retarding plate arranged as shown in Fig. 189. If the plate retards the stream an even number of half wave-lengths, we shall see in the instrument a single bright line with accompanying faint maxima and minima. If we now decrease the wave-length a trifle the line will move a little to the right, if we disregard the action of the prism. We can verify this experimentally by removing the prism and viewing the slit directly with the telescope. The slit can be illuminated with a monochromatic illuminator (spectroscope with narrow slit in place of the eye-piece), the slit of the instrument being substituted for the slit of the first spectroscope.

As we gradually decrease the wave-length by turning the prisms of the illuminator, we shall see the line become double and single in succession, the doubling being accomplished by the march of the fringes to the right and their fluctuating intensity in the manner already described. It is a little difficult to put into words the changes which accompany the alteration of wave-length. The appearance is much like that presented by a picket fence in motion viewed through a narrow vertical aperture, *i.e.* the pickets are only visible as they pass across a narrow vertical region. If we imagine

the visibility of the pickets to be a maximum as they cross the centre of the aperture and least when at the edges, we shall have a fairly accurate conception of the appearance of the moving maxima. Suppose that a change of 100 Ångström units in the wave-length is sufficient to change the single line to the double line. The changes which take place in the focal plane of the telescope (prism removed) as we decrease the wave-length over this range are indicated in Fig. 189 *a*. To avoid confusion I have represented each successive appearance a little lower down in the figure.

Now consider the prism in place, turning the telescope so as to view the deviated image. As we decrease the wave-length there will be a shift to the *right* as before, resulting from the retardation of the plate, and a shift to the *left*, due to the increasing deviation produced by the prism. If the compensation is perfect, the prismatic dispersion will shut up the picket-fence arrangement of lines in the diagram into single lines; in other words, if we illuminate the slit simultaneously with all of the wave-lengths in the given range of 100 Ångström units, we shall not see a short continuous spectrum, but single bright lines, to the intensity of which all the waves contribute. These lines are of course much narrower than the corresponding range of the continuous spectrum which would be formed in the absence of the plate. We thus see that the retardation has the effect of compressing a narrow region of the spectrum into a much narrower one which constitutes one of the bright Talbot bands. If we consider what happens to the entire spectrum, we can perhaps obtain a still clearer idea of how the bands are formed.

Consider a large number of equidistant monochromatic constituents of the spectrum: call these the lines A. Midway between them is another set of lines B. The distance from a line A to B is such that the difference in the retardation for the two lines is half a wave-length. If the retardation for the A lines is an even number of half-waves, they will remain single and fixed in position. The lines B will, however, double and fall upon the neighboring lines A. The regions previously occupied by the lines B are the dark Talbot fringes. It may at first sight appear as if the same result would be obtained regardless of the side on which the retarding plate was introduced. It obviously would for the wave-lengths A and B; but if we consider the elements of the spectrum between the A and B lines, we shall see that in one case they are shifted so as to fall upon the stationary lines A, while in the other they are moved into the region between. In the latter case we have a continuous spectrum if we consider the widths of the regions A and B infinitely narrow. In the foregoing treatment we have considered the condition as that of the "best thickness" which obtains when shifts due to retardation and prismatic dispersion exactly compensate. It is clear also that to obtain the bands the aperture must be somewhat restricted so as to obtain an appreciable doubling of the lines for which the retardation is an odd number of half-waves.

In this treatment we have considered white light as made up of an infinite number of monochromatic constituents, the older view.

Professor Schuster has given a very neat explanation of the failure of the bands to appear, when the plate is introduced from the wrong side, considering white light as consisting of irregular pulses without periodicity, the *periodicity* or regularity of light in the spectrum resulting from the dispersing apparatus which produces it. This idea was first advanced by Lord Rayleigh in considering the action of the diffraction grating when analyzing white light; a matter which we shall investigate more in detail when we come to the chapter on White Light. Suppose we have a source of white light at S (Fig. 190) which is emitting non-periodic impulses.

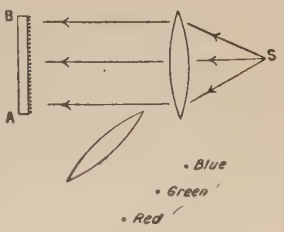


FIG. 190.

Let one of these impulses fall upon the grating. If our source is at a great distance, or if we use a collimating lens, secondary waves will leave the various elements of the grating at the same moment. Obviously we can choose three points, designated "Blue," "Green," and "Red," respectively, such that the separate impulses from the grating elements will pass through them (in succession) with the periodicity of blue, green, and red light. We thus see why the red, with its slow periodic impulses, is further removed from the normal to the grating than the blue, or, in other words, *why the grating constructs* red light at a point further removed.

Our light will not be monochromatic at the three points unless we add a lens to the system, for the inclination of different parts of the grating to the lines joining them with the points will be different; that is, different parts of the grating construct different colors at a given point. With the lens added, we have, however, monochromatic light at points lying in the focal plane of the lens.

If now we retard the impulses coming from one-half of the grating half a wave-length, they will destroy the impulses coming from the other half, *provided* the two sets traverse the point simultaneously. Clearly we must introduce our plate on the *red* side if we are to accomplish this, for if we introduce it on the other side, we retard a set of impulses which is already behind the set with which we wish to make it interfere. This is equivalent to introducing it on the blue side, if we put the plate between the spectrum and the eye, as can be easily seen by constructing a diagram illustrating the formation of the image of the spectrum. The rays cross at the point, and the bundle which we must retard, which was originally on the red side, is now found on the opposite side. If the retarding plate is placed between the grating and the telescope lens of the spectroscope, it must be introduced from the *red* side, instead of the blue, as is easily proved by experiment. The best thickness of the plate is such as will divide the whole series of impulses into two equal portions, which arrive at the point in pairs; *i.e.* an impulse from the near edge of the grating and one from the central element should reach the point simultaneously. If N be the total number of lines of the grating, the best retardation is therefore $\frac{1}{2} N\lambda$,

and the plate should intercept exactly one-half of the beam. The value λ here means the wave-length of the light which the grating constructs at the point. If the retardation is greater or less, some of the impulses arrive either too soon or too late to interfere with others, and the bands are not as clearly defined. The impulses coming from the various elements of the grating need not even be considered as "to-and-fro" to account for interference in the manner supposed. Consider them all in one direction, *i.e.* half-waves, and let the retardation be such as to cause one set to fit exactly halfway between those belonging to the second set. The resulting disturbance at the point would have a periodicity twice as great as it had before; there is therefore light at the point, but it is light which belongs to the overlapping spectrum of the second order. As regards the wave-length λ under consideration there is darkness.

It will be remembered that, in the previous treatment, we saw that the bands were most distinct when the interference maxima were the same distance apart as the diffraction maxima. The same thing can be shown by Schuster's treatment. Quoting from his paper:

"If at a certain point of the spectrum corresponding to wave-length λ there is a *maximum* of light, the relative retardation of the two interfering impulses must be equal to $m\lambda$, m being an integer, the next adjoining band towards the violet will appear at wave-length λ' such that

$$m\lambda = (m+1)\lambda'.$$

"Hence for the distance between the bands

$$\frac{\lambda - \lambda'}{\lambda'} = \frac{1}{m},$$

with the best thickness of the interposed plate $m = \frac{1}{2}N$, and hence

$$\frac{\lambda - \lambda'}{\lambda'} = \frac{2}{N},$$

where λ' in the denominator may with sufficient accuracy be replaced by λ. If λ'' be that wave-length nearest to λ at which there is a minimum of light, it follows that

$$\frac{\lambda - \lambda''}{\lambda} = \frac{1}{N}."$$

This equation shows us that, *under the conditions of* "best thickness" the difference between the wave-lengths of a maximum and its neighboring minimum, divided by λ, is equal to $\frac{1}{N}$. We shall find the same to be true for the diffraction maxima and minima.

"If a linear homogeneous source of light of wave-length λ be examined by means of a grating, the central image extends to wave-length λ_1, such that

$$\frac{\lambda - \lambda_1}{\lambda} = \frac{1}{N},$$

where N as before is the total number of lines on the grating."

In this expression $\lambda - \lambda_1$ is the distance of the first minimum from the central maximum, of the diffraction pattern of a linear source of monochromatic light seen with a grating of N lines, which we have fully considered.

Schuster also shows that if we are using a plate which is not of the " best thickness," we may restore the original contrast between the maxima, by screening off the portions of the beam which are not interfering.

In the case when the spectrum is formed by a prism, the method is not so obvious, as there is apparently no periodic structure to build up the colored light. There is, however, no especial difficulty in this case, as we shall see when we take up the subject of White Light.

Anomalous Propagation of Waves in the Vicinity of Foci. — The curious discovery was made by Gouy in 1890, that in its passage through a focus a wave gained a half wave-length, as if its velocity of propagation was accelerated in the vicinity of the focus.[1] He employed a pair of Fresnel mirrors, one of which was concave, the light falling upon them at nearly normal incidence.

The arrangement of the apparatus is shown in Fig. 191 *a*. An illuminated pin-hole at *S* sends light to the two mirrors at *C* and *D*. *C* is a concave silvered mirror made by cutting a small square from a spectacle lens, *D* is a plane silvered mirror of the same size. The distance of the source *S* from the mirrors is 1.5 metres and the mirror *C* brings the rays to a focus at *F*, about 35 or 40 cms. from *C*. The fringes are observed at *O* with an eye-piece, or better with a microscope of low power (30–40 diameters). If there were no advance of the wave in its passage through the focus, we should have a fringe system with a white centre bordered on each side by colored fringes, as with the ordinary mirrors. What we see is system with a black centre, with lateral fringes of colors complementary to those usually observed.

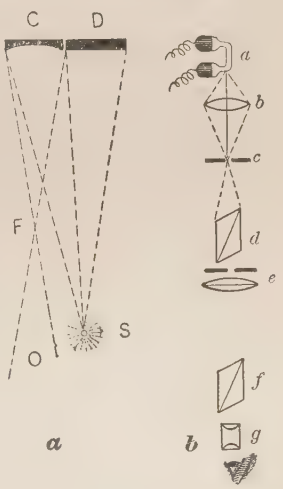


Fig. 191.

The fringes are circular since the two sources, one at *F*, the other at distance *SD* behind the plane mirror, are practically in line. Gouy's experiments were repeated by Fabry [2] and later by Zeeman, who used a lens made of Iceland spar, placed between two Nicol prisms. This lens was doubly refracting and gave two real images of the source, at different distances. A similar arrangement subsequently used by Sagnac and by Reiche is shown in Fig. 191 *b*. The light of a mercury lamp *a* is focussed upon a small aperture *c*, traversing the Nicol *d* and the doubly refracting lens *e*. The latter has focal lengths of 304 and 412 cms., and the

[1] Gouy, *Ann. de Chim. et de Phys.*, 6e Série, 24, pp. 145–213.
[2] Fabry, *Journ. de Phys.*, 3d series, 2, p. 22 (1892).

distance between c and e was about 7 metres. The Nicols are crossed and the lens turned until its principal section makes an angle of 45° with the planes of the Nicols. The circular fringes are observed with a powerful eye-piece. With this arrangement we can see the fringes, not only between the foci, but in front of and behind them. If the diaphragm in front of the lens e has a diameter of 3 cms., we observe a system with a white centre about .5 m. in front of the first focus, *i.e.* between the focus and the lens; on moving the Nicol eye-piece back the fringes become indistinct until we arrive at a point about .5 m. behind the focus, when they again appear, this time with a dark centre, owing to the change of phase produced by the passage of the waves through the first focus. Behind the second focus we again find rings with a white centre. If the diaphragm is contracted to a diameter of 1·cm., it is found that as the focus is approached periodic changes of the centre of the system occur, and the fringe system can be followed through the foci, where *no sudden change occurs*. Sagnac explains the apparent anomalous propagation as a result of diffraction, and shows that we may have a somewhat similar phenomenon, when plane waves pass through a circular aperture, without passing through a focus.[1]

We remember from our study of Huygens's principle that the resultant of the disturbances from a circular zone in the aperture at a point on the axis is retarded in phase with respect to a disturbance travelling from the centre of the aperture along the axis. The phase-difference between the two becomes less as we move the point away from the aperture. It is as if the phase of the resultant vibration from the circular zone was propagated along the axis with a velocity greater than that of a plane-wave, or greater than that of a disturbance from the centre of the aperture, for if it lags less and less upon the phase of the axial disturbance it must be travelling faster. This does not mean, however, that we are actually dealing with an abnormal velocity of light, for the direction of the axis does not coincide with the direction of propagation of the elementary waves from the circular zone. If we have a spherical wave starting from the circumference of the aperture we can regard the velocity with which its point of intersection with the axis travels, as the velocity of the resultant phase. It will be greater than the velocity of light in the vicinity of the aperture and equal to it at a great distance. Now the phase of the resultant at any given instant on the axis, of all of the disturbances coming from the aperture, depends upon the size of the aperture. The same is true of the illumination, which is zero for points so situated that the aperture contains 2, 4, 6, etc., Fresnel zones. If now we define the velocity of light as the velocity with which a given phase of vibration travels along the axis, it is clear that if we restrict the wave to the portion passed by the aperture, the velocity *as defined above* may be modified by diffraction.

We will now investigate exactly what happens in the case of a small aperture, and in the case of a wave passing through a focus.

Take first the case of a circular aperture, through which a wave-

[1] Sagnac, *Journ. de Phys.*, 2, page 721 (1903).

train has passed. If we are at a great distance from the aperture the resultant phase due to all of the disturbances will agree with that due to the one coming from the centre, since the path-differences are practically zero. This resultant phase will be accelerated 90° on the phase which we should have if the wave had not been restricted by the screen with its small aperture, as we saw in the Chapter on Huygens's Principle, for we must consider the secondary wavelets as starting with a phase one-quarter period in advance of the phase on the wave-front. If the phase which we should have at the point if the wave were unrestricted be represented by an arrow pointing vertically, the phase which we have with the small circular aperture can be represented by an arrow pointing to the right, retardations of phase being represented by counter-clockwise rotations of the vector.

If now we approach the aperture along the axis the illumination will increase, reaching its maximum value when we reach a point such that the aperture contains one Fresnel zone. The phase will drift back 90°, however, as a result of the $\frac{\lambda}{2}$ path-difference between the disturbances coming from the edge and centre elements, so that it agrees with the phase given by the unrestricted wave (vertical arrow). Moving still nearer to the aperture, the illumination decreases, reaching the value zero when the aperture contains two zones: the phase suffers a further retardation of 90°, as we can easily see by compounding the resultant by the graphical method, the elementary vector arrows forming a closed circle at this point. This first arrow, representing the effect from the central element, points to the right. Succeeding ones, with increasing phase retardations, represent resultant effects from circular zones surrounding the central element. Just before we reach the point of zero illumination, the vector which gives us the resultant phase and amplitude is a short arrow pointing to the left. Passing the zero point, the phase springs back suddenly to the value which it had at the most distant point, that is, it is accelerated 180°. This same thing occurs over and over again as we approach the aperture. We find, however, that as we approach the aperture the minima are no longer equal to zero, that is the vectors do not form a closed circle, but a portion of a spiral. The phase at these points is the same as at the maxima, that is it is represented by the short vertical arrow joining the beginning and end of the turn of the spiral. The maxima and minima gradually die out and we end up with a phase represented by a vertical arrow, that is retarded 90° on that of the distant point.

We thus see that as we pass from the aperture to the distant point we gain 90° in phase, the phase, however, oscillating back and forth through 180° a number of times, as was found in the experiment with the lens of Iceland spar.

This same thing takes place in the case of a wave passing through a focus. At the focus the path-difference between disturbances coming from the centre and edge of the wave becomes zero; that is, the focus corresponds to the distant point in the previous case,

and we have the phase represented by an arrow pointing to the right. As we pass through the focus the phase arrow begins to turn down and we have similar oscillations, ending up with an arrow pointing downwards when we get so far from the focus that the maxima and minima have disappeared. The passage of the wave through a focus is thus seen to result in an acceleration of phase of 180°; we can regard the decrease in area of the wave-front as we approach the focus as analogous to restricting it by a small circular aperture.

It is difficult to give a clear exposition of the subject in a limited space, and the reader is advised to refer to Sagnac's paper, and the more recent ones by Reiche (*Annalen der Physik*, 1909).

CHAPTER VIII

INTERFERENCE SPECTROSCOPES AND THE RESOLUTION OF SPECTRAL LINES

In the Chapter on Diffraction we have discussed the action of the diffraction grating, and we will now take up the subject of the more recently devised spectroscopes, which should have been treated in the Chapter on Interference, but which has been postponed for the reason that certain points cannot be well understood without previously considering the theory of the grating. We will begin with the Michelson interferometer, which is, perhaps, the best-known type.

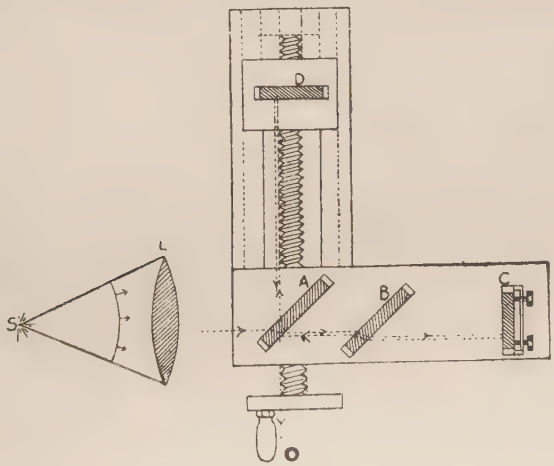


Fig. 192.

Michelson's Interferometer. — The essential parts of this instrument are four plates of glass arranged as shown in Fig. 192. Plates A and B are cut from the same piece of glass accurately plane parallel. Both may be transparent or A may be half silvered on the surface opposed to B. Plates C and D are heavily silvered on their front surfaces. Plate D is mounted on a carriage arranged so that it can be moved along parallel ways by means of a screw. The action of the apparatus is as follows: Light from a source S made parallel by a lens falls upon the plate A, the beam being divided into two portions by the half-silvered surface. One portion is reflected to the mirror D, the other transmitted through B to the mirror at C, which is fixed in position. The mirror D returns the light to A, a portion of it escaping through the half

silvered film and entering the observer's eye, which is located at O. The light reflected back from C is in part reflected from the silver film and enters the eye over the same path. If the path-difference is an odd number of half wave-lengths, these two streams will interfere destructively and we shall have darkness. The path-difference between the two rays can be altered by moving the mirror D by means of the screw. Consequently the point in question upon the half-silvered surface will appear alternately bright and dark as the carriage is moved along the ways. The plate B is not essential, and its object will be explained presently. We can get a better idea, perhaps, of the action of the instrument in the following way: The mirror C is seen by reflection in the half-silvered film in coincidence with the mirror D, if the optical paths are the same. The instrument is thus the equivalent of two parallel reflecting surfaces, the distance between which can be varied. The phenomena presented by the interferometer are thus similar to those shown by thin films, the difference lying in the fact that in the present case we may make the distance between the reflecting surfaces as great or as small as we please.

If our source of light is a point or narrow line, *e.g.* a vacuum tube, a lens must be employed as shown in the diagram. If, however, we use a broad source, such as a sodium flame, the lens may be dispensed with.

The plate B is called the compensator, and is introduced to make the two optical paths symmetrical. In its absence it is obvious from the diagram that one of the interfering beams which enters the eye has traversed the plate A three times, while the other has passed through it but once; the double transit of the latter ray through the compensator makes the two paths optically equivalent. The compensator has also another use, for by turning it slightly we can increase or diminish the optical path, thus compensating for and measuring a change produced in the other path, as, for example, by the introduction of a thin film, the refractive index of which we wish to determine.

Use of the Interferometer. — The following very explicit directions for using the interferometer are taken from Mann's *Manual of Advanced Optics*, in which various experiments with the instrument are described in detail.

Adjustment. — "Measure roughly the distance from the silver half-film upon the rear of the plate A to the front of the mirror C. Set the mirror D, by turning the worm wheel, so that its distance from the rear of A is the same as that of C from A. This need not be done accurately. It is suggested because it is easier to find the fringes when the distance between the mirror D and the virtual image of the mirror C is small. This distance will hereafter be called the distance between the mirrors.

"Now place a sodium burner, or some other source of monochromatic light, at S, in the principal focus of a lens of short focus. It is not necessary that the incident beam be strictly parallel. Hold some small object, such as a pin or the point of a pencil, between L and A."

A pin-hole in a card is preferred by the author, as the vertical and horizontal adjustments can be made with greater precision.

"On looking into the instrument from O, three images of the small object will be seen. One image is formed by reflection at the front surfaces of A and D; the second is formed by the reflection at the rear surface of A and the front surface of D; the third is formed by reflection from the front surface of C and the rear surface of A. Interference fringes in the monochromatic light are found by bringing this third image into coincidence with either of the other two by means of the adjusting screws upon which the mirror C rests. If, however, it is desired to find the images in white light, the second and third of these images should be brought into coincidence, because then the two paths of the light in the instrument are symmetrical, *i.e.* each is made up of a given distance in air and a given thickness of glass. When the paths are symmetrical, the fringes are always approximately arcs of circles as described above. If, however, the first and third images are made to coincide, then the two optical paths are unsymmetrical, *i.e.* the path from A to C has more glass in it than from A to D, and in this case the fringes may be ellipses or equilateral hyperbolae, because of the astigmatism which is introduced by the two plates A and B. It is quite probable that the fringes will not appear when the two images of the small objects seem to have been brought in to coincidence. This is simply due to the fact that the eye cannot judge with sufficient accuracy for this purpose when the two are really superposed. To find the fringes, then, it is only necessary to move the adjusting screws slightly back and forth. As the instrument has here been described, the second image lies to the right of the first.

"Having found the fringes the student should practise adjustment until he can produce at will the various forms of fringes. Thus the circles appear when the distance between the mirrors is not zero, and when the mirror D is strictly parallel to the virtual image of C. The accuracy of this adjustment may be tested by moving the eye sideways and up and down while looking at the circles. If the adjustment is correct, any given circle will not change its diameter, as the eye is thus moved. To be sure, the circles appear to move across the plates because their centre is at the foot of the perpendicular dropped from the eye to the mirror D, but their apparent diameters are independent of the lateral motion of the eye. For this reason it is advisable to use the circular fringes whenever possible.

"To find the fringes in white light, adjust so that the monochromatic fringes are arcs of circles. Move the carriage rapidly by intervals of a quarter turn or so of the worm wheel. When the region of the white-light fringes has been passed, the curvature of the fringes will have changed sign, *i.e.*, if the fringes were convex toward the right, they will now be convex toward the left. Having thus located within rather narrow limits the position of the mirror D, which corresponds to zero difference of path, it is only necessary to replace the sodium light by a source of white light, and move

the mirror D by means of the worm slowly through this region until the fringes appear."

A better control of the motion can be obtained by placing a small white gas flame behind the sodium flame. This gives us a white spot in the centre of the field, across which the fringes push when we reach the centre of the system.

"These white-light fringes are strongly colored with the colors of Newton's rings. The central fringe — the one which indicates exactly the position of zero difference of path — is, as in the case of Newton's rings, black. This black fringe will be entirely free from color, *i.e.* perfectly achromatic, if the plates A and B are of the same piece of glass, are equally thick, and are strictly parallel. If they are matched plates, *i.e.* if they are made of the same piece of glass and have the same thickness, their parallelism should be adjusted, until the central fringe of the system is perfectly achromatic. When this is correctly done, the colors of the bands on either side of the central one will be symmetrically arranged with respect to the central black fringe."

If the instrument is illuminated with sodium light it will be found that the fringes become invisible periodically as the mirror is moved, for reasons which have been given in the Chapter on Interference. It will be found instructive to illuminate the instrument with a lithium flame containing a little sodium, and note the shortness of the periods of indistinctness. In using the instrument to measure the refractive index or dispersion of a gas, the tube containing the gas can be closed with plates of thin plate glass, which, if of good quality, do not much affect the appearance of the fringes. The tube is highly exhausted and the gas then slowly admitted, the shift in the fringe system being determined by counting the number of bands which cross the hair in the telescope used to view them.

The interesting investigation by Johonnott (*Phil. Mag.*, 47, page 501, 1899) on the thickness of the "black spot" on soap films, is an example of the many interesting applications of the interferometer. If we know the thickness of a transparent plate we can measure its refractive index by inserting it in one of the optical paths of the instrument and measuring the fringe displacement. The white system must be used of course in conjunction with the sodium or other monochromatic system, as the central fringe is the only one that can be identified. The abnormal displacement of the central band referred to in the Chapter on Interference must also be remembered.

It is evident now that if the refractive index of a film is known the thickness can be determined. Johonnott found that, by employing a battery of 54 soap films mounted on frames, it was possible to get a measurable shift of the fringes even when the films were so thin that they refused to reflect light, *i.e.* showed the Newton black.

The thickness was found to vary between .00006 mm. and .0004 mm.

Determination of Refractive Index and Dispersion with the Interferometer. — The refractive index of a transparent plate and

its dispersion can be obtained by means of white light in combination with monochromatic light of a single wave-length. The determination of the dispersion is based upon the shift between the *true* and the *apparent* position of the centre of the system of fringes formed by white light, which we have just studied. The plate should be sensibly plane-parallel, and should be cut in two, the two portions placed in the paths of the interfering beams in such positions that they cover the same portion of the field. We may illuminate the instrument with a sodium flame backed by a candle flame. The two pieces of the plate should be so arranged that they can be rotated about vertical axes, one of them very slowly and uniformly, the angles of rotation being measured with a mirror and scale. We can set them normal to the rays, by turning them to the point where the direction of motion of the fringes resulting from the increase of path with increasing incidence angle reverses.

Adjust the instrument so that both the white-light fringes and the sodium fringes appear in the field. Then turn one plate through a convenient angle, which is read from the scale. Turn the other plate very slowly, counting the sodium fringes as they pass over the cross hair of the observing telescope until the white fringes again appear and occupy their former position. Let the angle through which the plates have been turned be i, the fringe count $2N$, the thickness of the glass t, its refractive index μ, and the wave-length of the sodium light λ, it can be shown that

$$\mu = \frac{(t - N\lambda)(1 - \cos i) + \dfrac{N^2\lambda^2}{2t}}{t(1 - \cos i) - N\lambda},$$

in which the term $\dfrac{N^2\lambda^2}{2t}$ is negligible. We now restore one plate to its original position, and move the interferometer mirror until the white-light fringes appear in their former position, counting the sodium fringes as they cross the hair. The number will be greater than $2N$, the difference, which we will call $2N'$, being due to the dispersion.

The Cauchy dispersion formula can be assumed, $\mu = A + \dfrac{B}{\lambda^2}$,

and we have $$N' = \frac{2Bt'}{\lambda^3},$$

in which t' is the thickness of the glass introduced by the rotation,[1] as was shown in the Chapter on Interference.

Light-Waves as Standards of Length. — Probably the most important use to which the interferometer has been put was the determination of the length of the standard metre in wave-lengths of the monochromatic radiations from cadmium. The invariableness of the wave-length of the radiation sent out from the atoms of a metal, brought to a state of luminescence by electrical discharges

[1] Mann's *Manual of Optics*.

in a high vacuum, suggests their adoption as a standard of length. This proposition was first made by Lamont in 1823, and subsequently by Dr. Gould about thirty years ago. At that time the interferometer in its present form was unknown, and the method proposed involved the use of the diffraction grating, the measurement of its width, and the determination of angles, all of which measurements would have entailed no very inconsiderable errors. Michelson suggested the use of his interferometer, and through the efforts of Dr. Gould, who represented the United States in the International Committee of Weights and Measures, was asked to carry out the experiments at the International Bureau at Sèvres in collaboration with Benoit. A very complete description of the method will be found in Prof. Michelson's book, *Light-Waves and their Uses* (Chicago University Press, 1903).

The general principle of the method can be briefly outlined as follows: —

The problem is to measure the distance between the two marks on the standard meter bar in terms of the wave-length of light, or, in other words, find out how many light-waves there are in a beam a meter long.

A bronze bar 10 cms. in length, of the form shown in Fig. 193, was prepared, on the ends of which two silvered-glass mirrors were mounted which could be made accurately parallel by observing the interference fringes, formed in the manner to be described presently. The principle consisted in finding the number of light-waves in a beam whose length was equal to the distance between the planes of the two mirrors, and then to find how many times this distance was contained in the meter. In a length of 10 cms., there are, however, roughly 300,000 light-waves, and the direct determination of this number by actual count would have involved too much labor and too great a risk of accidental mistakes. Nine other standards similar to the above were therefore prepared, each half as long as its predecessor, *i.e.* of lengths 10, 5, 2.5, 1.25, etc., cms.; the smallest unit had mirrors with reflecting planes only .39 mm. apart. The number of light-waves in this distance was first determined for the red, green, and blue radiations from a vacuum tube containing cadmium vapor. This was accomplished by putting the bar with its two mirrors in the place of one of the mirrors of the interferometer; the other mirror was then brought into such a position that the central fringe (white light) appeared in the field of, we will say, the lower mirror. By moving the mirror back the centre of the system could be made to appear in the upper mirror, and by counting the number of fringes which passed during this operation the number of wave-lengths in the distance through which the mirror moved could be determined.

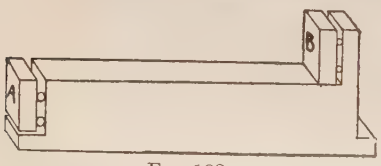


Fig. 193.

This first "*etalon*," as it was called, was next compared with the

second by mounting the two side by side, in place of the movable mirror of the interferometer. The field of view now consisted of four square areas corresponding to the four mirrors of the *etalons*. The longer of the two (No. II.) was fixed in position, while the shorter (No. I.) could be moved by turning the screw of the instrument. The reference plane (image of the interferometer mirror seen in the plate) was then brought into coincidence with the front surface of the lower mirrors of the two etalons, the plane R (Fig. 194, A), by moving the interferometer mirror until the colored fringes appeared. This mirror, which is usually fixed, in the present type of instrument could be moved along parallel ways. It was then moved back until the reference plane coincided with the upper mirror D of etalon I., the plane R'. The fringes passing during this motion of the mirror were counted, the number of course corresponding with the number previously determined. Etalon I. was now moved back until C came into coincidence with the reference plane R' (Fig. 194, b). The reference plane was now moved to R'', until it coincided with D' in its new position, and was within a few wave-lengths of the plane of B, the number being found by turning the compensating plate. The second etalon was then compared with the third, and so on, until finally the number of wave-lengths in the 10 cm. etalon had been determined. A mark on this etalon was then brought into coincidence with one of the end marks on the metre bar under the microscope, and the etalon was then progressively advanced, its front mirror being brought into coincidence with the plane previously occupied by the rear mirror, the reference plane then moved back and the process repeated. In this way the

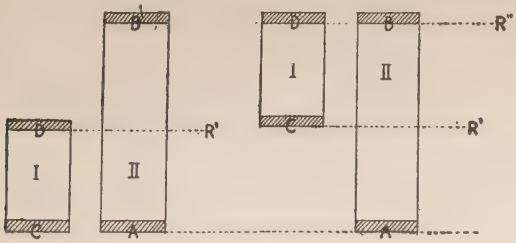


Fig. 194.

total number of waves in a length equal to the standard metre was determined. The final results were as follows, for 15° C. and 760 mms. pressure:

Red line 1 m. = 1553163.5 λ, *i.e.* λ = 6438.4722 *AE*,
Green line 1 m. = 1900249.7 λ, *i.e.* λ = 5085.8240 *AE*,
Blue line 1 m. = 2083372.1 λ, *i.e.* λ = 4799.9107 *AE*.

The values given by Rowland for these same lines are

6438.680, 5086.001, and 4800.097.

An idea of the accuracy of the work can be obtained by comparing

three independent observations, the first two by Michelson, the third by Benoit:

1553162.7, 1553164.3, 1553163.6.

In addition to recording the length of the standard metre in terms of an invariable unit, this remarkable piece of work has given absolute determinations of three standard lines, which will doubtless stand for a long time, if not forever, as the standards from which all other lines will be measured.

It may be well to point out here that it has been recently shown by Michelson, and proven experimentally by Kayser, that Rowland's coincidence method is not accurate. As a result of small errors of ruling, the second order ultra-violet line of wave-length 2. may not fall exactly upon a first order line of wave-length 4. The use of the grating is thus restricted to obtaining the wave-lengths of lines between fixed standard lines, by interpolation, at least if the greatest accuracy is required. Revision of the standard wave-lengths is in progress at the present time by interferometer methods.

The Visibility Curves. — As we saw in the Chapter on Interference, the fringe system formed with Newton's combination of a lens and flat plate, illuminated with sodium light, is not continuous. There are periodic regions of invisibility as we proceed outward from the centre, due to the fact that when the maxima of D_2 coincide with the minima of D_1, uniform illumination results. If now D_1 and D_2 were infinitely narrow lines and single, the fringes would be equally distinct when "in-step," regardless of the path-difference. If, however, this is not the case, the visibility will vary at the different points of maximum distinctness. Suppose, for example, that each line is a close double; with a sufficiently large path-difference, the two components of D_1 will get out-of-step, and we shall have uniform illumination and invisibility entirely independent of the light from D_2. Fizeau and Foucault, who may be regarded as the founders of interference spectroscopy, only recorded the successive recurrences of the fringes as the path-difference increased. Michelson went a step further, and measured the distinctness of the fringes at each reappearance. From these observations he was able to compute the nature of the lines, *i.e.* whether they were single or double, broad or narrow, etc. If J_2 denotes the maximum brightness of a fringe, and J_2 the intensity of the dark region between, Michelson calls

$$\frac{J_1 - J_2}{J_1 + J_2} = V,$$

the "Visibility" a quantity which represents the distinctness with which the fringes appear to the eye.

If we know the nature of the distribution of the light in the source, *i.e.* whether the lines are single or double, accompanied or not by fainter companions, etc., it is possible to construct a visibility curve in which the values of V are plotted as ordinates and the path-differences as abscissae.

Michelson commenced by calculating the visibility curves which would result from various types of single, double, and multiple lines.

Examples of such curves are shown in Fig. 195, the intensity curves of the spectrum lines being shown to the left of each. The curves shown are resultant curves formed by the superposition of wave-trains such as would emanate from sources having a distribution of intensity as figured. The visibility curves are obviously the envelopes of the above curves. Michelson next took up the subject of the construction of an intensity curve from a visibility curve, a much more difficult problem. His work in this line was much aided by the invention of his harmonic analyzer, a machine which separates out of a complex curve the simple harmonic curves of which it is formed; in other words, makes a Fourier analysis of it.

As Lord Rayleigh has shown (*Phil. Mag.*, 34, page 407, 1892), the rigorous solution of the problem is not possible, for, except in cases where there is symmetry in the group of lines, we may have a large number of different distributions of intensity, all of which give the same visibility curve. It is impossible, moreover, to decide from the visibility curve on which side of the principal line a fainter component lies. Michelson's predictions regarding the structure of many lines have been subsequently verified, however, and he is to be regarded as the pioneer in the field of investigations devoted to the minute study of spectrum lines.

The method has not been used to any great extent by other observers, partly from the great difficulty of estimating "visibilities" of the fringes, and partly from the difficulty in interpreting the results. Michelson's results were due to his great skill in this respect,

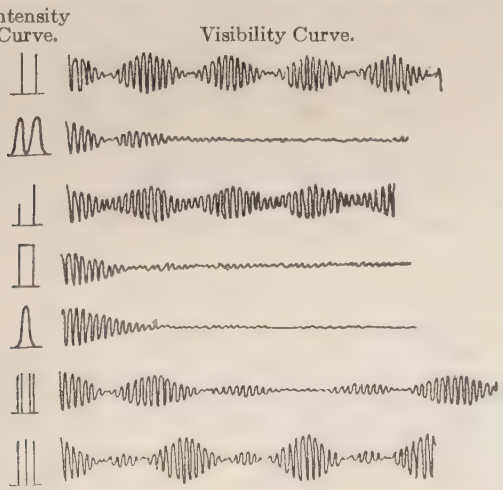


FIG. 195.

which resulted from long experience and familiarity with his instrument. The more modern interferometers show objectively what before could only be guessed at, that is, they actually separate the line into its components just as the prism and grating separate the originally composite light into a spectrum of lines.

Michelson's genius gave us the next instrument in the series which

we are considering, and we will now take up the subject of one of the most curious and interesting optical instruments ever devised, the echelon grating.

The Echelon Grating. — A remarkable type of grating was constructed by Michelson.[1]

As we have seen in the Chapter on Diffraction the resolving power of a grating is represented by mn, the product of the order of the spectrum and the number of lines. High resolving power had been secured previously by ruling a very large number of lines; Michelson attacked the problem in a new direction and constructed a grating for which m instead of n had a large value. The order of the spectrum is measured by the number of wave-lengths in the path-difference of disturbances coming from adjacent elements. If the path-difference can be made 1000 wave-lengths, we have a spectrum of 1000th order. Michelson accomplished this by building up a flight of steps of glass plates, all of exactly the same thickness and plane-parallel to within $\frac{1}{20}$ of a wave-length of sodium light. The plates were cut from a single disk, which was figured with the greatest care by Mr. Petitdidier, and mounted as shown in Fig 196. If light is sent through the series of plates in the direction indicated by the arrows, it is obvious that the streams emerging from the steps are retarded on each other by amounts depending on the thickness traversed, and the refractive index of the glass. Now the retardation by a plate 2 cms. in thickness is considerably over 20,000 wave-lengths; consequently we are dealing with a spectrum of the 20,000th order, if the plates have this thickness. The number of the plates cannot be increased above 30 to advantage, owing to the loss of light by absorption and reflection from the surfaces. Our resolving power is thus about 30 × 20,000 or roughly 600,000, or the grating should separate lines only $\frac{1}{600}$ of the distance between the D lines apart.

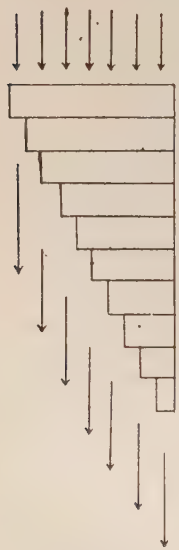


Fig. 196.

The echelon throws all its light into one, or at most two, spectra; consequently it is well adapted for the minute structure of faint spectrum lines. Its great disadvantage is the difficulty of interpreting the results obtained with it, and the impossibility of seeing more than a single line at a time. Even if sodium light is used nothing can be seen which can be interpreted. With certain thickness of plates the D_1 and D_2 spectrum lines may coincide, one being seen in, say, the 2000th order, and the other in the 2030th, owing to the difference of retardation. With plates of a different thickness the D_1 spectra may fall midway between those due to D_2. As the spectra of succeeding orders are very close together, it is obvious that, except when employing extremely homogeneous radiation, we shall have a confused jumble of lines.

[1] "The Echelon Spectroscope," *Astrophys. J.*, 8, page 36, 1898.

Only three different orders can be seen at one time, but by turning the echelon slightly others may be brought into view. We can set the echelon so as to have two adjacent orders of equal intensity, as in the first diagram of Fig. 197, or so as to have one bright line bordered by two faint ones. The latter condition is usually preferable.

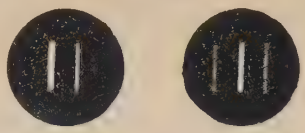


Fig. 197.

The light must undergo previous prismatic analysis before it enters the collimator slit of the echelon spectroscope, or we may illuminate the slit with the heterogeneous light and place a prism between the echelon and the telescope. The instrument is especially well adapted for the exhibition of the Zeeman effect, as it is compact, and extremely saving of light, and requires practically no adjustment if the plates are properly mounted in a metal case. The writer has had no difficulty in showing Zeeman effect with an improvised echelon made by standing four interferometer plates on the table of a small spectroscope. A five-element grating is secured in this way, since a stream of unretarded light can be passed by the edge of the first plate. The width of the steps should not exceed one or two mms. and a cardboard screen should be so arranged as to cut off all the light except that which comes through the steps, a clear space of equal width to one side of the first plate, and a strip of the same width at the edge of the last plate, *i.e.* the top step. In other words, when looking at the echelon from the direction of the telescope the screen should hide everything except five vertical elements of equal width, four of them glass and one air. A direct vision prism can be put between the plates and the telescope to separate the echelon spectra of the different lines in the spectrum under investigation. A mercury vacuum tube between the conical poles of a powerful electromagnet is a suitable source of light to work with, the green line splitting up as soon as the current is turned on.

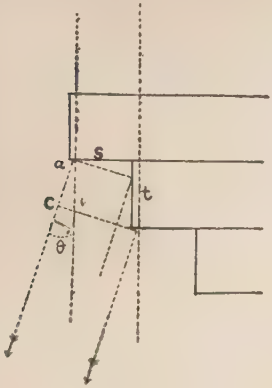


Fig. 198.

Theory of Echelon. — An elementary treatment has been given under Talbot's bands in the Chapter on Diffraction.

The theory as worked out by Michelson is as follows:

Let s be the width and t the height of each step, and θ the angle of diffraction (Fig. 180). If the order of the spectrum is designated by m, and the ref. index of the glass by μ, we have for the path-difference between the diffracted rays indicated by the arrows, $m\lambda = \mu t - ac$.

Now $ac = t\cos\theta - s\sin\theta$;

$\therefore m\lambda = \mu t - t\cos\theta + s\sin\theta$.

(1) $m\lambda = (\mu - 1)t + s\theta$,

since θ is very small, $\sin\theta = \theta$, $\cos\theta = 1$.

To find the dispersion which is represented by $\frac{d\theta}{d\lambda}$, we differentiate θ with respect to λ in the above equation, remembering that μ is also a function of λ,

$$\frac{d\theta}{d\lambda} = \frac{m}{s} - \frac{t}{s}\frac{d\mu}{d\lambda}.$$

Substituting for m its approximate value $(\mu-1)\frac{t}{\lambda}$, we obtain

$$(2) \quad \lambda\frac{d\theta}{d\lambda} = \frac{t}{s}\left[(\mu-1) - \lambda\frac{d\mu}{d\lambda}\right] = b\frac{t}{s},$$

if we represent the bracketed term by b.

The resolving power, which we define by $\frac{d\lambda}{\lambda}$, is by equation (2) $s\frac{d\theta}{bt}$. If $d\theta$ is the increment in the angle of diffraction necessary for resolution (Fig. 199), *i.e.* the angle between the principal maximum and the first minimum, we can write $d\theta = \frac{\lambda}{ns}$, as is made clear by the figure, in which we have

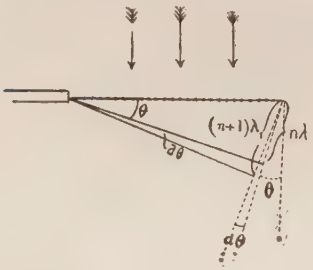


Fig. 199.

$$\frac{N\lambda}{ns} = \sin\theta, \frac{(N+1)\lambda}{ns} = \sin(\theta+d\theta),$$

in which N is some number depending on μ and t. It corresponds to the product mn used previously. (In Fig. 181 it is designated by a small n.)

Owing to the small value of θ, we get by subtraction of the first equation from the second

$$\sin d\theta = d\theta = \frac{\lambda}{ns},$$

in which ns is the total width of the grating.

This gives us for the resolving power,

$$(3) \quad \frac{d\lambda}{\lambda} = \frac{\lambda}{bnt}.$$

To find the distance between the spectra of succeeding orders, we differentiate equation (1) with respect to m,

$$\frac{d\theta}{dm} = \frac{\lambda}{s},$$

or if we put $dm = 1$ to obtain the change in θ, in passing from order m to order $m \pm 1$,

$$(4) \quad d\theta_1 = \frac{\lambda}{s},$$

in which $d\theta_1$ is the angle of diffraction between adjacent spectra.

We will now determine the change in the wave-length which gives $d\theta$ the same value as $d\theta_1$, that is, we will derive an expression which will enable us to compare the distance between the components of a double line, with the distance between the spectra. This we can easily do by substituting (4) in (2), which gives

$$(5)\quad \frac{d\lambda}{\lambda} = \frac{\lambda}{bt},$$

in which $d\lambda$ is the increment of wave-length necessary to produce the increment $d\theta_1$, that is the components of a double line, with a wave-length difference $d\lambda$ (as defined by (5)), will be separated by a distance exactly equal to the distance between the spectra of succeeding orders. Comparison of (3) with (5) shows us that the limit of resolution is $\frac{1}{n}$ of the distance between the spectra. "This," says Professor Michelson, "is a rather serious objection to this form of spectroscope. Thus in observing the effect of increasing density on the breadth of the sodium lines, if the broadening be of the order $\frac{\lambda}{bt}$ the two contiguous spectra of the same line will overlap. As a particular case let us take $t=7$ mms., then $\frac{\lambda}{bt} = \frac{1}{17000}$, and it will be impossible to examine lines whose breadth is greater than $\frac{1}{14}$ of the distance between the D lines. It is evidently advantageous *on this account* to make t as small as possible."

Obviously t has its smallest possible value (zero) in the case of the ordinary diffraction grating ruled on glass.

To get high resolving power, however, we must make either n or t large. The former quantity cannot be increased to advantage above 25 or 30, for reasons already specified, consequently t must be made large.

Michelson constructed three echelons, with plates 7, 18, and 30 mms. in thickness, having resolving powers equal to 210,000, 540,000, and 900,000 respectively, the smallest value surpassing that of the largest gratings ruled on speculum metal.

The distribution of intensity is deduced from the formula

$$A = \int_{-\frac{s}{2}}^{\frac{s}{2}} \cos px\,dx, \text{ where } p = \frac{2\pi}{\lambda}\theta.$$

Hence

$$I = A^2 \frac{\sin^2 \pi \frac{s}{\lambda}\theta}{\left(\pi \frac{s}{\lambda}\theta\right)^2},$$

which vanishes for $\theta = \pm\frac{\lambda}{s} = d\theta$, the distance between the spectra.

Two spectra will thus in general be visible, of unequal intensity, as shown in Fig. 200, but by inclining the echelon a trifle one of them can be brought into the position $\theta = 0$, when it reaches its maximum

intensity, and the two adjacent spectra, falling at π, practically disappear. Or we can so adjust the echelon as to have two spectra of equal intensity symmetrically placed. These two positions have been named positions of single and double order.

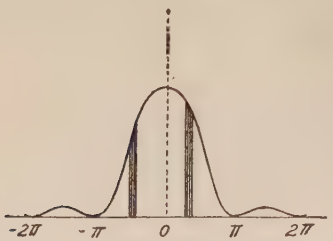


Fig. 200.

We can perhaps arrive at a better conception of the action of the echelon if we consider a type in which t, the thickness of the plates, is made very small. Such a grating can be constructed of mica films, .05 mm. or so in thickness. We can in this way obtain a type intermediate between the ordinary grating and the thick plate echelon of Michelson. With a grating of this description, the distance between two fairly close lines will not be greater than the distance between the spectra.

An echelon grating of 8 or 10 elements can be made of mica, which illustrates the principle, though of no use as a piece of optical apparatus. It is useful as coming midway between the ordinary grating and the echelon as usually constructed. By its aid lines can be separated which, with an ordinary grating of the same number of elements, would appear single. It shows spectra of the same general appearance as in the more powerful instruments, can be set for single and double order, and though useless as a tool for research, is almost as satisfactory for purposes of demonstration as the costly batteries of thick plates.

The very best quality of mica must be used, a sheet about .05 mm. thick being split off. This must be examined with an interferometer, and a portion picked out of uniform thickness, that is, an area must be found, across which the fringes run straight and unbroken. Possibly a simple examination of the film with a sodium flame would answer as well. A fuller description of the mica echelon constructed by the author will be found in the *Philosophical Magazine* for June, 1901, and may be helpful to those who wish to undertake the task of constructing a similar piece of demonstration apparatus.

The Interferometer of Fabry and Perot. — In the Michelson instrument, as we have seen, the fringes due to a double line disappear completely when out-of-step. It is obvious that if the ratio between the widths of the bright and dark bands could be decreased this disappearance would not occur, the bright fringes due to one line falling midway between those due to the other. An interferometer has been devised by Fabry and Perot in which the width of the bright band has been made many times smaller than the width of the dark band. To understand how this has been accomplished we must go back to the theory of the diffraction grating. It will be remembered that the intensity curve, in the system of fringes formed by light passing through two parallel slits, is similar to the curve which we have with the Michelson interferometer. Increasing the number of slits in-

creases the steepness of the curve, *i.e.* the maxima become narrower and the minima wider. Regarding the slits as similar sources of light, we see that an increase in the number of the sources results in a narrowing of the bright fringes. In the case of thin films or the Michelson interferometer, we are dealing essentially with light coming from two virtual sources, one immediately behind the other. If we can by any device increase the number of these sources, we shall in consequence decrease the width of the bright maxima without altering their distance apart. This can be done by availing ourselves of multiple reflections. We can illustrate the principle very easily by half-silvering two small squares of thin plate glass, by the process which will be presently described.

If they are placed with their silvered sides together, but separated by two narrow strips of thick writing paper, and held in front of a sodium flame, they will resolve the *D* lines, *i.e.* show a double set of fringes. By varying the thickness of the paper strips, com-

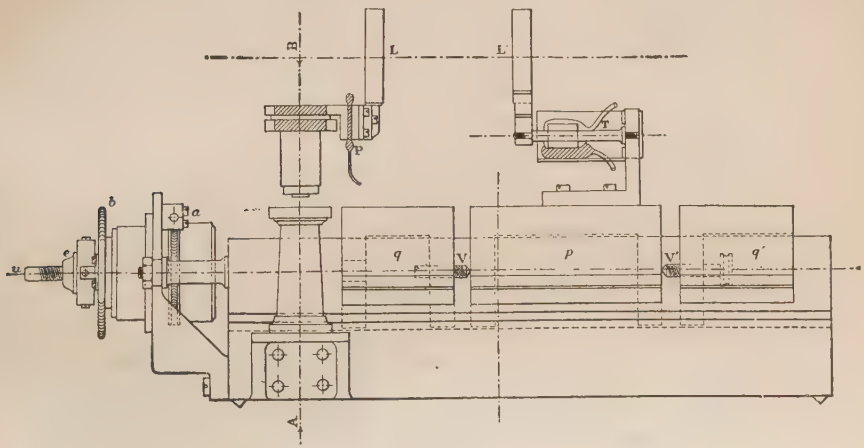


Fig. 201.

mencing with thin tissue, the process can be followed. The fringes appear at their best when the eye is normal to the plates. This can be accomplished by bringing the reflected image of the pupil of the eye over the sodium flame. The Fabry and Perot instrument is based upon this principle. It consists of two plates of plane-parallel glass, one stationary, the other movable in a direction perpendicular to its surface by a device similar to that employed by Michelson. The general arrangement of the apparatus is shown in Fig. 201. The paths of the rays through the half-silvered films are shown in Fig. 202, in which the source of the light is located at *S* and the eye at *O*. The opposed surfaces of the glass plates are half-silvered, and the distance between them can be varied from zero to 30 or 40 cms. Consider now a ray of light leaving the source and passing through the first silver film. A portion of it will be transmitted through the second film and a portion reflected back upon the

first, from which it will be returned, the same sequence of events occurring again. Owing to these multiple reflections we shall have a number of beams of light of decreasing intensity emerging parallel to one another from the second silvered surface, the beams coming from a number of virtual sources situated in a line behind the actual source. The distance between the sources will be double the distance between the silvered films, which will consequently be the path-difference between two streams of light coming from adjacent sources. The case may be regarded as analogous in some respects to a number of slits situated in a line behind one another. Starting with the plates close together and accurately parallel, we shall observe in the case of sodium light a system of circular fringes similar to those seen with Newton's glasses, except that the maxima are extremely narrow circles of light with broad dark regions between them. On increasing the distance between the mirrors the rings due to the two components of the sodium light will gradually get out-of-step; but instead of disappearing, as they do in the Michelson instrument, they merely become double in the present case. In other words, we can observe the separation of the D_1 fringes from the D_2 ones, the number of circles of light doubling. On further increasing the path the fringes will again coincide.

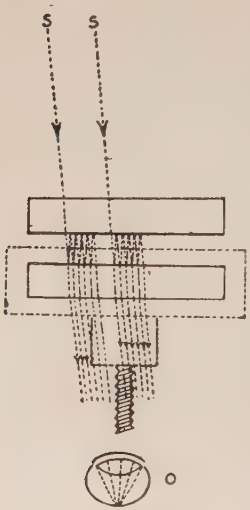


FIG. 202.

It is clear that in the present case we can observe *directly* that which we were obliged to infer from the visibility curves in the case of the Michelson interferometer. Faint components lying close to a bright line can be observed directly with this instrument by making the distance between the plates sufficiently great. Fabry and Perot have studied the minute structure of a large number of spectrum lines, and have obtained results which the Michelson instrument is incapable of yielding.

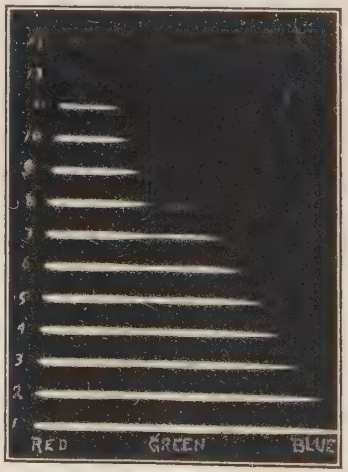


FIG. 203.

The resolving power of the Fabry and Perot instrument depends upon the number of reflections. It is thus important to have the thin silver films highly polished, and quite free from the yellow deposit which covers them when they come from the silvering solution. As a result of considerable

experimenting it was found by A. H. Pfund that it was possible to polish the thin, semitransparent films so highly that forty images could be counted when the source of light was very bright. If the plates are placed in opposition at a small angle the images lie side by side and can be counted. They are white at first, but speedily become reddish owing to the lower reflecting power of silver for the blue. If the row of images is viewed through a prism, the falling off of the blue end of the spectrum by repeated reflection is easily seen. A photograph of the phenomenon is reproduced in Fig. 203. The blue has practically disappeared at the 8th image and the green at the 10th. On this account the resolving power is greatest in the red and falls off rapidly with decreasing wavelength.

Silvering the Plates. — The simplest method is the one in which formaldehyde is used as a reducing agent.

Prepare the two following stock solutions.

A	B
5 grs. Silver nitrate. 500 cc. Dist. water.	50 cc. Formaldehyde (40 per cent solution). 500 cc. Dist. water.

The glass plates are placed in a small glass dish, and cleaned first by strong nitric acid, then by caustic potash, swabbed over the surface by small wads of cotton twisted around the end of a glass rod. Rinse them thoroughly by a stream of running water, lifting the edges with a rod, to allow the potash to escape from beneath them. Take about 50 cc. of Solution A in a beaker and add ammonia to it drop by drop, until the precipitate which forms is almost, *but not quite*, redissolved, the liquid having a light straw color. If too much is added the solution becomes clear, in which case add A until a slight *permanent* turbidity is produced.

Next mix equal parts of this solution and Solution B in a clean beaker, and pour them at once over the plates (first pouring off the water). The dish should be shaken gently and the process watched. Usually with the proportions given a pinkish film forms, which is remarkably uniform, and serves as a substratum for the thicker deposit. If this is the case, pour off the solution at once, rinse the plates, and then flow over a solution made up of three parts of the ammonia solution to one of B. This will give the required blue deposit.

In a minute or two the silvering will be complete, and the plates can be rinsed off with distilled water and dried. Pfund recommends following the water with alcohol and ether. He has found that to polish these thin films it is essential that they should be dried for an hour in a warm place. They can be polished by brushing them with a feather " powder-puff " charged slightly with opticians' rouge, or even with a piece of a clean kid glove, covered with rouge. Chamois skin should not be used. The polishing removes the yellow deposit. If it is desired to produce a thick opaque deposit,

the silvering process should be repeated several times. Thick deposits can be polished with less danger of damaging them.

A photograph by Fabry and Perot of the fringe system of the green line of mercury as seen in their instrument is shown in Fig. 204. The fainter companion lines are distinctly shown. The narrowness of the bright rings is to be noticed. Compare these photographs with the rings seen with Newton's arrangement of a plate and lens.

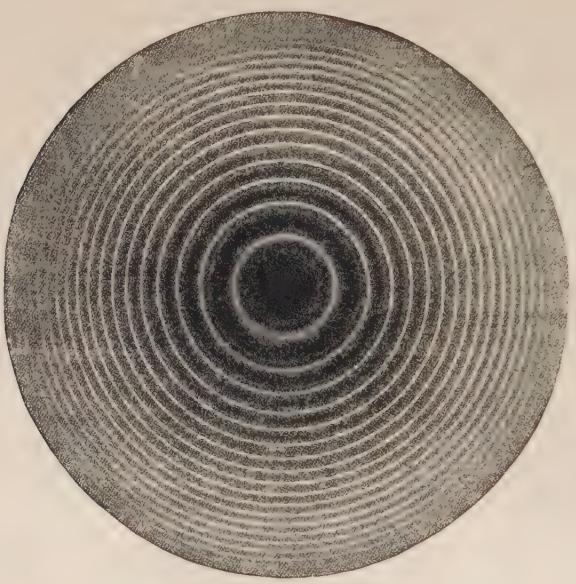


Fig. 204.

The Interferometer of Lummer and Gehrcke. — An extremely ingenious device for obtaining narrow interference fringes under conditions of large retardation of path was originated by O. Lummer and E. Gehrcke.[1] It resembles in some respects the echelon grating, but is in reality a Fabry and Perot plate in principle, the difference being that the angle of incidence, instead of being normal, is very nearly 90°, and the multiple reflections result from the ray being incident upon the inner surfaces at nearly the critical angle instead of from thin silver films. The light from a slit is rendered parallel by a lens and passed through a prism and a Nicol falling upon a wide slit at S, Fig. 205.

After passage through this slit, the plane waves enter a long slab of plane-parallel glass at such an angle that they meet the second surface at very nearly the critical angle. To prevent the large loss by reflection at the first surface which would occur at the very oblique incidence necessary, a small right-angle prism p is cemented to the upper surface.

[1] *Annalen der Physik*, 10, page 457, 1903.

Owing to the large angle of incidence only a small portion of the energy escapes along the path 1, the remainder being reflected to-and-fro within the plate. The function of the Nicol is to polarize the light in such a plane that it is most copiously reflected at the inner surfaces.

In the case of the echelon and the ordinary diffraction grating the width of the interference maxima increases as the slit is opened, and the spectrum lines are eventually obliterated by fusion if they are close together. We obtain the extended interference system by diffraction. With the Lummer and Gehrcke plate the width of the slit plays no part in the sharpness of the interference maxima. Diffraction plays no part here, and we must have a collection of plane waves of varying inclination to obtain an extended interference system. As a matter of fact the plate can be considered as a Fabry and Perot interferometer with the incidence angle very large.

After two internal reflections a second portion emerges along path 2. The path-difference between these two streams is evidently very large, for it is represented by $\mu(AB+BC)-AD$. Owing to the small amount of energy which escapes at each reflection a large

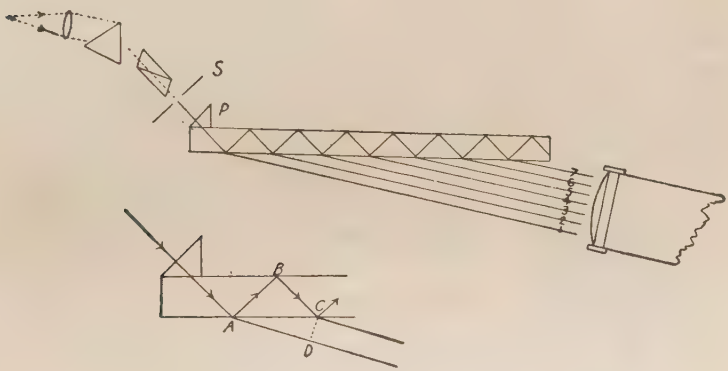


Fig. 205.

number of parallel beams, 1, 2, 3, 4, 5, etc., are obtained with path-differences increasing in arithmetical progression.

Lummer and Gehrcke studied the structure of the mercury lines with this instrument, obtaining results which have given rise to a good deal of discussion. The yellow line ($\lambda=579$) was found to consist of 12 components, the other yellow line ($\lambda=577$) of 11. The green line, $\lambda=546$, had 21, and the blue line ($\lambda=492$), 3 components.

These results have been criticised by Fabry and Perot,[1] on the ground that they do not agree with the results obtained with other instruments of equal power. Though Lummer and Gehrcke do not give the wave-lengths of the component lines, Fabry and Perot

[1] *Journ. de Phys.*, Jan. 1904.

estimate that the three brightest components of the green line, as figured by Lummer, ought to be easily visible with a small grating. They appear to be of nearly equal intensity, while other observers, working with instruments of different types, all agree upon one bright line with two faint companions. Lummer and Gehrcke state that the resolving power of their instrument is 400,000. The distance between the extreme bright lines in the set is about twenty times the distance between the closest pair, from which Fabry and Perot estimate that for them $\frac{\delta\lambda}{\lambda}$ must be $\frac{1}{200000}$, or in other words, that if they had a real existence, the distance between them would be only $\frac{1}{20}$ of the distance between the D lines.

The multiple lines observed by Lummer they regard as optical "ghosts," due to slight imperfections in the glass plates.

Interference Points. — Gehrcke then devised a method of ascertaining which of the lines were real and which ghosts. The light was reflected from two interference plates in succession, the plates being mounted in perpendicular planes, as shown in Fig. 206. This is the method of crossed gratings described in the section on "Ghosts" in the Chapter on Diffraction. The second plate introduces a set of parallel interference minima at right angles to the maxima and minima due to the first. The illumination is thus reduced to a series of points arranged in rectangular order. The companion points due to real lines will lie along the diagonals as described in the method of crossed gratings. Suppose we have a source of light emitting a strong line accompanied by a group of three close lines on one side.

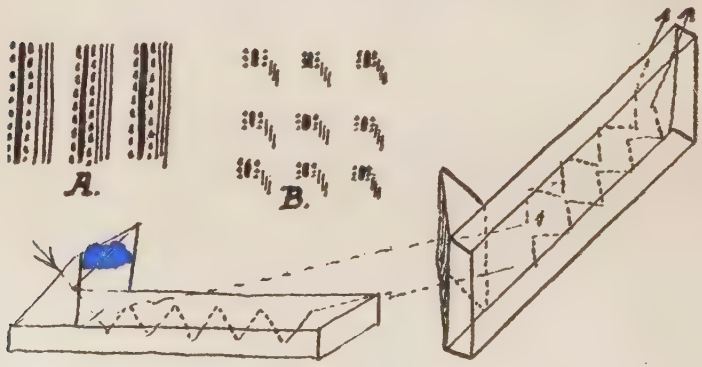


Fig. 206.

Suppose further that one of the interference plates is free from ghosts, while the other exhibits a faint one on each side of the strong line. The appearance of the maxima and minima due to this second plate acting alone will be as shown in Fig. 206, A, the ghosts being represented by dotted lines. Three different orders are shown. When the second plate is crossed with this one, it will introduce a second series of minima at right angles to the first. These hori-

zontal minima will have the same width in proportion to the maxima as the vertical minima; that is, they will break up the vertical lines into dots or interference points. They will have the same position for the primary line and its ghosts since λ is the same for all, but not for the three companion lines, as shown in Fig. 206, *B*. The result is that the "points" (short lines in the figure) representing the true companion lines lie in an oblique direction. The actual appearance of the interference points for the green mercury line 546 as photographed by Gehrcke is shown in Fig. 207. The companion lines, or satellites, as they are called, are represented by the oblique lines of dots. The principal dot is somewhat elongated, showing the finite width of the line. It is apparent that a true satellite dot may happen to fall both on the diagonal and on the vertical line of the next order, and be mistaken for a ghost of the principal line. By using a plate of a different thickness we can tell whether this is the case or not.

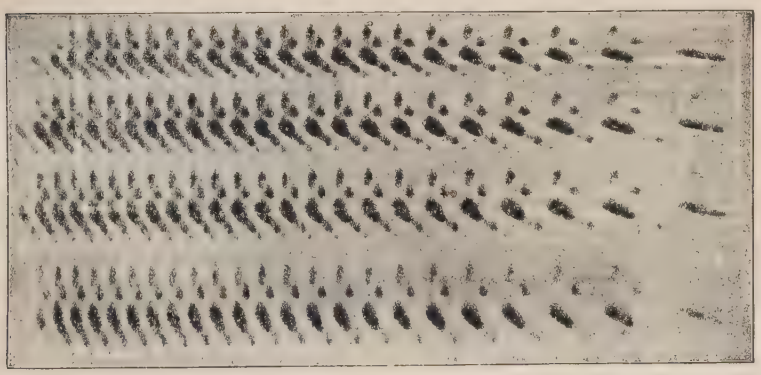


Fig. 207.

Resolving Power and Spectral Range. — The spectral range of an interference spectroscope is the range of wave-lengths which can be analyzed with it. This is always very small, and as has already been pointed out, the light must be previously analyzed with a prism and each line studied separately. If we call the spectral range $\Delta\lambda$, we have

$$\Delta\lambda = \frac{\lambda}{q},$$

in which q is the path-difference measured in wave-lengths or the "order" of the spectrum.

For example, the *D* lines of sodium form a fringe system which is in dissonance (out-of-step) with a path-difference of 500 waves, as we have seen. This means that a change of wave-length equal to 6 Ångström units causes a line seen in an interference spectroscope with path-difference of 500 waves to move a distance equal to one-half of the distance to the next fringe (order). If we have a path-difference equal to 1000 waves, the same change of wave-length will

cause it to move to the position of the fringe of next higher order. The spectrum range is therefore equal only to the range of wave-lengths represented by that change of wave-length necessary to cause a fringe to shift over one order.

For sodium light the wave-length is approximately .0006 mm. With a path-difference of 1000 waves we have

$$\Delta\lambda = \frac{.0006}{1000}, \text{ or 6 Ångström units,}$$

the distance between the D lines. For a plane-parallel glass plate .5 cm. in thickness of refractive index 1.5, half-silvered on each surface, we have at normal incidence a path-difference of 30,000 waves for green light, and

$$\Delta\lambda = \frac{.0005}{30000},$$

a range equal to $\frac{1}{3}\frac{1}{0}$ of the distance between the D lines. The resolving power $\frac{\lambda}{\delta\lambda}$ expresses the wave-length difference $\delta\lambda$ capable of separation by the instrument. To resolve the D lines $\frac{\lambda}{\delta\lambda} = 1000$.

From the analogy of the plate to the echelon we see that the resolving power will be the product of the order of spectrum (*i.e.* path difference between two adjacent emergent beams, and the total number of emergent beams) corresponding to elements of the grating. If the length of the plate is l and the thickness d, and we send in the light in such a direction as would correspond to an angle of refraction r in the absence of the small prism, we have for p, the number of emergent beams, $\frac{l}{2\,d\,\tan r} = p$, as can be easily seen by drawing a diagram of the plate. In Fig. 205 the distance $AC = 2\,d\,\tan r$, r being the angle between the ray AB and the normal. The order of spectrum or the retardation is $\frac{\lambda}{\Delta\lambda}$, as we have seen.

The resolving power is therefore pq.

Gehrcke has given the following table for the resolving power of various echelons and plane-parallel plates.

Echelon Grating (Michelson)

$d(mm)$	3	5	10	20
p	21	21	21	21
$q=\frac{\lambda}{\Delta\lambda}$	3000	5000	10,000	20,000
$l(mm)=pd$	63	105	210	420
$pq=\frac{\lambda}{\delta\lambda}$	63,000	105,000	210,000	420,000

LUMMER AND GEHRCKE PLATE

$d(mm)$	3	5	10	20
p	24	15	11	8
$q=\frac{\lambda}{\Delta\lambda}$	12,000	20,000	40,000	80,000
$l(mm)$	129	134	197	366
$pq=\frac{\lambda}{\delta\lambda}$	288,000	300,000	440,000	640,000

Michelson has recently ruled a ten-inch grating with 15,000 lines to the inch, the resolving power of which in the third order spectrum would be 450,000. Large gratings of high perfection will in all probability be the instruments of the future, for they can already be made with very nearly the resolving power of the interference spectroscopes, and have the great advantage of an indefinitely wide spectral range.

Quite recently Michelson has succeeded in ruling gratings 10 inches in diameter which appear to be more perfect than any previously ruled. With such gratings the companion lines of the green mercury line have been seen and photographed. A photograph of this line, taken by Gale and Lemon with one of Michelson's gratings, is reproduced in Fig. 208 (*Astrophys. J.*, January 1910). The upper spectrum line was yielded by a small arc, the lower by a Cooper-Hewitt lamp, — the latter shows a strong satellite line.

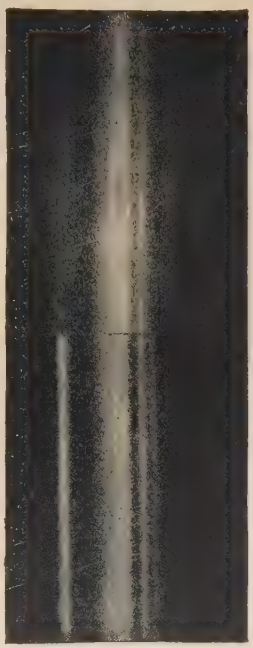


FIG. 208.

CHAPTER IX

POLARIZATION OF LIGHT

In the preceding chapters we have considered rays of light as symmetrical around their direction of propagation; this amounts to saying that the rotation of the ray about its line of propagation is wholly without influence upon the optical phenomena exhibited by it. Rays of light exist, however, which possess a one-sidedness and behave differently when differently orientated. For example, it is possible to obtain light which a glass or water surface refuses to reflect at a certain angle of incidence. Such light is said to be polarized, and is distinguished from ordinary light in that its vibrations are of a fixed type; that is, the ether particle travels in a fixed orbit. If the motion is back and forth along a line, the light is said to be plane-polarized, and it is with this type of polarization that we are concerned in the present chapter. Elliptically polarized light, when the ether particle moves in an elliptical orbit of fixed eccentricity and orientation in space, and circularly polarized light, where the orbit is a circle, will be subsequently dealt with.

The fact that light can be obtained having a lack of symmetry around the direction of propagation is one of the most direct and convincing proofs which we have of the transverse nature of the waves, for we cannot very well conceive of a pressural or longitudinal wave, having different properties in the different directions perpendicular to the line of propagation.

Discovery of Polarization. — The polarization of light was discovered by Huygens in 1690, while experimenting with Iceland spar. He found that a ray of light was, by passage through the crystal, divided into two separate rays of equal intensity, except when the light traversed the crystal in a direction parallel to the crystallographic axis. He found furthermore that if one of these emergent rays was passed through a second crystal, it was divided into two rays of equal or unequal intensity, or not divided at all, according to the orientation of the crystal. Though this single experiment was sufficient to establish the existence of light which was asymmetrical around its line of propagation, and though many other crystals exist having similar properties, Huygens was ignorant of the nature of the phenomenon, and the discovery remained an isolated fact for more than a century. The polarization is in this case produced by double refraction, which we shall study in detail in a subsequent chapter.

Polarization by Reflection. — The discovery was made by Malus in 1810 that light, which had suffered reflection at a certain angle

from a surface of water or glass, exhibited the same peculiarities, which had previously only been observed in the case of light after its passage through a crystal of Iceland spar. The polarization of light by reflection can be exhibited by means of the easily improvised apparatus shown in Fig. 209. The reflectors A and B are made of ordinary plate-glass, the backs being coated with black paint or asphalt varnish. The polarizing plate A is mounted on an iron stand on a hinged support so that it can be set at the polarizing angle. The other reflector is mounted on the vertical axis of an ordinary turn table, in such a way that the light reflected down from the polarizer meets the surface of the glass at an angle of 57° with the normal, *i.e.* the plate must make an angle of 33° with the axis. A cylindrical ring of parchment paper or other translucent medium surrounds the revolving plate, and receives the light reflected from it. The upper plate is so adjusted that its plane is parallel to the plane of the lower plate, in which position it will be found to reflect light capable of reflection from the latter: if, however, the lower plate is turned through an angle of 90°, it will be found that the light is no longer reflected from it, while in intermediate positions of the plate the reflection is partial. If the plate be set in rapid revolution, a ring of light is seen on the translucent screen with two maxima and two minima, corresponding to the positions in which the light is most and least copiously reflected.

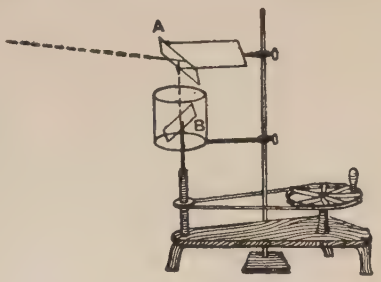


FIG. 209.

Angle of Polarization. Brewster's Law. — If the angle of the upper mirror is varied, it will be found that the reflected light is less completely polarized, and the maxima and minima obtained by revolving the lower plate are less marked. In general, as we increase the angle of incidence from normal to grazing, the polarization increases, passes through a maximum, and then decreases. The angle at which the polarization is most complete varies with the nature of the substance, and is known as the polarizing angle. Jamin found that only a few substances with a refractive index of about 1.46 completely polarize the reflected light. For all other substances the polarizing angle is merely the angle at which the polarization is a maximum.

The relation between this angle and the refractive index of the substance was investigated by Brewster, who discovered the remarkable law that the index of refraction was the tangent of the angle of maximum polarization. When the light is incident at this angle, the refracted ray makes an angle of 90° with the reflected ray, for

$$\frac{\sin i}{\sin r} = n = \tan i = \frac{\sin i}{\cos i};$$
$$\therefore \cos i = \sin r \text{ and } i + r = 90°.$$

If this law is true, the angle of maximum polarization will be different for the different colors owing to dispersion. In the case of most transparent media the dispersion is too small greatly to affect the angle, as can be shown by examining the image of the sun reflected in a glass plate through a Nicol prism so orientated as to cut off most completely the reflected light. The image of the sun appears uncolored, which would not be the case if the angle of polarization was very different for different parts of the spectrum. The Nicol prism, which, will be presently described, takes the place of the second reflector, having the property of completely cutting off light polarized in a certain plane, and transmitting with greater or less facility light polarized in all other planes. In the case of substances having very high dispersion, the variation of the angle with change of wave-length becomes very marked.

The organic compound nitroso-dimethyl aniline, which has been found by the author to have the highest dispersion, in the brighter parts of the visible spectrum, of any known substance, is admirably adapted for the exhibition of what may be termed the dispersion of the angle of polarization. A little of the substance is fused on a glass or metal plate, or better in a small brass cell heated by steam (Fig. 210).[1]

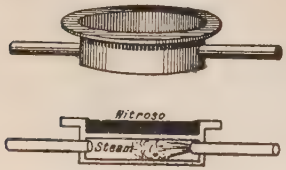


FIG. 210.

A brilliant source of light of small dimensions — nothing is better than a Nernst lamp — is arranged so that its light is reflected from the liquid surface at an angle which can be varied.

On examining the light reflected at a fairly large angle with a Nicol prism, it will be found to vary from light blue to deep violet and purple, as the angle of incidence is increased, the Nicol being held in such a position as to refuse transmission to the light polarized by reflection. If a small direct-vision spectroscope is placed behind the Nicol, a dark band will be seen crossing the spectrum, which shifts its position as the incidence angle varies. The centre of this dark band evidently marks the wave-length for which the angle of incidence happens to be the angle of maximum polarization, or in other words the refractive index of the substance for this wave-length is the tangent of the angle of incidence. In the case of glass and substances of low dispersion, the different colors are polarized at nearly the same angle, *i.e.* very little color effect is observed when the reflected light is examined with the Nicol. In these cases the dark band is so broad as to occupy practically the entire visible spectrum. In the case of a substance with as high a dispersion as that of the nitroso, the angle of maximum polarization is quite different for the different colors; consequently the Nicol prism only extinguishes a portion of the spectrum for a given angle of incidence. This gives us a reflection method of determin-

[1] Nitroso-benzyl aethyl aniline, which can be obtained from the Berlin Aniline Co., is better than the nitroso-dimethyl compound, as after fusion it remains liquid for some hours at ordinary temperatures.

ing the refractive index of a substance, for by determining the angle of incidence for which the centre of the dark band is at a given point in the spectrum, we have only to look up the tangent of the angle in order to get the refractive index for the wave-length in question. The band will be found to be very sharp and quite narrow when it occupies a position in the green and greenish blue, but on attempting to drive it into the red, we shall find that it broadens and becomes much less sharply defined. This is of course due to the fact that the dispersion is much less in the red and orange portion of the spectrum. If the nitroso cannot be obtained, selenium plates, made by pressing the molten substance between glass plates, which are to be separated by a blow from a hammer when cold, can be used for the exhibition of the dark band in the spectrum.

Plane of Polarization. — The plane of polarization is defined as the particular plane of incidence in which the polarized light is most copiously reflected. Referring to Fig. 209, we will determine the plane of polarization of a ray reflected from the upper mirror. The ray meets the lower mirror at the polarizing angle, but as the mirror turns the plane of incidence changes, and the particular plane of incidence which we have when the light on the translucent screen is a maximum, is the plane of polarization of the ray. Since the mirrors are parallel (or turned through 180° from the parallel position) when this occurs, the plane of polarization of light polarized by reflection is obviously the plane of reflection. In the case of light polarized by some other method, we can determine its plane of polarization by reflecting it at the polarizing angle from a glass plate so oriented as to give maximum reflection. The plane of polarization of the ray is then the plane of incidence.

This definition of the plane of polarization is rather unfortunate, for, as we shall see later on, the vibrations of plane polarized light are in a direction at right angles to this plane, and the plane of vibration is the one in which we are chiefly interested, for it is the one in which something is taking place. It would have been preferable if what we now believe to be the plane of vibration had been called the plane of polarization, but the definition was given before any very definite ideas were held regarding the direction of the vibration.

Nörrenberg's Reflecting Polariscope. — The reflecting polariscope of Nörrenberg is a convenient and very easily constructed piece of apparatus for the study of polarized light. It consists of a wooden base (Fig. 211) with two vertical supports which carry the hinged polarizing mirror A_1 made of plate glass, and the two circular collars C and R, the latter fitted with a glass plate, upon which the object to be examined is laid. The upper collar carries the analyzing mirror, A_2, which is mounted on a revolving collar fitting concentrically into the other. This mirror is made of plate glass, backed with black varnish, and is hinged like the polarizer. If the upper collar is roughly graduated, the utility of the apparatus is increased, for it may then be used for measuring the rotation of the plane of polarization, which occurs when light passes through certain substances. On the base of the instrument, between the two

vertical supports, a small circular mirror of silvered glass is cemented. The entire apparatus can be made of wood, if facilities for metal turning are not available.

A beam of sunlight, coming in a direction $a'a$ such that it makes an angle of 66° with the vertical, is reflected from the mirror at an incidence angle of 57°, the mirror being adjusted at an angle of 33° with the vertical, so that the reflected ray is thrown down normally upon the silvered reflector. It is then reflected back, and for the most part passes through the mirror A_1 and the circular glass table, falling upon the upper mirror A_2, which is also set at an angle of 33° with the vertical. The upper mirror we shall call the analyzer, since it is used for the study of the light which comes from the polarizer. On looking down into this mirror at an angle of 66°, we see the reflected image of the sun or other source of light, the intensity, depending on the position of the revolving collar, being greatest when the mirrors are parallel or turned through 180° with respect to this position, and practically zero when the planes of reflection are perpendicular to each other.

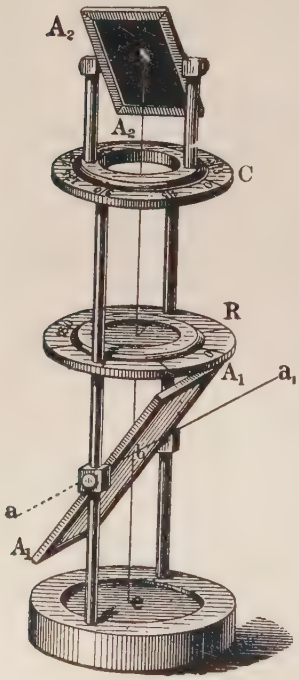


FIG. 211.

The proper angles can be found by making small adjustments of the mirrors when they are in the latter position, until the reflected image disappears entirely. The instrument will be found useful in the study of the colors of thin crystalline plates, and the phenomena of circular and elliptical polarization, which we shall take up in subsequent chapters.

The polarized light from the reflector passes through the object under examination once or twice, according as it is laid on the glass table or the silver mirror; placing the object in the latter position is equivalent to doubling its thickness. On this account the instrument is sometimes called the Nörrenberg doubler.

Polarization by Refraction. — If we examine the light transmitted through a plate of glass placed at the polarizing angle, we shall find that the light is partially polarized; *i.e.* its intensity varies slightly when examined by means of an analyzer. Arago discovered that the reflected and refracted portions of the light contained equal quantities of polarized light, and that the planes of polarization were at right angles.

The greater intensity of the transmitted light is responsible for the incompleteness of the polarization. If the light transmitted through a plate placed at the polarizing angle is received upon a second plate, the unpolarized portion suffers a further resolution into two polarized components, one of which is reflected out through

the upper plate and the other transmitted. By increasing the number of plates we can increase the intensity of the reflected polarized light, and consequently the completeness of the polarization of the transmitted light, seven or eight being sufficient to give us nearly complete polarization in the transmitted, as well as in the reflected beams.

A simple polariscope can be constructed on this principle, which has an advantage over the reflecting instrument in that it can be directed towards the source of light, and requires no adjustment. The glass plates used should be as thin as possible, in order to avoid loss of light by absorption. The large sized rectangular cover glasses used for microscopical preparations are best for the purpose, though the thin glass employed for lantern slides is almost as good. The plates should be carefully cleaned and freed from dust, and mounted in two piles, of eight plates each, in tubes of wood or pasteboard at an angle of about 33° with the axis of the tube. It is best to determine the angle experimentally, as it varies slightly with the nature of the glass. The two piles of plates should almost completely cut off light when the planes of incidence are at right angles. If this is not the case, a few more plates can be added, Bundles of plates thus mounted form very fair substitutes for the more expensive Nicol prisms, and are well adapted to lantern experiments. It may appear at first sight as if each successive plate in the pile would diminish the intensity of the polarized portion coming through the one next to it by reflection, but it must be remembered that *transmitted* polarized light is polarized in a plane at right angles to the plane of the reflected portion, and consequently is incapable of reflection. After passage through a sufficient number of plates the light is completely polarized, and the addition of more plates does not decrease the intensity except by absorption. This is of importance in connection with the calculation of the loss of light by reflection from the surfaces of prism-trains in spectroscopes. Transmission through the oblique surfaces polarizes the light, and we must take this into account in calculating the loss at each surface, the amount reflected decreasing as the polarization becomes more complete. After passage through five prisms there is practically no further loss by surface reflection, and the spectrum is almost completely polarized.

Law of Malus. — The law which governs the decrease in the intensity of the light, as the upper mirror of the Nörrenberg polariscope is revolved, was formulated by Malus. It may be stated thus: When a beam of light, polarized by reflection at one plane surface, is allowed to fall upon a second, at the polarizing angle, the intensity of the twice reflected beam varies as the square of the cosine of the angle between the two planes of reflection. The assumption was made that the incident vibration, polarized in a plane making, say, an angle θ with the plane of incidence, was resolved into two components, one perpendicular, the other parallel to the plane of incidence, the former being partially reflected, and the latter wholly transmitted. This will make the reflected amplitude $a \cos \theta$, if a is the reflected amplitude when $\theta = 0$, and the

intensity will be $a^2\cos^2\theta$, or the maximum reflected intensity multiplied by the $\cos^2$ of the angle between the plane of polarization and the plane of incidence.

The law of Malus is therefore simply a statement of the resolution of a vibration into two rectangular components, the direction of the vibration being considered perpendicular to the plane of polarization.

It is important to distinguish between the behavior of vibrations parallel to the plane of incidence, and vibrations perpendicular to the plane, when they meet a reflecting surface at the polarizing angle. If the light is so polarized that the vibrations are perpendicular to the incidence plane, *i.e.* parallel to the reflecting surface, a portion of the energy will be reflected, and a much larger part transmitted, the directions of the vibrations remaining parallel to the incident vibration (Fig. 212, *a*). If, on the other hand, the direction of vibration is parallel to the incidence plane, practically no energy is reflected, the light being refracted without loss of intensity (Fig. 212, *b*). If now the vibration takes place in a direction making, say, an angle of 45° with the plane of incidence, it will be resolved into two components, one parallel to the reflecting surface and the other parallel to the plane of incidence.

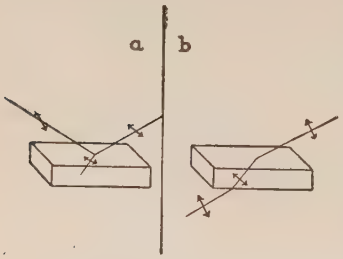


FIG. 212.

Let $ABCD$ (Fig. 213) represent a portion of the wave-front of the incident beam, which is coming towards us, the direction of vibration being AC. We have resolution into the components AB parallel to the incidence plane, and AD parallel to the reflecting surface. The former is wholly transmitted ($A'B'$), the latter in part reflected and in part transmitted ($A'D'$). The reflected light is therefore polarized with its vibration parallel to the surface, since only this component is reflected; the refracted light is made up of the completely transmitted component $A'B'$, and the partially transmitted component $A'D'$, their resultant being $A'C'$, a polarized vibration, rotated counter clockwise with respect to the incident vibration AC. If now the resultant $A'C'$ be received on a second reflecting surface, the same resolution will take place, and there will be a further rotation of the plane. The effect of a pile of plates will therefore be to bring the plane of vibration of the transmitted light into coincidence with

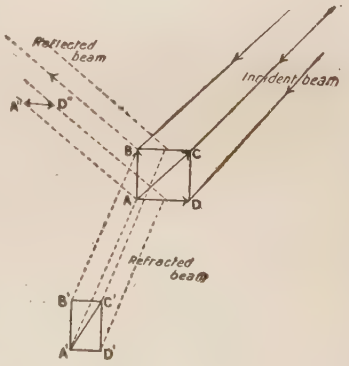


FIG. 213.

the plane of incidence, since $A'B'$ is transmitted each time without loss, while $A'D'$ is reduced in intensity by the partial reflection. This rotation of the plane of polarization is clearly the result of the reduction in the intensity of one of the rectangular components, and may be shown best by means of a pair of Nicol prisms, so oriented as to refuse transmission. If a glass plate, or better, a pile of three plates, be placed between the prisms at the polarizing angle and so oriented that the plane of incidence is inclined at 45° to the principal planes of the Nicols, there will be a partial restitution of light, and the analyzing Nicol will have to be turned counter clockwise to produce complete extinction.

It is clear now why the transmitted light is only partially polarized, when a ray is refracted at a single surface, and completely polarized by refraction at a large number of parallel surfaces. We may consider ordinary light as consisting of vibrations polarized in all possible planes. Each vibration is therefore transmitted with a slight rotation towards the plane of incidence, and the light will not differ greatly in its properties from ordinary light. By every succeeding surface there is a further rotation, and eventually all are brought into the plane of incidence and the transmitted light is plane-polarized. If the vibrations of the incident light be represented by A (Fig. 214), the effect of successive refractions may be represented by the succeeding diagrams B, C, D, E, the transmitted light being plane-polarized in the latter case. As we shall see in a subsequent chapter, the probable condition in ordinary light can be more nearly represented by considering it plane-polarized light, the plane of poplarization changing with great rapidity. This conception will not alter the above representation when the condition present during a finite time is under consideration. The foregoingelement ary treatment of polarization by reflection will suffice for the present. The theoretical treatment of the subject will be given in a subsequent chapter. We will now consider some of the other methods by which polarized vibrations can be obtained.

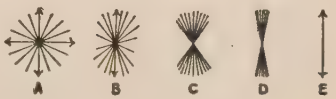


Fig. 214.

Polarization by Double Refraction. — The double refraction of light by crystals of Iceland spar was first noticed by Erasmus Bartholinus, and subsequently more completely investigated by Huygens. It occurs whenever light enters the crystal in a direction not parallel to the optic axis, and is due to the fact that the incident vibration is decomposed into two mutually perpendicular components which travel through the crystal with different velocities. One of the two rays obeys the ordinary laws of refraction and is called the ordinary ray, while the other behaves in a most peculiar manner and is called the extraordinary ray, for it is bent away from the normal *even at perpendicular incidence*. At first sight this appears impossible, for even if the two rays have different velocities there seems to be no reason why there should be a change of *direction* for normal incidence. As we shall see when we come to the

Chapter on Double Refraction, this is due to the fact that the wave-front of the extraordinary disturbance is not spherical but ellipsoidal. For the present, however, we are not concerned with this question, and merely consider the crystal as a means of resolving ordinary light into two polarized beams. One of these may be cut off by any suitable device, leaving us plane-polarized light. Crystals of tourmaline have the remarkable property of absorbing the ordinary ray and transmitting the extraordinary, consequently a thin section of a crystal transmits only polarized light, and may be used either as a polarizer or analyzer. The sections are cut parallel to the optic axis, and when superposed with their axes parallel transmit light quite freely. If one is rotated through a right angle, the combination becomes opaque, since the polarized vibrations transmitted by one are absorbed by the other. The tourmaline polariscope is a very simple instrument, consisting of a pair of crystal sections mounted in a pair of wire tongs in such a way that one of them can be rotated in front of the other. The object to be examined, for example, a mica or selenite film, is placed between the two tourmalines, and the instrument directed towards a bright light; owing to the deep color of the tourmaline crystals, this form of polarizer and analyzer is very inferior to the Nicol prism, which only cuts down the intensity of the original light one-half.

The Nicol Prism. — Iceland spar, on account of the large size in which the crystals occur, and their great transparency, is especially suitable for the construction of polarizing prisms. Since both the ordinary and extraordinary rays are transmitted with equal facility, it becomes necessary to eliminate the one or the other by some optical contrivance. The method employed by Nicol was to stop the ordinary ray within the crystal by total reflection. A rhomb of spar is cut in two along a plane, as indicated in Fig. 215, and the two halves cemented together with Canada balsam, the two oblique surfaces having first been polished. The refractive index of the balsam is intermediate between that of the spar for the ordinary and extraordinary rays, and the former meeting the balsam film at an angle greater than the critical angle is totally reflected to one side and absorbed by a coating of black paint.

Fig. 215.

The prism therefore only transmits the extraordinary ray, which is plane-polarized, the direction of the vibration being parallel to the short diagonal of the prism, *i.e.* perpendicular to the balsam surface. In some prisms it is impossible to tell by mere inspection in which direction the emergent light vibrates, owing to the way in which the crystal has been cut. It is always possible, however, to determine in a moment the direction in question. We have only to reflect ordinary light from a glass surface in the neighborhood of the polarizing angle and examine it through our prism, holding it in such a position that the transmitted light has its maximum intensity. The vibrations of the light are parallel to the glass surface, consequently the diagonal of the prism which is parallel to this surface is the direction in which

the transmitted light vibrates. If any difficulty is experienced in fixing in the mind the direction of vibration in the case of reflected and transmitted light, the following analogy may prove useful. If a cylindrical wooden rod is thrown in an oblique direction and with great force upon the surface of water, it will bounce off if parallel to the surface, the rod being supposed to move in a direction perpendicular to its length. If, however, the rod is perpendicular to the surface, the lower end will enter the water first and the rod "cut down" into the fluid, without suffering reflection. Consider our polarized vibrations as parallel to the rod, and we have the optical analogy, which is only of use, however, in enabling us to remember the direction of the vibration in the reflected and refracted components.

The Nicol prism is sometimes made with end surfaces perpendicular to the axis. This necessitates a more oblique section, and consequently a greater length in proportion to the width, the ratio being nearly 4 : 1. The Foucault prism is similar to the Nicol, except that the balsam film is replaced with an air film. This device reduces the ratio of length to width to 1.5 : 1, but the prism is less efficient than the Nicol, owing to multiple reflections in the air film. It is useful for work in the ultra-violet region, for balsam absorbs these rays.

Angular Aperture of Polarizing Prisms. — Since many experiments require the passage of a convergent or divergent beam through the prism, it is of some interest to consider the maximum angular aperture of the prism, or maximum divergence which a cone of rays may have, and still be completely polarized by passage through a prism of given type. If the divergence exceeds a certain amount, it is obvious that some of the ordinary rays will not suffer total reflection. The type of prism originally designed by Nicol was made by grinding down and polishing the ends of the rhomb, by an amount sufficient to reduce the angle between the end surfaces and the sides from 72° to 68°, and make the section in a plane perpendicular to the end surfaces and the plane containing the optic axis and long axis of the crystal. The angular aperture of a prism of this type is about 30°, while that of the Foucault prism is only 8°. A prism was devised by S. P. Thompson [1] in which the optic axis was perpendicular to the long axis of the prism, which had an aperture of 39°. Glazebrook [2] constructed one along similar lines, but with end surfaces perpendicular to the long axis.

The question of the best construction for a prism with large aperture and end surfaces perpendicular to the long axis was investigated by Feussner,[3] who found that the optic axis should be perpendicular to the section, and the refractive index of the cementing film the same as that of the crystal for the extraordinary ray. Such a prism has an aperture of 42° and a ratio of length to width of 4 : 1.

In another and quite different type of polarizing prism, the doubly refracting substance acts as the rarer medium, the extraordinary ray being totally reflected from a thin plate of Iceland spar im-

[1] *Phil. Mag.*, 5, 12, page 349, 1881. [2] *Phil. Mag.*, 5, 15, page 352, 1883.
[3] *Zeitsch. für Instrkdé*, 4, page 41, 1884.

mersed in a liquid of higher refractive index. The first prism constructed on this principle was made by Jamin, who immersed a thin plate of spar in a glass trough filled with carbon bisulphide. Zenker improved the device by substituting prisms of flint glass for the liquid, while Feussner suggested the use of a plate of sodium saltpetre instead of Iceland spar, on account of the greater difference between the ordinary and extraordinary refractive index. Such a prism would have an aperture of 56°.

Detection of Polarized Light. — If the amount of polarized light present in a beam is too small to be detected by the slight changes in intensity produced by passing it through a slowly revolving Nicol, some more delicate method must be adopted.

The Nicol used alone will not give evidence of the presence of less than about 20% of polarization, consequently in cases where the polarization is not considerable (the solar corona, for example), we cannot rely upon its testimony. By the use of the so-called "bi-quartz," which is described in the Chapter on Rotatory Polarization, the presence of 5 or 10% of polarization may be detected by the slight coloration of the two segments of the plate when it is placed in front of a Nicol and directed towards the light.[1]

Savart's plate and Babinet's compensator are still more sensitive detectors of small amounts of polarization, and are generally employed in the study of the polarization of the sun's corona during total eclipses. The presence of polarized light is shown by a system of colored fringes which appear when the plate is used in the same manner as the bi-quartz. Savart's plate is made by cutting a plane-parallel plate from a quartz crystal at an angle of 45° with the optic axis. The plate is then cut into two halves, which are mounted the one above the other, but rotated through 90° with respect to each other. The sensitiveness is at a maximum, *i.e.* the fringes are most distinct when the direction of the polarized vibration is perpendicular to the fringes, the instrument then being capable of showing the presence of 1% of polarization.

Determination of the Percentage of Polarized Light. — In the case of light partially polarized it is often of importance to determine the percentage polarized. This may be accomplished in a number of ways. A method frequently used in studying the polarization of the solar corona consists in compensating the polarization by means of one or more inclined glass plates. The same method was used by the author in studying the polarized fluorescence of sodium and potassium vapor. One or more glass plates are placed between the partially polarized source of light and the Savart plate with its analyzing Nicol. The plates are rotated until the Savart fringes disappear, *i.e.* until the polarization produced by transmission through the oblique plates exactly compensates the opposite polarization originally present in the source. The angle through which the plates have been turned is read.

If the angle of incidence upon the inclined plate is Φ, and the

[1] An excellent account of various methods of detecting polarized light and measuring the percentage will be found in the "Report of the U S. Naval Obs of the Total Eclipse of July 29, 1878."

refractive index of the glass is n, we can calculate χ, the angle of refraction from $\frac{\sin \Phi}{\sin \chi} = n$.

From the theory of reflection, we have for the ratio of the amplitude of the vibration *perpendicular* to the plane of incidence to that *in* the plane, for transmission through the two surfaces of a plate,

$$\cos^2(\Phi - \chi).$$

The intensity ratio will be the square of this, or $\cos^4(\Phi - \chi)$.

If we use m plates, we raise $\cos^2(\Phi - \chi)$ to the mth power. For two plates at angle $\Phi = 52°$, which was the angle found in the case of the sodium vapor polarization, $\chi = 31° 10'$,

and $$\cos^2(\Phi - \chi) = .873.$$

We square this again for the intensity ratio, and again since we have two plates and obtain for our ratio .58, that is, 58 parts of the light may be regarded as polarized perpendicular to the plane of incidence, and 100 parts in the plane. The percentage of polarization is the difference between these quantities (total light) divided by their sum,

or $$\frac{42}{158} = .27.$$

Therefore we have 27 % of polarization.

The following method is due to Cornu. It cannot be used, however, unless the percentage of polarization is above 25. A screen perforated with a rectangular opening measuring about 2×4 mms. is placed in front of the source of light, and a double image prism oriented in such an azimuth that one of its planes of vibration is parallel to the polarized vibration which is to be measured. It frequently happens that we know the position of this plane beforehand; for example, if we are studying light reflected from transparent media, we know that the plane of vibration of the polarized portion of the reflected light is parallel to the reflecting surface. The screen and prism are to be placed at such a distance apart that the two images of the aperture just touch one another. One of these images will be found to be brighter than the other, since *all* of the polarized light is present in it, while the unpolarized light is divided equally between the two. We now compensate the intensities by means of a Nicol prism mounted on a graduated circle.

In Fig. 194, let AB be the plane of vibration of the polarized portion of the light. Then the right-hand image of the aperture will be the brighter, if the planes of vibration of the double image are as indicated by the arrows a and b. Let a and b equal the amplitudes of the vibrations, a^2 and b^2 the intensities. The proportion of polarized light will then be given by

$$p = \frac{a^2 - b^2}{a^2 + b^2}.$$

Call ω the angle between the transmission plane (short-diagonal) of the Nicol and the plane of the vibration a. Then the intensities of the two images seen through the Nicol will, by the law of Malus, be $a^2 \cos^2 \omega$ and $b^2 \sin^2 \omega$. If we orient the Nicol so that equality is established, we can equate these two quantities

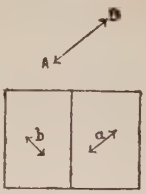


Fig. 216.

$$a^2 \cos^2 \omega = b^2 \sin^2 \omega \text{ or } \frac{a^2}{b^2} = \frac{\sin^2 \omega}{\cos^2 \omega}.$$

The proportion of polarization is given by

$$p = \frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin^2 \omega - \cos^2 \omega}{\sin^2 \omega + \cos^2 \omega} = -\frac{\cos^2 \omega - \sin^2 \omega}{1} = -\cos 2\omega.$$

It will be at once seen that 2ω will be over 90°, since $\omega = 45°$ when a and b are originally equal, *i.e.* when the light contains no traces of polarization. If the plane of partial polarization is not known, we may take a reading with the double image prism in any position, and then rotate the whole instrument through 90° and take a second reading. The proportion of polarized light will then be given by

$$p = \sin(\omega_2 - \omega_1).$$

The Eye as an Analyzer: Haidinger's Brush. — The polarization of light can sometimes be detected by the eye alone. If we look through a Nicol prism at a white cloud, and slowly revolve the prism, a faint blue and yellow double brush appears at the point upon which the eye is fixed, which revolves with the prism. It is not easy to see it at the first attempt, but once noticed, it is easily recognized on subsequent occasions. It consists of four quadrants, colored blue and yellow alternately, and is usually very faint. Various explanations of the phenomena have been given, most of them based on the laminary structure of the lens of the eye. If this were the case, however, the centre of the brush ought to appear a little to one side of the point observed, since the point on the retina which receives the image of the point upon which the eye is fixed lies to one side of the axis of the eye. This fact led Helmholtz to look for the cause of the phenomenon in the structure of the "yellow spot" of the retina, which is the point of the retina alluded to above. He found that the brush was due to the fact that the yellow elements of the spot were doubly refracting, and that the extraordinary rays of blue color were more strongly absorbed than the ordinary rays. For a more complete treatment of the subject the reader is referred to Helmholtz's *Physiological Optics*.

Polarization by Oblique Emission. — Arago found that the light emitted in an oblique direction from the white hot surface of a solid or liquid was partially polarized in a plane perpendicular to the plane of emission, *i.e.* it resembled light refracted at an oblique surface. From this we may infer that the emitted light comes not only from the surface molecules, but also from those lying below the surface, and that the polarization is due to the refraction of this light when it emerges into the air. The light from an incandes-

cent gas is not usually polarized, owing doubtless to the low refractive index, and the fact that we seldom have a sharply defined surface.

The state of polarization of the light coming from an oblique surface of a body gives us a clew as to the origin of the light and the state of the body. The light of the moon, for example, is partially polarized in a plane passing through the sun, moon, and earth, which shows us that the moon shines by reflected light.

If it shone by emission the light would be polarized slightly in the opposite plane. The absence of polarization in the light coming from near the edge of the sun led Arago to infer that it was emitted by an incandescent gas, which is in accord with Schmidt's theory of the solar disk. (See Chapter on Refraction.)

Stationary Polarized Waves. — Wiener found, in repeating his experiments with plane-polarized light, that if the light was incident at an angle of 45° the effects of stationary waves were obtained only when the plane of polarization was parallel to the plane of incidence. Stationary waves can be formed only when the effective vector in the reflected disturbance is parallel to the vector of the incident light, from which we infer that the vector which is effective in producing photographic action is perpendicular to the plane of polarization. The same was found to hold true for fluorescence action. Wiener having already determined that a node of the stationary wave system occurred at the reflecting surface, the inference to be drawn from the experiments with oblique light was that the electric vector was the one concerned in photographic and fluorescent action, and that it was perpendicular to the plane of polarization. This will be made clearer in the Chapter on the Theory of Reflection.

Landolt's Fringe. — If a brilliant source of light is viewed through a pair of Nicol prisms, so oriented that their principal planes are at right angles, it will be seen that the whole field does not become absolutely dark when exact adjustment is reached, but that the darkened field is crossed by a back fringe which changes its position if either prism is rotated ever so slightly. This fringe, which was first noticed by Landolt, was explained by Lippich,[1] who showed that it was due to the fact that the directions of vibration in the different parts of the field were not strictly parallel, a circumstance resulting from the varying obliquity of the rays. Lippich showed further that in the case of polarizing prisms with end faces perpendicular to the prism axis the direction of vibration was represented by a system of converging lines which met in a point outside of the prism, as shown in Fig. 217, *a*. Complete darkness will occur with crossed Nicols only in those parts of the field where the directions of vibra-

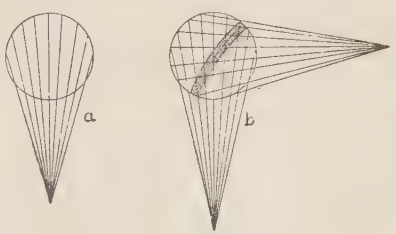


FIG. 217.

[1] F. Lippich, *Wien. Akad. Ber.*, III., Bd. lxxxv., page 268, 1882.

tion in the two prisms are perpendicular. If we draw two fields similar to the one represented by *a*, at right angles to each other, we shall find that the small areas formed by the intersection of the converging lines are in general diamond shaped, but that they are approximately rectangular along a curved line represented by the dark band in Fig. 217, *b*. This is the region where the directions of vibration are strictly perpendicular, and it in consequence appears black. A slight rotation of either field will obviously change the position of this locus, the squares becoming diamond shaped, and the adjacent diamonds straightening out into squares. Rotation of one of the prisms through 180° will be found to give a locus of squares perpendicular to the one shown in the figure, and since the fringe moves broadside across the field, the directions of motion in the two cases are perpendicular.

Lippich made use of the fringe in the construction of a polarimeter, in which the position of the fringe was determined with reference to a pair of cross-hairs. It was possible in this way to set the analyzing Nicol with an error not exceeding two or three seconds of arc. A very brilliant source of light must be used, the sun for example.

CHAPTER X

DOUBLE REFRACTION

We have seen in the Chapter on Polarization that crystals of Iceland spar have the property of dividing a ray of light into two plane-polarized rays, the directions of polarization being at right angles to each other. In the present chapter we shall study in detail the propagation of light in crystalline media, and the laws which govern it.

The division of a ray of light by a crystal of Iceland spar, or double refraction, was observed in 1669 by Erasmus Bartholinus, and the polarization of the two rays was subsequently discovered by Huygens, though he was unable to explain the phenomenon, since at the time light was supposed to consist of waves in which the displacement was parallel to the direction of propagation. It was not until Young and Fresnel introduced the idea of transverse waves that the true nature of polarization was understood. Double refraction can be easily observed by laying a crystal of Iceland spar over an ink dot on a sheet of paper. Two images are seen which can be quenched in succession by the rotation of a pile of glass plates held at the polarizing angle. On revolving the crystal of spar, one of the images is seen to remain stationary, while the other revolves around it. The distance between the two images is independent of the position of the eye, showing that the rays of light, after refraction through the crystal, emerge parallel to one another, as shown in Fig. 218. At first sight it may appear strange that rays of light, incident normally upon the refracting surface, should be deviated away from the normal, since we are sometimes accustomed to associate the bending due to refraction with oblique incidence, the light-waves turning through an angle as they enter the denser medium. We shall see presently, however, that the phenomenon is easily explained by Huygens's principle, when applied to the peculiar type of waves which we have in doubly refracting media.

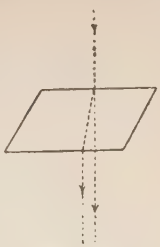


Fig. 218.

Experiments have shown that in crystals belonging to the hexagonal and tetragonal system one of the rays obeys the ordinary laws of refraction, *i.e.* the refracted ray lies in the plane of incidence, and the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction. This ray is called the ordinary ray; the other ray in general conforms to neither of these two laws, though in certain cases it may conform to one or to both of them. This ray is called the extraordinary ray. In the case of all other crystals except those of the cubic system, neither of the two refracted rays

conforms of necessity to the ordinary laws of refraction. Crystals belonging to the cubic system do not exhibit the phenomenon of double refraction, the light being propagated as in isotropic media.

Physical Explanation of Double Refraction. — On the elastic solid theory we can explain double refraction in crystalline media by assuming that there are three directions called axes of elasticity, which have the distinctive properties of the two planes of vibration of Blackburn's pendulum, which consists of a weight suspended by strings as shown in Fig. 219, *a*. If the weight is displaced either in, or perpendicular to, the plane of the paper it will oscillate in a straight line, the period being greater for vibrations perpendicular to this plane than for those parallel to it. If, however, it is displaced in an oblique direction, the force acting upon it will no longer be directed towards the position of equilibrium, and the weight will move in a curved orbit. In the case of crystals a particle displaced parallel to any one of the axes of elasticity will be acted upon by a force directed towards the equilibrium position, and the vibration will be plane-polarized. If displaced in any other direction and released, it does not return to its original position, but moves in a curved path in a manner analogous to that of the pendulum. We require an explanation of the splitting of a beam of light into two polarized components, and for their unequal velocities of propagation. The vibrations of a cylindrical rod form a useful analogy. In this case the elasticity is the same in all directions, and traverse vibrations in all planes are transmitted with the same velocity. Suppose the rod to be struck in a very brief time in every possible direction, then each particle will move in an orbit which is the resultant of all these impulses. The waves transmitted along the rod in this case are analogous to the waves of light in isotropic media. Consider now that the rod has an elliptical cross section (Fig. 219, *b*). The elasticity is now not the same in all directions, being greatest in the plane of the major axis, and least in the plane of the minor. Wave-motion will traverse it with greater velocity if the direction of vibration is parallel to the longer diameter than if the direction is perpendicular. If an attempt is made to transmit vibrations making an angle with the axes of the elliptical cross section, by striking the end of the rod in a direction other than that parallel or perpendicular to the major axis, the vibration will be decomposed into two components which travel along the rod with different velocities. The rod, in other words, is incapable of transmitting vibrations which make an angle with the axes.

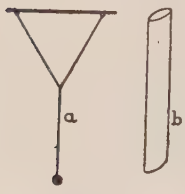


FIG. 219.

We have in doubly refracting media a somewhat similar condition, the elasticity being different in different directions. Luminous vibrations will be decomposed into two polarized components which traverse the crystal with different velocities.

Wave-Surface in Uniaxal Crystals. — The fact was established by Huygens that, in isotropic media, the form of the wave-surface was spherical, and as one of the rays in Iceland spar was found to obey

the ordinary laws of refraction, he assumed that the corresponding wave was a sphere. In the case of the extraordinary ray, which does not obey such simple laws, he made the assumption that the wave-surface was a spheroid, *i.e.* an ellipsoid of revolution. The velocity of the extraordinary ray in any direction is therefore given by the following construction: "Let an ellipsoid of revolution be described around the optic axis, having its centre at the point of incidence, and let the greater axis of the generating ellipse be to the lesser, in the ratio of the greater to the lesser index of refraction. Then the velocity of any ray will be represented by the radius vector of the ellipsoid which coincides with it in direction."

The optic axis may be defined as the direction in the crystal in which a ray of light may be propagated without double refraction. The law just given was found to apply to Iceland spar and many other crystals, but in all of these there was but a single optic axis. Brewster, however, discovered that in many crystals there were two directions in which light could be propagated without double refraction. Such crystals are termed biaxal, and the law of Huygens was found not to apply in these cases. Fresnel then established a theory which not only conformed to all of the known facts, but made possible predictions which were afterwards verified by experiment. This theory we shall take up a little later.

According to the theory of Huygens the wave-surface in uniaxal crystals consists of two sheets, one a sphere, the other a spheroid, which touch each other at two points. The direction of the line joining these points of contact is called the optic axis of the crystal. This conception applies, however, only to uniaxal crystals. In the case of Iceland spar and all so-called negative crystals, the sphere lies within the spheroid. In such crystals the angle of refraction of the extraordinary is greater than that of the ordinary ray. In the case of quartz and other positive crystals, the spheroid lies within the sphere, and the angle of refraction of the extraordinary ray is less than that of the ordinary. This will be clearer when we come to the construction of the refracted ray.

Huygens's Construction. — Suppose a luminous disturbance to start within a uniaxal crystal. The wave will spread out in two sheets, a sphere and a spheroid, which touch each other at two points. In the direction of the line joining these two points both waves travel with the same velocity. If we apply Huygens's construction to crystalline refraction, giving to the secondary wavelets, which originate on the refracting surface, the forms of spheres and ellipsoids, we can account for, and calculate the position of the two refracted rays. In all other directions the velocities will be unequal and we shall have a division of the ray, as may be seen by Huygens's construction. Consider a wave-front AB incident in an oblique direction upon the surface of a uniaxal crystal (Fig. 220). The direction of the optic axis is represented by the dotted line. The point A becomes the centre of two secondary wavelets which are propagated with different velocities. Making use of the same construction which we applied in the case of isotropic media (see Chapter IV.), we draw tangent planes from the point C to the two

wave-surfaces; the directions of the refracted rays are given by joining the point A with the points of tangency. In the case of the ordinary wave the refracted ray lies in the plane of incidence. This is also the case with the extraordinary ray, *provided the optic axis lies in the plane of the paper*. If, however, the optic axis is not in the plane of the paper, the point of tangency for the extraordinary wave will lie above or below the plane of the paper, and the refracted ray will no longer be in the plane of incidence. In the latter case neither of the ordinary laws of refraction is obeyed, for the sine relation only holds when the section of the secondary wave is circular. If the optic axis is perpendicular to the plane of incidence, the section of the spheroid is equatorial and therefore circular, the extraordinary refracted ray in this case lying in the plane of incidence and obeying the sine law. The ratio of the sines of the angles of incidence and refraction in this case is termed the extraordinary index of refraction.

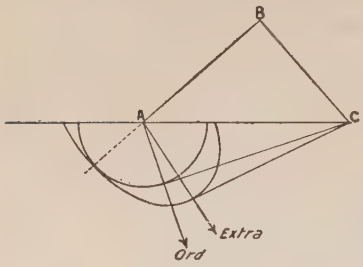


FIG. 220.

We will next consider the case in which rays of light are incident in a normal direction upon the crystal. As we have seen, double refraction occurs in this case, one of the rays passing straight through, while the other is deflected away from the normal. It is obvious that we cannot apply in this case the simple explanation of refraction which assumes successive portions of the wave-front retarded upon entrance into the refracting medium. For the wave-fronts originally parallel to the surface must remain so after refraction. What we have actually, if our original waves be plane, are two plane-waves travelling through the crystal with unequal velocities but parallel always to the surface. The deflection of the extraordinary ray is obvious if we apply Huygens's construction to the present case. Assume that the points on the surface of the crystal become simultaneously the centres of ellipsoidal wavelets as indicated in Fig. 221. If the incident wave-front is limited to the region AB, the refracted wave-front will be the tangent plane of the ellipsoidal wavelets, and the refracted rays will be the lines AA', BB'. What actually happened may be described as follows: The refracted wave-front travels in the medium in a direction normal to its surface, but any limited portion of it bears away constantly to one side, and the ray is defined as the direction in which a limited portion of the wave-front travels. We see in this

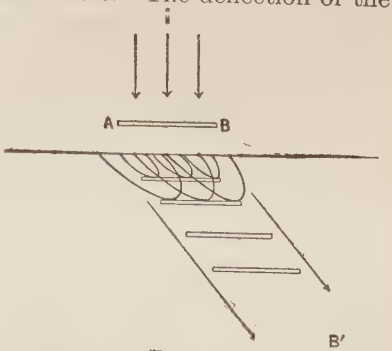


FIG. 221.

case that the ray is not perpendicular to the wave-front, which is, in general, the case in doubly refracting media.

Verification of Huygens's Construction. — The assumptions made by Huygens regarding the form of the wave-surfaces in uniaxal crystals were speedily verified by experiment. That the ordinary wave-front is a sphere, was shown by constructing a prism formed of pieces cut in all possible directions from a crystal of Iceland spar and cemented together. The spectra formed by the extraordinary rays were deviated by different amounts, whereas a single spectrum only was formed by the ordinary rays which traversed the different elements of the prism. To verify the construction of the extraordinary wave-front we will consider several cases.

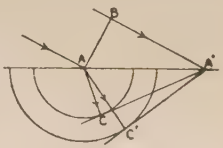


FIG. 222.

(1) **The refracting face is parallel to the optic axis, and the plane of incidence perpendicular to it** (see Fig. 222). — In this case the axis is perpendicular to the plane of the paper. The sections of the two wave-surfaces will in this case be circles, as we have seen. The tangent planes touch the sphere and spheroid at C and C'. Let the velocity in air be 1, then the velocity of the ordinary and the extraordinary rays will be proportional to b and a the radii, and the refractive index of the extraordinary ray will be

$$\frac{\sin i}{\sin r} = \frac{1}{a} = \mu_e.$$

By cutting a prism of Iceland spar with its refracting edge parallel to the optic axis we obtain two spectra, and by measuring the deviation with a spectrometer we can calculate in the usual manner the refractive indices μ_o and μ_e for the ordinary and extraordinary rays. It can easily be shown that both rays are propagated through the prism according to the same law which holds in the case of a glass prism. This indicates that the section of the wave-surface is a circle for both rays, the radius for the extraordinary ray being $\frac{1}{\mu_e}$ and for the ordinary $\frac{1}{\mu_o}$. The extraordinary wave is therefore a surface of revolution around the optic axis, and to determine the form of the generating curve we shall consider the refraction which takes place under different conditions.

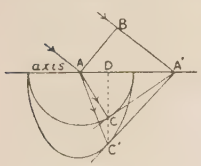


FIG. 223.

(2) **Optic axis parallel to the surface of the crystal and to the plane of incidence.** — The sections of the wave-surfaces in this case are shown in Fig. 223. *Assume* the extraordinary wave-section to be an ellipse, the minor axis of which lies in the surface. The section of the sphere will be a circle touching the ellipse at the extremities of the minor axis. Drawing tangent planes from A', as before, to the two wave-surfaces, and joining the points of tangency with A, we obtain the refracted rays. A line joining the two points of contact and produced will cut the minor

axis at a right angle, since the polar of any point in the chord of contact of a circle and ellipse having double contact is the same with regard to both curves. We have then

$$\frac{\tan r}{\tan r'} = \frac{AD}{CD} \times \frac{C'D}{AD} = \frac{C'D}{CD} = \frac{a}{b} = \frac{\mu_o}{\mu},$$

or the ratio of the tangents of the angles of refraction is equal to the ratio of the two indices of refraction.

This relation, which was deduced on the assumption that the wave-section was an ellipse, was verified by Malus in the following manner: Two scales AC and BC (Fig. 224) were engraved on a

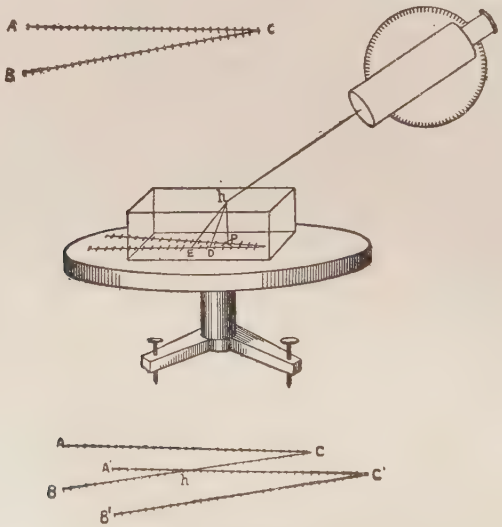


Fig. 224.

plate of polished steel, and a thick plate of crystal with its faces parallel to the optic axis was laid on the scale and viewed through a telescope mounted on a graduated vertical circle. The crystal was brought into the horizontal position by means of levelling screws, the correct position being that in which the image of a distant point of light was not changed by rotation of the platform. Two images of the scale were seen in the telescope, and if we denote these by AC, $A'C'$, BC, $B'C'$, there will be some point of BC coinciding with some point of $A'C'$. We call this point h. If the axis of the telescope is directed towards this point it will cut the surface of the crystal at h, the position of which can be determined with reference to the scales. The divisions at E and D which appear to coincide can be read off and the distance ED determined by actual measurement. If e is the thickness of the crystal, we have

$$ED = EP - DP = e\,(\tan r' - \tan r),$$

in which $\tan r$ is known, for the angle of incidence (considering the ray reversed) is equal to the angle which the axis of the telescope makes with the vertical. Moreover, $\sin i = \mu_o \sin r$ (since the ordinary ray obeys both laws of refraction for all conditions), therefore r is known, and r' may be determined by the above formula. If the value of r' thus found agrees with the value determined by the formula $\frac{\tan r'}{\tan r} = \frac{\mu_e}{\mu_o}$, the experiment will have proved that the section of the extraordinary wave is, in the present case, an ellipse. Inasmuch as we have already proven that the wave front is a surface of revolution, the experiment will prove that it is a spheroid of axes A and B. The experiments made by Malus completely verified this theory.

The method employed by Malus was not very accurate. A method based upon prismatic refraction was used by Stokes, which gave results correct to the fourth place of the decimal, verifying Huygens's construction, and completely disproving the law resulting from the theory that the double refraction results from a difference of inertia in different directions.

In Fresnel's theory it is the elasticity which is supposed to be variable. Lord Rayleigh many years ago suggested that the inertia might vary with the direction. It is hard to see how the actual density could vary with the direction, but the same difficulty does not hold in the case of the effective density or inertia.

Lord Rayleigh takes, as an analogy, the vibration of a disk held in position by springs and immersed in water; its period would be very slow when vibrating in a direction perpendicular to its plane. As Stokes pointed out, however, experiment disproves this theory. We will now take up Fresnel's theory.

Fresnel's Theory of Double Refraction. — In the foregoing discussion we have considered only uniaxal crystals, making certain assumptions regarding the form of the wave-surfaces, and showing that certain relations deduced from them were verified by experiment.

We will now consider the phenomenon of double refraction in its more general aspect, following the treatment of Fresnel.

As we have seen, the velocity of a transverse wave in an isotropic medium is proportional to $\sqrt{\frac{e}{d}}$, in which e is the elasticity of the medium.

In doubly refracting media e is assumed to vary with the direction of the displacement, and there will be two directions in every possible plane for which e has its maximum and minimum values. The corresponding velocities of propagations $\sqrt{\frac{e_1}{d}}$, $\sqrt{\frac{e_2}{d}}$ are for vibrations parallel to these two directions. If the displacement is in any other direction, the wave is not propagated with an intermediate velocity, as might at first be supposed, but is decomposed into two waves, which travel with the above velocities, the directions of their vibrations being perpendicular to each other. If we are dealing with

trains of waves, as is always the case, the actual motions of the vibrating particles will not be along straight lines, for they are the resultants of the two sets of disturbances which are travelling with different velocities. Until the rays become completely separated by the double refraction, we must regard the vibration as changing its type from point to point, changing from plane to elliptical and circular, and then back again to plane, as the relative phases of the two perpendicularly polarized disturbances alter.

If the direction of displacement coincides with one or the other of the two directions of maximum or minimum elasticity, a single plane-polarized wave will be propagated in the medium. From this it is clear that in the case of the changing type of vibration alluded to above, the vibration along a line will never occur in either one of these directions, for if it did, it would be propagated from that point on, as a plane-polarized vibration without further change.

Fresnel arrived at a conception of the wave-surface by considering it as the envelope of an infinite number of plane-waves, which have passed simultaneously in all possible directions, through a given point in the doubly refracting medium.

Consider now the following construction. Through the point in question imagine an infinite number of planes, in all possible orientations, and draw through the point, on each plane, two lines at right angles to each other, and coinciding with the directions of maximum and minimum elasticity, and of lengths proportional to velocities of propagation of disturbances vibrating parallel to the lines in question.

If the two lines are made to bisect each other at the point, the terminal points of the lines for all the planes taken collectively will lie upon an ellipsoid. This fact can be deduced theoretically, by making certain specifications regarding the medium, but as the deduction will not help us much in understanding the phenomena, we will simply consider it as representing experimental facts. Having the ellipsoid given, it is possible to find the direction of vibration and the velocities of propagation of a plane-wave, by drawing a central section of the ellipsoid parallel to the plane wave-front.

The ellipsoid is called the ELLIPSOID OF ELASTICITY.

Let its equation be $a^2x^2+b^2y^2+c^2z^2=V^2$, in which V is the velocity of light in vacuo.

The constants, a, b, and c, are related to the elastic properties of the medium, and represent the velocities of waves vibrating parallel to the axes of elasticity, which may be defined as the three directions at any point, along which we can displace the ether, and have the force of restitution parallel to the displacement. In any given *plane* there are but two such directions; in space, however, there are three.

If we take as our unit of time the time occupied by a wave in travelling unit distance in vacuo, then $V=1$. If we put $x=0$ in our equation, we obtain the equation of the intersection of the ellipsoid with the yz plane, which is an ellipse having $\frac{1}{b}$ and $\frac{1}{c}$ as semi-axes, and a plane-polarized wave will be propagated along the x

axis with a velocity b if the direction of vibration is parallel to y, or with a velocity c if it is parallel to z.

The reciprocals $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ correspond to refractive indices, and are called the principal refractive indices. If we designate them by μ_1, μ_2, μ_3, we can write the equation of the ellipsoid in the form

$$\frac{x^2}{\mu_1^2}+\frac{y^2}{\mu_2^2}+\frac{z^2}{\mu_3^2}=1.$$

The deduction of the equation of the ellipsoid from a consideration of the elastic properties of the medium is generally accomplished by considering the potential of the medium. The following simple method is taken from Schuster's *Optics:*

"Fresnel's method of treating double refraction which led him to the discovery of the laws of wave-propagation in crystalline media, though not free from objection, is very instructive, and deserves consideration as presenting in a simple manner some of the essential features of a more complete investigation. Consider a particle P attracted to a centre O with a force a^2x when the particle lies along OX, and a force b^2y when it lies along OY. The time of oscillation, if the particle has unit mass, is $2\pi/a$ or $2\pi/b$ according as the oscillation takes place along the axis of X or along the axis of Y. When the displacement has components both along OX and OY, the components of the force are a^2x and b^2y, and the resultant force is

$$R=\sqrt{a^4x^2+b^4y^2}.$$

"The cosines of the angles which the resultant makes with the coordinate axes are a^2x/R and b^2y/R. The direction of the resultant force is not the same as that of the displacement, the direction cosines of which are x/r and y/r. The cosine of the angle included between the radius vector and the force is found in the usual way to be

$$\frac{a^2x^2+b^2y^2}{Rr},$$

and the component of the force along the radius vector is

$$\frac{(a^2x^2+b^2y^2)}{r}.$$

"If we draw an ellipse $a^2x^2+b^2y^2=k^2$ (Fig. 225), where k is a constant having the dimensions of a velocity, the normal to this ellipse at a point P, having coordinates x and y, forms angles with the axes, the cosines of which are in the ratio a^2x to b^2y, hence the force in the above problem acts in the direction of ON of the line drawn from O at right angles to the tangent at P. The component of the force along the radius vector is k^2/r, and the force per unit distance is k^2/r^2, so that if the particle were constrained to move on the radius vector OP, its

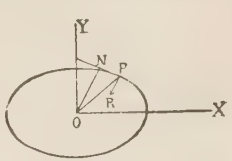


FIG. 225.

period would be $2\pi r/k$. Since the ratio r/k depends only on the direction of OP, our result is independent of the particular value we attach to k.

"If we extend our investigation to three dimensions, the component of attraction along OZ being c^2z, we obtain the same result, and the component of force acting along any radius vector OP per unit length is k^2/r^2, where r is the radius drawn in the direction of OP to the ellipsoid

$$a^2x^2+b^2y^2+c^2z^2=k^2."$$

For a plane-wave to be propagated without alteration it is essential that the effective force of restitution shall be parallel to the displacement.

Though in general this force does not even lie in the plane of the wave-front, we can always resolve it, however, into two components, one in, the other perpendicular to the front. Fresnel neglected the latter component, as it contributes nothing towards the propagation of a transverse wave. The longitudinal disturbance which, in the case of elastic solids, is produced by the normal component, is considered as non-existent in the case of light, owing to the incompressibility of the medium.

The direction of the component of force parallel to the wave-front is along the radius vector of the ellipsoid which is perpendicular to the section conjugate to the direction of the displacement. This will be made clearer by reference to Fig. 226. Let $abcd$ be a plane-wave travelling within the crystal, the direction of the displacement being parallel to ab. The ellipsoid is assumed constructed around a point lying on the wave-front, which cuts it in the elliptical cross-section as indicated. The displacement is along AO, which we will assume to be the semi-major axis of the ellipse, while the direction of the force of restitution is along the radius ON, perpendicular to the plane BOC. If the projection of ON on the plane of the wave-front coincides with the direction of the displacement OA, the plane AON must be perpendicular to the wave-front, and since ON is perpendicular to OB, OB must be perpendicular to OA; in other words, OA and OB are the axes of the elliptical section. This is the condition which we assumed at the start. If the direction of the displacement is not along one of the axes, the effective force of restitution will not be directed parallel to the displacement, and *two* plane-polarized waves will result as we have seen. Two sections of the ellipsoid will be circular, and plane-waves parallel to these sections will be propagated without alteration, whatever may be the direction of the displacement, though there may be a division of the ray, as we shall see presently. These circular sections of the

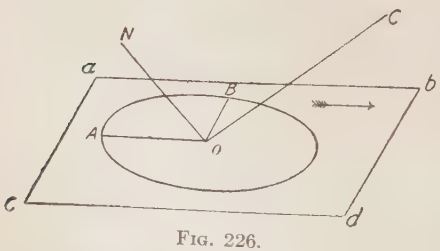


Fig. 226.

ellipsoid of elasticity are perpendicular to the optic axes of the crystal. We may sum up as follows:

In any given direction in the crystal two systems of plane-waves can be propagated normally, the vibrations being along the axes of the elliptical cross-section, and the velocities of normal propagation inversely proportional to the lengths of the axes. Two directions exist, however, in which but a single wave-front is propagated, known as the axes of single wave-velocity or optic axes. In these directions the velocity of normal propagation of a plane-wave is independent of the direction of vibration, although the direction in which a limited portion of the wave-front travels (the ray direction) depends upon the nature of the vibration, for the ray is not necessarily perpendicular to the wave-front in crystalline media.

We will now investigate the form of the wave surface, which we can do by considering a geometrical construction known as the normal velocity surface.

The Normal Velocity Surface. — Around any point O within a crystal construct the ellipsoid of elasticity, and consider a system of plane-waves passing simultaneously through O in all possible directions. We have seen that, in general, a crystal has the property of transmitting only vibrations polarized in a definite direction, and that all other types of vibrations are resolved into two components which travel with unequal velocities. We shall thus have two systems of plane-waves passing through the point. To determine the velocities of these waves in different directions we proceed as follows. Let any one of the plane-waves passing through O cut the ellipsoid in the section AOB (Fig. 227), of which the axes are OA and OB. Draw a normal to the plane at O and measure off on it distances ON and ON', inversely proportional to the axes OA and OB. If now planes are drawn through N and N' parallel to the original plane of the section, they will represent the positions of the two waves which passed through the point O simultaneously, the one having its vibrations parallel to OA and the other parallel to OB. If we rotate the plane AOB around O in every possible direction, the points N and N', as defined above, will trace out a surface consisting of two sheets termed the surface of normal velocities, any radius vector of which determines the normal velocity of the plane-wave propagated in that direction. Since for two positions of the plane AOB the section of the ellipsoid is circular, it is obvious that the points N and N' will coincide, when the waves are parallel to these sections. In other words, the inner sheet will touch the outer at four points.

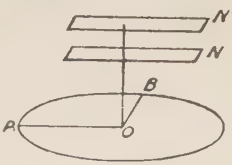


Fig. 227.

This surface is not, however, identical with the wave surface, which is the surface enveloped by the plane-waves which we have just considered. This family of planes is represented by the equation

$$lx + my + nz = v,$$

in which l, m, n are the direction cosines of the direction in which the wave travels with a velocity v, which is, however, a function of l, m, and n. We require a relation connecting these quantities. The following treatment is taken from Rayleigh's *Wave-Theory:*

"If v be the velocity of propagation in the direction l, m, n, the wave-surface is the envelope of planes $lx+my+nz=v$, where v is a function of l, m, n, the form of which is to be determined. If $(\lambda\mu\nu)$ be the corresponding direction of the vibration, then

$$l\lambda+m\mu+n\nu=0."$$

According to the principles laid down by Fresnel, we see at once that the force of restitution $(a^2\lambda,\ b^2\mu,\ c^2\nu)$ corresponding to a displacement unity is equivalent to a force v^2 along $(\lambda\mu\nu)$ together with some force (P) along (lmn).

Resolving parallel to the coordinate axes, we get

$$lP=a^2\lambda-v^2\lambda, \qquad mP=b^2\mu-v^2\mu, \qquad nP=c^2\nu-v^2\nu,$$

or

$$\lambda=\frac{lP}{a^2-v^2}, \qquad \mu=\frac{mP}{b^2-v^2}, \qquad \nu=\frac{nP}{c^2-v^2}.$$

Multiplying these by l, m, n respectively, and remembering the relation $l\lambda+m\mu+n\nu=0$, we obtain

$$\frac{l^2}{a^2-v^2}+\frac{m^2}{b^2-v^2}+\frac{n^2}{c^2-v^2}=0,$$

an equation which we shall use presently.

The Wave-Surface. — If for every possible position of the section AOB in the construction which we have just considered, we construct planes through N and N' parallel to the section, these planes will envelop a surface which consists of two sheets, and resembles in its general appearance the normal velocity surface which we have just considered. The surface thus defined is the true wave-surface, representing the form of the wave which we should have if a luminous disturbance started within the body of the crystal.

The equation which represents the system of plane-waves which envelop the wave-surface is

$$lx+my+nz=v,$$

in which l, m, n, and v are subject to the conditions

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0 \text{ and } l^2+m^2+n^2=1.$$

The equation of the wave-surface was found by Archibald Smith (*Phil. Mag.*, 1838, page 335), in the following manner:

By differentiation of the three equations above regarding l, m, n as variables, we obtain

$$xdl+ydm+zdn=dv,$$

$$\frac{ldl}{v^2-a^2}+\frac{mdm}{v^2-b^2}+\frac{ndn}{v^2-c^2}-\left\{\frac{l^2}{(v^2-a^2)^2}+\frac{m^2}{(v^2-b^2)^2}+\frac{n^2}{(v^2-c^2)^2}\right\}vdv=0,$$

$$ldl+mdm+ndn=0,$$

whence by indeterminate multipliers we obtain

$$(1)\ x=Al+\frac{Bl}{v^2-a^2},\quad (2)\ y=Am+\frac{Bm}{v^2-b^2},\quad (3)\ z=An+\frac{Bn}{v^2-c^2},$$

$$(4)\ Bv\left\{\frac{l^2}{(v^2-a^2)^2}+\frac{m^2}{(v^2-b^2)^2}+\frac{n^2}{(v^2-c^2)^2}\right\}=1.$$

Multiplying the first three of these equations by l, m, and n, and adding, we obtain

$$(5)\ v=A.$$

By transposing the third terms, squaring, and adding, we get, since

$$r^2=x^2+y^2+z^2,$$

$$r^2+A^2=2vA+B^2\left\{\frac{l^2}{(v^2-a^2)^2}+\frac{m^2}{(v^2-b^2)^2}+\frac{n^2}{(v^2-c^2)^2}\right\},$$

which by (4) and (5) gives us

$$B=v(r^2-v^2).$$

We now substitute these values of A and B in equation (1) and obtain

$$x=lv+lv\frac{r^2-v^2}{v^2-a^2}=lv\frac{r^2-a^2}{v^2-a^2};\ \therefore\ l=\frac{v^2-a^2}{r^2-a^2}\cdot\frac{x}{v},$$

and similarly $$m=\frac{v^2-b^2}{r^2-b^2}\cdot\frac{y}{v},\quad n=\frac{v^2-c^2}{r^2-c^2}\cdot\frac{z}{v}.$$

Substitution of these values in $lx+my+nz=v$, the equation of the plane-wave system, gives us the equation of the wave-surface,

$$x^2\frac{v^2-a^2}{r^2-a^2}+y^2\frac{v^2-b^2}{r^2-b^2}+z^2\frac{v^2-c^2}{r^2-c^2}=v^2=\frac{v^2x^2}{r^2}+\frac{v^2y^2}{r^2}+\frac{v^2z^2}{r^2},$$

whence $$\frac{x^2a^2}{r^2-a^2}+\frac{y^2b^2}{r^2-b^2}+\frac{z^2c^2}{r^2-c^2}=0,$$

or $$(r^2-b^2)(r^2-c^2)a^2x^2+(r^2-a^2)(r^2-c^2)b^2y^2+(r^2-a^2)(r^2-b^2)c^2z^2=0.$$

Multiplying out and dividing by r^2, we obtain

$$r^2(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2 -c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

We are now in a position to determine the general form of the wave-surface, which we can do by studying its sections with the planes xy, xz, yz. This we can do by making $x=0$, $y=0$, $z=0$ in succession in the equation of the wave-surface, when we obtain the equations of the curves of section. Assume $a>b>c$.

If we make $z=0$, we get at once

$$(x^2+y^2)(a^2x^2+b^2y^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2+a^2b^2c^2=0,$$

or $$(x^2+y^2-c^2)(a^2x^2+b^2y^2-a^2b^2)=0,$$

which is separately satisfied by

$$x^2+y^2=c^2, \text{ a circle of radius } c.$$

and $$a^2x^2+b^2y^2=a^2b^2, \text{ an ellipse of semi-axes } a \text{ and } b.$$

The circle lies wholly within the ellipse, since we have assumed c less than either a or b. Making $x=0$, we find the section with the yz plane to be

$$y^2+z^2=a^2, \text{ a circle of radius } a,$$

and $$b^2y^2+c^2z^2=b^2c^2, \text{ an ellipse of semi-axes } b \text{ and } c.$$

In this case the ellipse lies within the circle.
For $y=0$, the section with the xz plane,

$$x^2+z^2=b^2, \text{ a circle of radius } b,$$

$$a^2x^2+c^2z^2=a^2c^2, \text{ an ellipse of semi-axes } a \text{ and } c.$$

In this case the circle meets the ellipse at four points. The three sections are shown in Fig. 228.

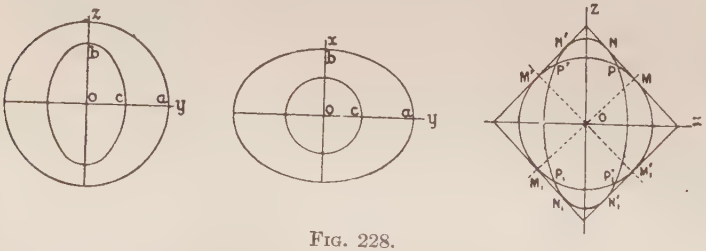


Fig. 228.

A model of the surface can be made by cutting the sections out of cardboard and fitting them together in three perpendicular planes, which can easily be done by cutting them up in a suitable manner, and fastening them together again by means of strips of gummed paper. Such a model is shown in Fig. 229. The inner surfaces should be blackened on both sides of the section as indicated. By a little exercise of the imagination it is easy to see the general form of the inner and outer sheets, though a still better idea can be obtained from the wire or plaster models, which can be procured from dealers in physical apparatus. The outer sheet has the general form of an ellipsoid with four depressions or pits similar to the pit found on an apple around the point where the stem is inserted, only much shallower. At these four points the two sheets come in contact, and some very remarkable optical phenomena are associated with this peculiar condition, which we will now investigate.

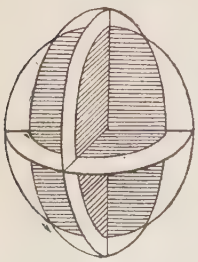


Fig. 229.

The Optic Axes or Axes of Single Wave-Velocity. — Consider now the xz section of the wave-surface, in which the curves intersect at

four points and have four common tangents, one of which is represented by MN (Fig. 230). Planes passing through these tangents and perpendicular to the plane of the section, are tangent planes to the wave-surface. They touch the surface, moreover, not at two points, as was imagined by Fresnel, but all around a circle of contact, a condition which can be represented by pressing a flat card against the dimple on an apple. This was first proved by Sir William Hamilton, who predicted from it the remarkable phenomena of internal conical refraction, which we shall consider presently. The lines OM, OM' perpendicular to the tangent planes are the directions in which a single wave only is propagated, for the planes MN and $M'N'$ touch both sheets. These directions are therefore the optic axes of the crystal.

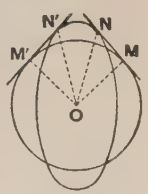


Fig. 230.

Internal Conical Refraction. — Huygens's construction may be applied to determine the direction of the refracted rays, the points on the surface of the crystal becoming centres of wave-surfaces of the form which we have just studied. If light is incident upon the crystal in such a direction that the refracted wave-front is parallel to MN or $M'N'$ (Fig. 230) any line joining the centre O with the circle of contact of MN with the wave-surface is a possible direction of the refracted ray. The direction of the refracted ray will depend on the direction of the vibration in the incident wave-front. The type of the vibration will not be altered by the crystal, since the wave-front is moving parallel to an optic axis, but the direction of the ray will depend on the plane of polarization. If the incident light is polarized in all possible planes, *i.e.* unpolarized, the ray upon entering the crystal will open out into a cone, each elementary ray of the cone being plane-polarized. This result was predicted from theory by Sir William Hamilton and verified by Lloyd (*Trans. Roy. Irish Ac.*, vol. xvii., page 145, 1833) with a plate of aragonite cut so that its faces were equally inclined to the two optic axes.

A divergent cone of light from a screen AB (Fig. 231) perforated with a very small hole, upon which sunlight is concentrated by means of a lens, is intercepted by a second perforated screen CD. This screen can be moved about over the surface of the crystal, and serves to isolate a narrow pencil from the divergent cone. In general, if the transmitted light is received upon a screen at E, two spots of light appear, but by moving the screen CD about, it is possible to find a position such that the two spots run together into a ring of light, the diameter of which is independent of the distance of the screen E from the lower face of the crystal plate. This proves that the rays leave the plate in a parallel direction, notwithstanding their strong divergence within the crystal. The angle of the cone was found to be 1° 50′, while the value calculated was 1° 55′, a very close agreement between theory and experiment.

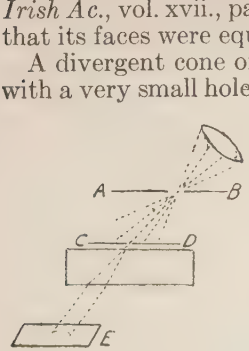


Fig. 231.

Axes of Single Ray Velocity. External Conical Refraction. — The directions determined by joining the point O (Fig. 230) with the points at which the two sheets of the wave-surface meet are termed the axes of single ray velocity. At each one of the conical points or pits an infinite number of tangent planes can be drawn to the surface, which collectively form a tangent cone. A rough model of such a cone can be made by cutting a paper circle along a radius and then pasting the edges together, making them overlap a little. This cone fits into the conical depressions of the wave-surface.

Suppose now that a ray is travelling within the crystal along the axis of single ray velocity, and emerges from the surface of the crystal. The direction of the ray after refraction out into the air is determined by the position of the plane tangent to the element of wave-surface corresponding to the ray. For example, suppose we are dealing with a simple spheroidal wave starting at O within the crystal (Fig. 231). We wish to determine the direction of the ray OB after emergence. This direction will be that traversed by a plane-wave CD tangent to the spheroid at B. In other words, the small element of the wave at B can be considered as a portion of the tangent plane. The direction of the refracted ray is thus seen to be determined by the position of the plane tangent to the wave-front at the point where it intersects the surface.

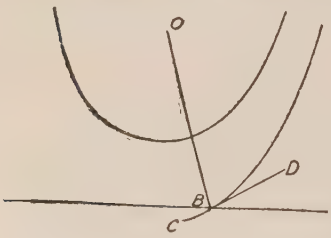


Fig. 232.

Now a ray travelling along an axis of single ray velocity has an infinite number of tangent planes which envelop a cone, and the refracted ray may pursue a direction determined by any one of them. It will therefore open out into a hollow cone, and if the light be received upon a screen we shall see a ring which increases in diameter as the distance from the crystal face is increased. The phenomenon is exhibited by concentrating a pencil of rays upon the surface of the crystal. This converging system of rays contains the hollow cone of rays which we should have if we transmitted a ray up through the crystal along the axis of the single ray velocity. The cone is indicated by solid lines (Fig. 233), the superfluous rays which pursue other paths in the crystal being indicated by dotted lines. A screen perforated with a small hole limits the emergent light to the ray which has traversed the plate in the direction of the single ray axis, and if the beam which issues is received upon a white screen it is seen to have the form of a hollow cone.

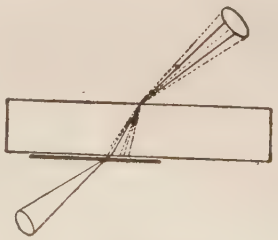


Fig. 233.

Crystal Plates in Convergent or Divergent Polarized Light. Colors of Thin Crystal Plates. — In the Chapter on Polarization we have seen that a beam of plane-polarized light falling upon a crystal plate

cut perpendicular to the optic axis (for example, a film of mica or selenite) is in general doubly refracted, that is, the incident vibration is resolved into two mutually perpendicularly vibrations, one polarized in, and the other at right angles to the principal plane. The two disturbances traverse the crystal with different velocities, and consequently emerge with a difference of phase depending upon the thickness of the crystal plate. The plane vibration on entering the medium becomes transformed into an elliptical vibration, owing to the different velocities of the two rectangular components. As the disturbance proceeds its type changes, becoming circular, elliptical, and plane in succession, each plane phase being turned through 90° with respect to the phase immediately preceding or following. It is obvious that if the plate is thick, and the two rays become separated by double refraction, each ray will be plane polarized, that is, we shall no longer have a circular and an elliptical type. If white light falls upon the plate the difference of phase at emergence of the two components will vary with the wave-length, certain colors, for example, emerging plane polarized parallel to the original plane of polarization; other colors polarize at right angles to it. Certain colors will, therefore, be absent when the emergent light is examined with Nicol prism held with its principal plane parallel to the principal plane of the polarizing Nicol. On rotating the Nicol through 90° each color changes to its complementary tint for obvious reasons. The state of polarization for waves of length intermediate between those specified will, in general, be of the circular or elliptical type. We will now derive an expression for the intensity of the illumination as a function of the position of the polarizer and analyzer and the phase-difference between the two emerging streams. Let the principal plane of the polarizer be parallel to OD (Fig. 234), and the principal plane of the analyzer parallel to OA. Since the vibrations are parallel to the principal plane of the Nicol it is clear that the incident light will vibrate parallel to OD. On entering the plate it is resolved into two components vibrating at right angles along OX and OY, where OX and OY are the two directions in the crystal in which vibrations may occur without change of type. Let the incident vibration be represented by $Y = A \sin \omega t$. The vibrations in the crystal will then be $A \cos \alpha \sin \omega t$ and $A \sin \alpha \sin \omega t$, along OX and OY, where $DOX = \alpha$.

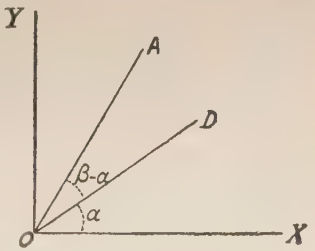


FIG. 234.

These two disturbances on emerging will have a difference of phase which we will represent by δ. The vibrations, therefore, take the form $A \cos \alpha \sin \omega t$ and $A \sin \alpha \sin (\omega t + \delta)$. The analyzing Nicol resolves these vibrations parallel to its principal plane OA, transmitting one component and suppressing the other. If $AOX = \beta$ we have two vibrations parallel to the plane of the analyzer, one along OA represented by

$$A \cos \alpha \cos \beta \sin \omega t,$$

contributed by the OX component, and another also along OA represented by

$$A \sin \alpha \sin \beta \sin (\omega t + \delta).$$

These two combine into the resultant vibration

$$y = A \cos \alpha \cos \beta \sin \omega t + A \sin \alpha \sin \beta \sin (\omega t + \delta).$$

The intensity is represented by the sum of the squares of the coefficient of $\cos \omega t$ and $\sin \omega t$

$$I = A^2 \left\{ \cos^2(\alpha - \beta) - \sin 2\alpha \sin 2\beta \sin^2 \frac{\delta}{2} \right\},$$

where $\alpha - \beta$ is the angle between the principal planes of the polarizer and the analyzer.

If we are working with white light δ will vary with the wave-length, and if A also varies with the wave-length the general expression for the intensity is

$$I = \cos^2(\alpha - \beta) \Sigma A^2 - \sin 2\alpha \sin 2\beta \Sigma A^2 \sin^2 \frac{\delta}{2}.$$

The first term is seen to be independent of δ the phase-difference, and will therefore have no effect in producing color in the image; and the transmitted light will therefore consist of two parts, one of which is white, depending on the first term, and the other colored to a greater or less extent, depending on the second. If we rotate the plate around its normal, the Nicols remaining fixed, the colors will be affected in the same proportion, and the tint of the emerging light therefore remain unaltered, except that it will be diluted to a greater or less extent with white light arising from the first term. The colors will be most intense when $\alpha - \beta = 90°$ and least intense when $\alpha - \beta = 0$, the former case corresponding to crossed Nicols and the latter to parallel. In both cases the effects are most pronounced when $\alpha = 45°$; in other words, when the principal plane of the polarizer and analyzer bisect the angle between the principal planes of the plate.

Colors of Crystal Plates in Convergent and Divergent Polarized Light. — A remarkable series of phenomena are presented when we examine crystal plates, cut in various ways, in a beam of strongly convergent or divergent light. Colored fringes of varied forms appear crossed by dark crosses and brushes, the variety being almost as great as in the kaleidoscope. A complete investigation of the forms which occur under all possible conditions is hardly profitable, and we will examine a few typical cases only.

The simplest form of polariscope for viewing the rings and crosses in convergent light is the tourmaline tongs. When the crystal plate is placed between the tourmalines, and the eye brought close to the apparatus, which is directed towards a brilliant light of large size, such as the sky, the rays which enter the pupil have traversed the crystal in the form of a cone of wide aperture, as shown

in Fig. 235. Tourmaline crystals are, however, usually so strongly colored, that only an imperfect idea of the color distribution can be obtained in this way, though the general form of the fringes can be made out. It is therefore customary to use some such arrangement as that shown in the lower part of the figure.

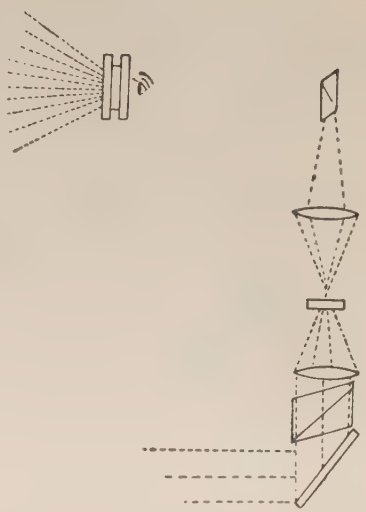


Fig. 235.

Uniaxal Crystal Cut Perpendicular to Axis. — Consider what happens when a cone of plane-polarized rays diverging from S' (left hand, Fig. 236) passes through a crystal plate, the central ray SO of the cone coinciding with the optic axis. Consider the source S' in front of the plane of the paper, and let the vibrations be vertical. The ray incident at O passes through the plate in the direction of the optic axis, and its vibration plane remains unaltered. Other rays in general will suffer double refraction, and emerge with a phase-difference between the components of the vibration. This will not be true, however, for certain rays. Consider the ray $S'P$. The direction of vibration is in the principal plane $S'OP$, *i.e.* the plane

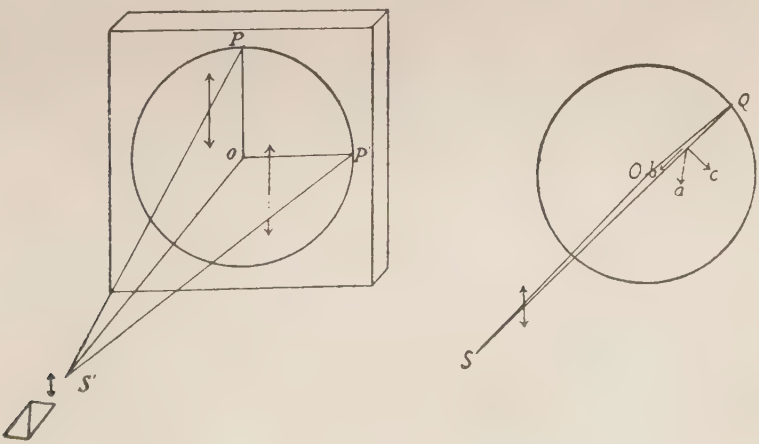


Fig. 236.

containing the ray and the optic axis, and it will therefore be transmitted by the crystal without resolution. The same is true for the ray OP', since in this case the vibration is perpendicular to the principal plane. Hence all rays striking the crystal plate along OP

or OP' or their prolongations, will not suffer double refraction, and will be wholly transmitted or completely stopped by a Nicol held behind the plate, according as its principal plane is vertical or horizontal. Consider now a ray incident at some other point, say Q (right hand Fig. 236). The vibration a will be resolved into two components, b and c, one lying in the principal plane SOQ and the other perpendicular to it. The vibrations will traverse the crystal with different velocities, and emerge with a difference of phase, which will depend upon the thickness traversed, and also upon the wave-length of the light. Now the thickness traversed will increase as we pass from O to Q, owing to the increasing obliquity of the rays. The phase-difference of the emergent components will therefore vary along the line OQ, and the emergent light at some points on this line will be polarized in the same plane as the incident light, at other points in a plane perpendicular to it. The analyzing Nicol will quench one or the other, according to its position. By symmetry the conditions of equal phase-difference will occur along concentric circles with a common centre at O. We shall accordingly see bright and dark circles surrounding O if the light is monochromatic, and colored fringes if it is white. These circles will, however, be interrupted along the lines OP and OQ (for reasons above specified) by a cross which appears bright or dark according to the position of the analyzing Nicol (Fig. 237).

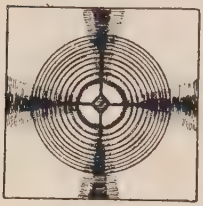


Fig. 237.

Isochromatic Surfaces. — The characteristic of a fringe is that the retardation δ is constant along its length, and the locus of points in space for which δ is constant is called an isochromatic surface. For every value of δ there will be a corresponding surface, and if we describe these surfaces around S as an origin, with retardations of 1, 2, 3, 4, etc., half wave-lengths, the intersections of these surfaces with the second surface of the crystal will determine the isochromatic lines or fringes. The form of the surface was worked out by Bertin[1] in the following way:

We may suppose the source from which the rays diverge located on the surface of the crystal.

Let O be the source: then the time occupied by the two disturbances in traversing OP will be $\frac{OP}{v_o}$ and $\frac{OP}{v_e}$ for the ordinary and extraordinary disturbances. The time retardation is therefore

$$t_o - t_e = OP\left(\frac{1}{v_o} - \frac{1}{v_e}\right),$$

and the phase retardation

$$\frac{2\pi}{T}(t_o - t_e) = \frac{2\pi}{T}\left(\frac{1}{v_o} - \frac{1}{v_e}\right) OP.$$

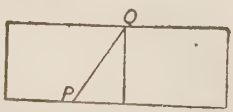


Fig. 238.

The wave-surface consists of a sphere of radius b, and a spheroid

[1] *Ann. de Chim. et de Phys.*, lxiii., page 57, 1861.

of which the generating curve is the ellipse

$$a^2x^2+b^2y^2=a^2b^2.$$

If r be a radius vector of this curve, we have v_0 proportional to b, and v_e proportional to r, and the time retardation is, for a thickness ρ,

$$\delta=\rho\left(\frac{1}{b}-\frac{1}{r}\right)=\rho\left(\mu_o-\frac{1}{r}\right).$$

If we write the equation of the ellipse in the form

$$\mu_o^2x^2+\mu_e^2y^2=1,$$

we have

$$\frac{1}{r^2}=\mu_o^2\cos^2\theta+\mu_e^2\sin^2\theta,$$

which, if we combine with the equation for δ, gives us

$$\frac{1}{r^2}=\left(\frac{\delta}{\rho}-\mu_o\right)^2,$$

$$\left(\frac{\delta}{\rho}-\mu_o\right)^2=\mu_o^2\cos^2\theta+\mu_e^2\sin^2\theta,$$

$$(\delta-\rho\mu_o)^2=\mu_o^2x^2+\mu_e^2y^2,$$

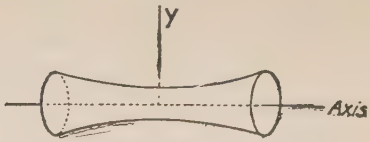


FIG. 239.

and since $\rho^2=x^2+y^2$,

$$\{(\mu_e^2-\mu_o^2)y^2-\delta^2\}^2=4\mu_o^2\delta^2(x^2+y^2),$$

which is the generating curve of the isochromatic surface, which we form by rotating the curve around the optic axis. Its general form is shown in Fig. 239. Its sections with the surface of a plate cut perpendicular to the axis are circles, with a plate parallel to the axis hyperbolae.

Isochromatic Surfaces in Biaxal Crystals. — The form of the surface in biaxal crystals is shown in Fig. 240. A section parallel to

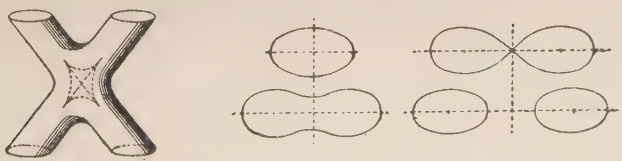


FIG. 240.

the plane containing the axes gives us curves closely resembling hyperbolae. A section perpendicular to the bisector of the angle between the optic axis gives us a family of lemniscates. Sections in planes along a, b, c, d give us fringes of the form shown on the right hand side of the figure. These different curves correspond to successive values of δ, and they may all be seen simultaneously, as in Fig. 241.

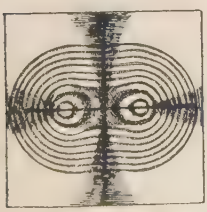


FIG. 241.

The region of constant illumination, which in the case of uniaxal crystals had the form of a

cross, in the case of biaxal crystals presents the appearance of a double brush of hyperbola form. The subject of the various modifications which the fringes and brushes may undergo is a very large one, but its study teaches us very little regarding the phenomenon of double refraction, the problems being purely geometrical. We shall examine but one other case, the remarkable transformation of a biaxal into a uniaxal crystal resulting from an elevation of temperature.

Position of Axes as a Function of Temperature. — A remarkable phenomenon occurs when certain biaxal crystals are heated, for as the temperature rises, the angle between the optic axes becomes decreased until the axes finally coalesce, the crystal becoming uniaxal. An oblique section of selenite is usually used for exhibiting the phenomenon. As the plate is warmed the lemniscates close in, the centres approaching, and presently meeting, at which stage the isochromatic fringes are circles crossed by a rectangular cross. A further elevation of temperature causes the axes to cross one another, so to speak, the crystal becoming again biaxal. The experiment makes one of the most beautiful lantern demonstrations ever devised.

Phenomena exhibited by Twin Crystals. — Calcite is sometimes found with one or more layers crystallized in opposite directions. Such crystals sometimes show the rings and crosses without either polarizer or analyzer, the front and back parts of the crystal taking their place, and the oppositely crystallized plane serving as the thin film. A slice from a nitre crystal frequently exhibits four systems of rings.

If a crystal possesses rotating power still further complications result, notable among which are the beautiful spirals described by Airy and named after him. They appear when plates of right- and left-handed quartz cut perpendicular to the axis are superposed and viewed in convergent light. Or a single plate may be made to exhibit them if it is placed on the lower mirror of the Nörremberg polariscope, on account of the reversal of the rotation.

Convergent Circular Light. — If a quarter-wave plate is interposed between the first Nicol and the crystal plate in a converging polariscope, the appearances are completely altered. As we should expect, the black cross disappears almost completely, the arms being replaced by thin lines of nebulous gray, which rotate with the analyzer without changing in appearance. The rings in adjacent quadrants are dislocated as shown in Fig. 242, the light rings in one quadrant being opposite the dark ones in the next.

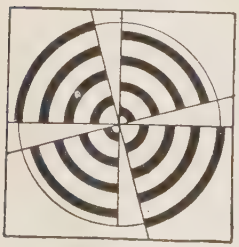


Fig. 242.

The explanation of this can easily be found by working out the resolution of the circular vibration in two opposite quadrants, taking care to distinguish between components parallel to the radii and those perpendicular to them.

Double Refraction in Non-Crystal Media. — Many of the phe-

nomena of double refraction can be observed in isotropic substances subjected to strain, or to sudden differences of temperature. Glass plates squeezed in a vise (Fig. 243) and viewed between crossed Nicols exhibit most beautiful colored fringes, the lines of strain being clearly indicated.

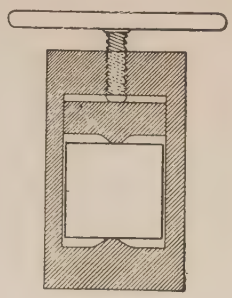


Fig. 243.

Tyndall found that a long strip of glass thrown into sonorous vibration restored the light when placed between crossed polarizing prisms. The experiment has since been modified in a beautiful manner, the transmitted light being examined in a revolving mirror and found to be restored periodically, the band appearing broken up into beads, showing that the double refraction was coincident with the vibration. On inserting a selenite plate the band was found to vary in color.

A permanent strained condition can be established by heating a block of glass nearly to a red heat and cooling it suddenly. Polarized light is an extremely sensitive test for imperfect annealing. Prince Rupert drops make excellent objects. They can be easily prepared by melting the end of a glass rod in a powerful blast-lamp, and allowing the drop to fall into a bowl of water with some filter paper on the bottom. Four out of five fly to pieces, but with a little practice a number can be prepared in a short time. They are best viewed by immersing them in a small rectangular cell of glass filled with a mixture formed by dissolving about ten parts of chloral hydrate in one part of hot glycerine. This mixture has the same refractive index as the glass.

CHAPTER XI

CIRCULAR AND ELLIPTICAL POLARIZATION

In the case of plane-polarized light the vibration of the ether is linear, as we have seen. We will now consider another type of polarization, in which the ether particle moves in a circular or elliptical orbit. Such a vibration results when two rectangular vibrations, of the same period, but differing in phase, are simultaneously impressed upon a point.

If the amplitudes are the same and the phase-difference an odd number of quarter periods, we shall have a circular vibration which is right- or left-handed according to the circumstances. This can be easily shown by means of the circular pendulum: suspend a weight by a string and strike the weight a blow in any direction: a linear vibration results. Strike a second blow, at right angles to the direction of the first, and a quarter period later, *i.e.* when the weight has reached its position of greatest displacement, and the linear vibration will be replaced by a circular one. If we delay the second blow until a half period has elapsed, the resultant motion will be linear, but in a direction making an angle of 45° with the original direction, while if we wait until three quarters of a period have elapsed, we again get the circular vibration, but in an opposite direction.

We have cases precisely similar to the above when plane-polarized light is transmitted through a thin crystalline plate which is doubly refracting. The incident vibration is in general decomposed into two rectangular vibrations which traverse the plate with different velocities, and consequently emerge with a phase-difference depending on the thickness of the plate. If the plate is very thick, the two components are completely separated and emerge plane-polarized, but in the case of very thin plates the components emerge without appreciable separation, and compound into a vibration which may be circular, elliptical, or linear according to the path-difference within the plate, and the amplitudes of the two components. The circular vibration results only when the amplitudes are equal and the path-difference is $(2n+1)\frac{\lambda}{4}$.

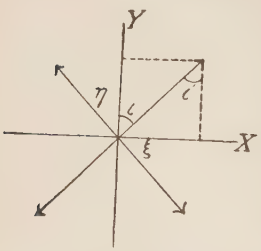


Fig. 244.

Let the X and Y axes (Fig. 244) represent the directions of the vibrations of the extraordinary and ordinary ray in the crystal plate, and let the incident vibration of amplitude a be represented by the arrow making an angle i with the Y axis. The incident vibration is represented by the equation $\sigma = a \sin 2\pi \frac{t}{T}$, and the projections of σ,

the displacements along the X and Y axes, by

$$\xi = a \sin i \sin 2\pi \frac{t}{T},$$

$$\eta = a \cos i \sin 2\pi \frac{t}{T}.$$

The projections on X and Y after passage through the plate are given by

$$\xi = a \sin i \sin 2\pi \left(\frac{t}{T} - \frac{E}{\lambda}\right),$$

$$\eta = a \cos i \sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda}\right),$$

in which E and O are the reduced paths, *i.e.* the thicknesses of the two air films which would be traversed in the same times by the extraordinary and ordinary rays, as the times occupied by the rays in traversing the crystal plate.

These equations can be written in the form

$$\xi = a \sin i \sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda} + \frac{O-E}{\lambda}\right),$$

$$\eta = a \cos i \sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda}\right).$$

Plane Polarization of the Emergent Light. — The light on entering the plate is decomposed into the components parallel to x and y. For the resultant to be plane-polarized $\frac{\eta}{\xi}$ must be a constant, *i.e.* independent of the time. This occurs for any thickness of plate when $i = 0$ and when $i = 90$, the disturbance being propagated in these two cases without change. For all other values of i we have the condition for plane-polarized emergent light given by the equation

$$\frac{\sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda} + \frac{O-E}{\lambda}\right)}{\sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda}\right)} = K,$$

or

$$\sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda}\right) \cos 2\pi \frac{O-E}{\lambda} + \cos 2\pi \left(\frac{t}{T} - \frac{O}{\lambda}\right) \sin 2\pi \frac{O-E}{\lambda}$$
$$= K \sin 2\pi \left(\frac{t}{T} - \frac{O}{\lambda}\right).$$

This equation is satisfied for all values of t only when

$$\sin 2\pi \frac{O-E}{\lambda} = 0 \text{ or } O - E = n\frac{\lambda}{2},$$

the path-difference being a whole number of half wave-lengths.

If n is even $\frac{\eta}{\xi} = \cot i$, and the emergent light is polarized in a plane parallel to the original plane of vibration.

If n is odd $\frac{\eta}{\xi} = -\cot i$, and the emergent light is plane-polarized in azimuth $2i$, the vibration being represented by the double-headed arrow.

Circular Polarization of the Emergent Light. — This occurs when $i = 45°$ and $O - E = (2n+1)\frac{\lambda}{4}$.

This makes

$$\sin i = \cos i = \tfrac{1}{2}\sqrt{2},\ \cos 2\pi\left(\frac{O-E}{\lambda}\right) = 0,\ \text{and}\ \sin 2\pi\left(\frac{O-E}{\lambda}\right) = 1.$$

Substituting, we have

$$\xi = \pm\frac{a}{2}\sqrt{2}\cos 2\pi\left(\frac{t}{T} - \frac{O}{\lambda}\right),$$

$$\eta = \frac{a}{2}\sqrt{2}\sin 2\pi\left(\frac{t}{T} - \frac{O}{\lambda}\right).$$

Squaring and adding, these two equations give us $\xi^2 + \eta^2 = \frac{1}{2}a^2$. The equation of a circle of radius $\sqrt{\frac{a^2}{2}}$ = projection of original amplitude on X and Y.

Intensity of Circularly Polarized Light. — The intensity of plane-polarized light is as the square of the amplitude. We will now find an expression for the intensity of circularly polarized light. As we shall see presently, when plane-polarized light is transformed into circular light the intensity remains unaltered. This means that the intensity is measured by twice the square of the radius of the circle as defined above, or by twice the square of the amplitude of one of the plane-polarized components. When therefore we add two rectangular disturbances together to produce circular light we get double illumination, exactly as when we add the effects of two independent sources of light.

Production and Properties of Circular Light. — The easiest method of producing circularly polarized light is by means of a plate of mica of such thickness that the path-difference between the ordinary and extraordinary rays is a quarter of a wave-length, the proper thickness for yellow light being .032 mm. Such a plate is called a **quarter-wave plate**, which we shall hereafter speak of as a $\lambda/4$ plate. These plates can be prepared without difficulty by splitting a good quality of mica by means of a needle into the thinnest possible sheets, and selecting such as completely restore the light when they are placed in the proper azimuth between a pair of crossed Nicols.

The thickness of the plates should be measured with a micrometer caliper or spherometer, as otherwise the mistake may be made of getting the plate three times too thick, the optical effects produced (with sodium light) by a $\frac{3}{4}\lambda$ plate being similar in appearance. The two directions on the plate parallel to vibrations which are propagated without change should be marked. They can be easily found by holding the plate between two crossed Nicols, in such an

azimuth that the field appears dark. The directions in question will then be parallel and perpendicular to the longer diagonal of the field of the analyzing Nicol. It is also important to know which of the two directions corresponds to the greater retardation. Singularly enough this point has been very generally neglected by text-books. In fact I have failed to find any mention of it anywhere.

If the plate is mounted with its principal directions vertical and horizontal respectively, in front of one of the silvered mirrors of a Michelson interferometer and the fringes found with white light, it is not difficult to determine the direction corresponding to the faster propagation. The central black fringe is brought upon the cross-hair of the telescope in which the fringes are viewed, and the light passed through a Nicol before it reaches the instrument. It will be found that a shift of $\frac{1}{2}$ a fringe width occurs when the vibration plane is changed from horizontal to vertical. If this shift is in the same direction as the shift originally produced by the introduction of the mica plate, it means that the retardation has been increased by changing the direction of the vibration from horizontal to vertical, consequently the vertical direction in the plate is the direction in which the slower component vibrates. This direction should be marked "Slow," the other "Fast."

As this method involves some trouble, the following, based on observations made with a plate previously tested as above, will be found simpler:

A Nicol prism is mounted in front of a sodium flame with its short diagonal turned in the direction in which the hands of a clock move, through an angle of 45° from the vertical. The light polarized in azimuth 45° is then reflected from a polished metal surface, *e.g.* silver or speculum metal, at an angle of about 60°, which introduces a phase-difference between the components of nearly a quarter of a period (the component perpendicular to the plane of incidence being retarded). If the light is then passed through the quarter-wave plate and an analyzing Nicol, it will be found that it can be extinguished by the latter; that is, the quarter-wave plate reduces the nearly circular vibration into a plane vibration. The plane of this vibration, which is given by the *long diagonal* of the analyzing Nicol when set for complete darkness, makes an angle of 45° with the two directions of vibration which we have marked on the $\lambda/4$ plate; this direction is indicated by the dotted arrow in Fig. 245, the directions of vibration of the fast and slow disturbances being as shown. We have then merely to fit this diagram to our $\lambda/4$ plate, making the dotted arrow coincide with the direction of vibration of the plane-polarized emergent ray.

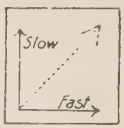


FIG. 245.

For the present we will not concern ourselves with the explanation of why this method enables us to distinguish between the ordinary and extraordinary rays. The reason will become clear after the study of the direction of revolution of circularly polarized light. If plane-polarized light is passed through the $\lambda/4$ plate, with its plane of vibration making an angle of 45° with the two principal directions, it

will be found to suffer very little change in intensity when examined with a slowly rotating analyzer. In this respect it resembles ordinary unpolarized light. It may be distinguished from the latter, however, by passing it through a second $\lambda/4$ plate, which, by bringing the retardation between the components up to $\lambda/2$, converts it into plane-polarized light, which can be extinguished with a Nicol. It also shows brilliant colors in the designs made up of thin flakes of selenite, when the latter are viewed through a Nicol or other analyzer.

Circular Polarization by Total Reflection. Fresnel's Rhomb. — When light polarized in a plane making an angle of 45° with the plane of incidence is totally reflected at an angle of 54°, the two reflected components have a phase-difference of one-eighth of a period (for glass-air reflection). Two such reflections give the required $\lambda/4$ difference, and produce circular polarization. In the case of total internal reflection, the phase of the component of vibration parallel to the plane of incidence is retarded 135°, or a total retardation of 270° for two reflections. This is virtually the equivalent of an *acceleration* of 90°, and we can so consider it in all experimental work. (See Lord Kelvin's *Baltimore Lectures*, page 401.)

Fig. 246.

This phenomenon will be more completely discussed in the article on the theory of reflection, and for the present we shall merely assume the fact to be true. Fresnel constructed a rhomb of glass to verify his calculations of the effect of total reflection upon plane-polarized light, and found that after two internal reflections at an angle of 54°, as shown in Fig. 246, the light emerged circularly polarized.

A rhomb of this description can be easily made out of a rectangular piece of thick plate glass, the dimensions of which should be in about the proportion 1 : 2 : 3. The plate glass employed should be as thick as possible. It is usually possible to get strips of glass an inch or two in width and an inch thick, which have been trimmed from large plates. These make excellent rhombs, though equally good results on a smaller scale can be obtained with pieces cut from quarter-inch plate. The ends of the block are to be ground down on a grindstone to an angle of 54°, as shown by the dotted lines in Fig. 246. With a quarter-inch plate this can be done in a short time, but if the very thick plate is employed it is better to saw off the ends with a mineralogist's saw, as the slow grinding is very tedious. Small pieces of thin plate glass, cemented to the rough ground ends of the rhomb with " boiled-down " Canada balsam, make an excellent substitute for polished faces, and save several hours of labor.

If the light entering one of the oblique faces of the rhomb is polarized in a plane making an angle of 45° with the plane of incidence, the emergent light will be freely transmitted by a Nicol in every azimuth. If, however, thin mica or selenite films are interposed between the rhomb and the analyzer, they will show brilliant colors, which is not the case when ordinary light is used.

The Fresnel rhomb has an advantage over the $\lambda/4$ plate, for the phase-difference between the rectangular components is nearly independent of the wave-length, which is not so in the case of the mica film.

Elliptical Polarization of the Emergent Light. — Suppose i to have some value between 0 and 45°, and $O-E=(2n+1)\frac{\lambda}{4}$. This is the case of the quarter-wave plate, with the light polarized in such a plane as to give neither plane nor circularly polarized light.

The components of displacement along the x and y axes are then

$$\xi = \pm a \sin i \cos 2\pi\left(\frac{t}{T}-\frac{O}{\lambda}\right),$$

$$\eta = a \cos i \sin 2\pi\left(\frac{t}{T}-\frac{O}{\lambda}\right),$$

which gives by squaring and adding

$$\frac{\xi^2}{\sin^2 i}+\frac{\eta^2}{\cos^2 i}=a^2,$$

the general equation, the two which we have already discussed being special cases.

This equation shows us that the vibration is an elliptical one, the axes of the ellipse being parallel to x and y, and proportional to $\sin i$ and $\cos i$ respectively.

Let $i=30°$, then $\sin i=\frac{1}{2}$ and $\cos i=\frac{1}{2}\sqrt{3}$, and we have

$$4\,\xi^2+\tfrac{4}{3}\,\eta^2=a^2.$$

The major and minor are then $\sqrt{\frac{3}{4}}a^2$ and $\sqrt{\frac{a^2}{4}}$ respectively, and since the sum of their squares is equal to a^2, and the intensity of the plane-polarized light is not changed by converting it into elliptical light, we have the *intensity* represented by the sum of the squares of the major and minor axes. The more general equation we obtain by considering our plate of any thickness, in which case we have the components

$$\xi = a \sin i \sin 2\pi\left(\frac{t}{T}-\frac{O}{\lambda}\right)\cos 2\pi\,\frac{O-E}{\lambda}$$

$$+a \sin i \cos 2\pi\left(\frac{t}{T}-\frac{O}{\lambda}\right)\sin 2\pi\,\frac{O-E}{\lambda},$$

$$\eta = a \cos i \sin 2\pi\left(\frac{t}{T}-\frac{O}{\lambda}\right).$$

Eliminating t from this equation gives us

$$\frac{\eta^2}{a^2\cos^2 i}+\frac{\left(\xi\cos i-\eta\sin i\cos 2\pi\frac{O-E}{\lambda}\right)^2}{a^2\cos^2 i\sin^2 i\sin^2 2\pi\frac{O-E}{\lambda}}=1,$$

or $$\xi^2 \cos^2 i + \eta^2 \sin^2 i - 2\,\xi\eta \sin i \cos i \cos 2\pi \frac{O-E}{\lambda} = a^2 \sin^2 i \cos^2 i \sin^2 2\pi \frac{O-E}{\lambda},$$

the equation of an ellipse of which the axes are parallel and perpendicular to the principal section *only when* $O-E=\left(2n+1\right)\frac{\lambda}{4}$. The use of a mica plate of some other thickness gives us an elliptical vibration, the axes of which are inclined to the original direction of vibration.

Production and Properties of Elliptical Light. — Elliptically polarized light can be produced in a number of ways: by the transmission of plane-polarized light through a quarter-wave plate, the plane of vibration making an angle of say 20° with the principal plane of the plate; by decreasing this angle the ellipse becomes more eccentric; degenerating into a line when $i = 0$. By increasing i the ellipse becomes less eccentric, and passes through the circular condition when $i = 45°$. If we use a plate of some other thickness, we obtain an ellipse with its axis inclined to the original direction of vibration.

Elliptically polarized light, when examined through a Nicol, shows fluctuations in brilliancy as the prism is rotated, the change in intensity becoming more marked as the eccentricity of the ellipse is increased. It thus resembles partially polarized light, but can be distinguished from it by introducing a retardation of a quarter of a period by means of a $\lambda/4$ plate, which converts it into plane-polarized light. The directions of the axes can be determined by the $\lambda/4$ plate, for they are parallel and perpendicular to the principal section of the plate when it is so oriented as to give plane-polarized light.

The ratio of the axes can be determined by observing the angle between the principal plane of the analyzing Nicol when it extinguishes the light, and the principal plane of the $\lambda/4$ plate. The tangent of this angle is the ratio of the axes of the ellipse, for when two rectangular vibrations compound into a linear vibration the tangent of the angle which the resultant makes with one of the components is the ratio of the components.

When the $\frac{\lambda}{4}$ plate and the Nicol are in such positions as to extinguish the light, we have the arrangement shown in Fig. 247, in which the elliptical disturbance A (with components a and b) approaches the observer, passing through the $\frac{\lambda}{4}$ plate B, which decomposes it into its components. On emerging, the resultant linear vibration c is extinguished by a Nicol oriented as shown, the tangent of the angle c giving us the ratio $\frac{b}{a}$. The constants of elliptical

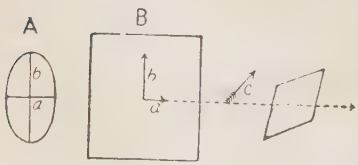


FIG. 247.

polarization cannot be very accurately determined with the $\frac{\lambda}{4}$ plate, owing to the difficulty of making accurate settings of the mica plate and Nicol. A better contrivance is Babinet's compensator, which has been adapted by Jamin to the study of elliptically polarized light. It consists of two acute prisms of quartz, which, when placed in contact, form a plate the thickness of which can be varied by sliding the prisms. The optic axes are parallel to the surfaces of the plate, but perpendicular to each other, as shown in Fig. 248. If plane-polarized light falls normally on the face of the compensator, the plane of vibration not coinciding with either of the principal planes, it will be broken up into two components parallel and perpendicular to the optic axis. When these vibrations enter the second prism, their directions will remain unaltered, but they will exchange velocities; *i.e.* the ordinary ray in one becomes the extraordinary ray in the other. If μ_e and μ_o be the refractive indices for the two polarized disturbances, and if a ray traverses a thickness ϵ in one prism, the relative retardation of the two disturbances is $\epsilon(\mu_e - \mu_o)$, and for a thickness e' in the second prism it is $-\epsilon'(\mu_e - \mu_o)$, for the disturbance which is the faster in the first prism is the slower in the second. The retardation produced by the plate as a whole at the point in question is obviously

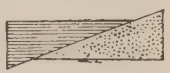


Fig. 248.

$$(\epsilon - \epsilon')(\mu_e - \mu_o).$$

The retardation is zero for the central ray, for at this point $\epsilon = \epsilon'$, and the light emerges polarized in the original plane. On either side of this point we shall have points at which the relative retardation is π, 2π, 3π, etc.; the plane of polarization of the light emergent at the points at which the retardation is an even multiple of π is parallel to the original plane of the incident light. At intermediate points, where the retardation is an odd multiple of π, the transmitted light will be polarized in a plane inclined to the original plane by an amount $2a$, where a is the angle between the plane of the original vibration and the plane of vibration of the retarded component. There will thus be a system of lines across the face of the compensator along which the light is polarized in the original plane, and another system midway between them where the light is polarized at angle $2a$ with the original plane.

If the incident vibration makes an angle of 45° with the principal planes of vibration of the compensator, the plane of vibration along this second set of lines will be at right angles to the vibration along the first system. At points between the lines the light will be elliptically or circularly polarized, the condition over the surface of the compensator being roughly represented in Fig. 249. If the surface of the compensator be viewed through a Nicol with its planes of vibration parallel to the planes of the linear vibrations along the surface,

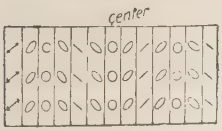


Fig. 249.

the light will be extinguished along one set of lines, and the field will appear traversed by equidistant dark bands.

The bands of circular polarization may be detected by bringing a quarter-wave plate between the compensator and the analyzing Nicol; this gives us plane polarization along the lines which were previously circularly polarized, and a new system of dark bands results. The mica plate should be so oriented as not to affect the appearance of the systems of plane-polarized lines.

The dark bands previously alluded to are of course most distinct when $a=45°$.

Determination of the Constants of Elliptical Polarization. — When the elliptical polarization is produced by a quarter-wave plate we can calculate the position and ratio of the axes, but in cases where the ellipticity is the result of reflection, it becomes necessary to determine the constants experimentally. These determinations are of importance in connection with the theory of reflection, as we shall see in a subsequent chapter.

The compensator in its original form was provided with a fine cross wire moved by a micrometer screw, by means of which the distance between the bands could be measured, and the displacement of the bands determined. As modified by Jamin for the study of elliptically polarized light, the instrument has a fixed cross wire, one of the quartz wedges being moved by the screw. The relative retardation $e-e'$ is increased or diminished at a given point according to the direction of the motion of the wedge, consequently the dark bands are displaced by a corresponding amount.

The wedge must, however, be moved through double the distance moved by the wire in the old form of instrument, in order to displace the system by the width of a band, since in this case e varies while e' remains constant, while in the case of the movable wire both e and e' vary, the one increasing and the other diminishing; the difference between e and e' consequently increases twice as rapidly with a moving wire as it does with a fixed wire and moving wedge. Let $2a$ be the distance between two dark bands as measured by the wire, and $2b$ the distance through which the wedge is moved in order to produce the same shift — from the foregoing $b=2a$.

The retardation δ at distance x (measured by moving wedge) from the central band is $\delta=\frac{x}{b}\frac{\lambda}{2}$, since moving the wedge a distance b changes the retardation by $\frac{\lambda}{2}$. We have now the necessary data for the study of an elliptical vibration, and will first determine the phase-difference between the two components. The components into which the incident vibration is resolved at the quartz surface will differ in phase by an amount $\alpha-\beta$, if we represent them by

$$x=A\cos(\omega t+\alpha),\ y=B\cos(\omega t+\beta).$$

Transmission through the plate alters this phase-difference by an amount $\delta=\frac{2\pi}{\lambda}(e-e')(\mu_e-\mu_o)$, and there will be a system of lines

along which the total phase-difference $\alpha-\beta+\delta$ will be multiples of π and the transmitted light plane polarized.

We first adjust the wedges so that with plane-polarized light to start with the central dark band is bisected by the cross wire. The phase-difference at this point is zero. Substituting elliptically polarized light we find the central band shifted to a point, so situated that the phase-difference between the components of the elliptical vibration is compensated exactly by δ, the phase-difference resulting from transmission through the plate. The quartz wedge is now to be moved by the micrometer screw until the central band is again bisected by the wire. If this distance is denoted by x, we have

$$\frac{\alpha-\beta}{\pi}=\frac{x}{b} \quad \text{or} \quad \alpha-\beta=\pi\frac{x}{b},$$

b having been previously determined.

Position of the Axes. — The phase-difference of the component vibrations along the axes is 90°. We set the compensator as before, so that with plane-polarized light the central band falls under the wire, and then move the wedge a distance $\frac{1}{2}b$. There is now a phase-difference of 90° along the line under the wire. Substituting the elliptical light we rotate the compensator until the central band is again bisected by the wire. The axes of the elliptical vibration are now parallel to the axes of the quartz wedges.

Ratio of the Axes. — If the compensator is set so that its axes are parallel to the axes of the elliptical vibration, the tangent of the angle between one of its principal planes and the principal plane of the analyzer is the measure of the ratio of the axes. The compensator acts in this case in the same way as the $\frac{\lambda}{4}$ plate, the use of which in the determination of the ratio of the axes has already been given.

Elliptical Polarization by Reflection. — We have seen that when plane-polarized light is twice internally reflected at an angle of 54°, it emerges as circularly polarized light if the original plane of polarization made an angle of 45° with the plane of incidence; each reflection in this case introduces a phase-difference of $\frac{1}{4}\pi$ between the reflected components, consequently a single internal reflection from glass will give us elliptically polarized light. This can be shown with an ordinary right-angle prism. In general, when plane-polarized light is reflected at an azimuth of 45°, *i.e.* with its plane of vibration inclined at 45° to the plane of incidence, the reflected light will be, to a greater or less extent, elliptically polarized. In the case of glass and other transparent media the eccentricity of the ellipse is very great; in other words, the reflected light is very nearly plane-polarized, but in the case of metals the elliptical polarization is very marked. If plane-polarized light is reflected from a silvered mirror it will be found to be quite freely transmitted by a Nicol prism in all positions, if the plane of polarization originally made an angle of 45° with the plane of incidence. These cases will be more fully discussed in the chapter on the Theory of Reflection.

Direction of Revolution in Circularly Polarized Light. — The direction of revolution of the circular vibration depends on the thickness of the crystalline plate, its orientation, and its nature, *i.e.* whether it is a positive or negative crystal. The positions of the ordinary and extraordinary ray, and the directions of vibration in each, are shown for positive and negative crystals in Fig. 250. In the former the extraordinary component travels slower than, and consequently lags behind, the ordinary; in the latter the reverse is true. We will now take the case of a $\frac{\lambda}{4}$ plate of mica, which is a negative crystal, and determine the direction of revolution for two different orientations. First suppose the direction of vibration of the incident light to make an angle of $+45°$ with the principal section (optic axis in Fig. 251). It is decomposed into the components O and E, the former lagging behind the latter by one quarter of a wavelength. The E component consequently carries the ether particle to the right, and when it is at its point of greatest displacement, the O component acts in a vertical direction, consequently the direction of rotation is from right to left, as shown by the arrow. If we now rotate the plane of vibration of the incident light through 180°, making the angle between it and the principal section $-45°$, we have the condition shown in B, and applying the same reasoning we find that the direction of rotation is now from left to right. In the case of positive crystals we apply the same construction, considering, however, that the O component acts first, since it is in advance of the other. The directions of revolution will be found to be the reverse of those in the former cases.

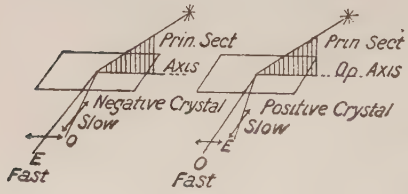


Fig. 250.

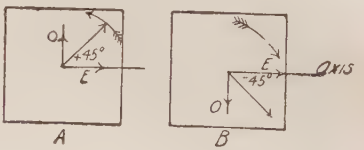


Fig. 251.

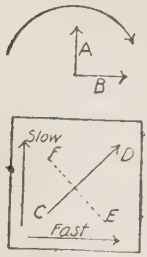


Fig. 252.

We can determine experimentally the direction of revolution with the quarter-wave plate. Suppose the light to be coming towards us, and the direction of revolution clockwise. It can be decomposed into two rectangular components A and B, B being a quarter of a period behind A. We will now suppose it transmitted through the $\frac{\lambda}{4}$ plate (placed as shown in Fig. 252) and examined with an analyzer. The component A, which is a quarter period ahead, will traverse the plate at the slower velocity and be brought into the same phase as the component B, the resultant plane vibration having the direction CD. If the direction of revolution be re-

versed, component A will be a quarter period behind B, and will experience a further quarter period relative retardation in traversing the $\frac{\lambda}{4}$ plate, the resultant having the direction EF. The direction of revolution is thus determined by observing whether the plane vibration makes an angle of $+$ or $-45°$ with the direction designated "fast."

If we have a Nicol prism and quarter-wave plate so oriented as to give us a right-handed circular vibration, by turning the Nicol through 180° we reverse the direction of rotation. This can be readily understood by constructing two diagrams representing the two conditions.

Let us now return to the method which we use for determining the "fast" and "slow" directions in our $\frac{\lambda}{4}$ plate. With the arrangement of the Nicol and metallic reflector which we employed, we obtained a clockwise circular vibration. Obviously, if the direction of rotation is known, the fast and slow directions of the $\frac{\lambda}{4}$ plate can be determined by observing the direction in which the plane-polarized disturbance vibrates on leaving the plate.

Direction of Revolution in the Case of Fresnel's Rhomb. — As has been stated before in the case of total reflection, the component perpendicular to the plane of incidence virtually lags behind the other. If the rhomb is placed in a vertical position, and the incident light polarized in a plane turned clockwise 45° from the vertical, the direction of revolution will be clockwise. A convenient way of determining the direction of revolution when the rhomb is set for circular polarization is to notice the direction in which it must be turned in order to bring the plane of incidence into coincidence with the plane of vibration. This is the direction of revolution of the circularly polarized light.

Natural and Partially Polarized Light. — By natural light we mean ordinary unpolarized light, which is characterized by showing no change of intensity when passed through a $\frac{\lambda}{4}$ plate and Nicol prism, no matter how oriented, and by being doubly refracted by certain crystals, the intensities of the two refracted rays being independent of the orientation of the crystal. Partially polarized light, such as we obtain by reflection from a glass surface at some other angle than the polarizing angle, is characterized by showing fluctuations of intensity when it is examined through a rotating Nicol, never being completely extinguished, however. In this respect it resembles elliptically polarized light, but the difference between the two can be shown with the $\frac{\lambda}{4}$ plate, which converts the latter into plane-polarized light. Partially polarized light is doubly refracted by crystals, the relative intensities of the two rays varying with the orientation.

We have now to consider the simplest forms of vibration which

are consistent with the above results, and we will begin with a discussion of natural or unpolarized light. Brewster explained natural light by assuming it to be made up of two plane-polarized disturbances perpendicular to each other and independently propagated. A disturbance of this nature, if it could exist, would undoubtedly have the properties of natural light, but there are mechanical objections to the conception of a disturbance in which it is necessary to assume that the adjacent ether particles on the wave-front move in totally different directions. Fresnel accordingly, in 1821, advanced another hypothesis, namely, that natural light was in reality plane-polarized light, the azimuth of which changed with exceeding rapidity. Fresnel considered that a ray which came from a single centre of disturbance was plane-polarized, but that the azimuth varied rapidly. If we could isolate such a ray and experiment with it, we should find that it was alternately transmitted and cut off by a Nicol prism in a fixed azimuth. The isolation of a ray coming from a single centre of disturbance is, however, impossible, and even if it could be done there would be no way of verifying the hypothesis experimentally unless the changes took place so slowly that the fluctuations in the intensity of the light, after passage through the Nicol, could be followed by the eye. In every source of light we have a vast number of independent centres of disturbance, and the joint effect of all at a given moment, on a given point in the ether, will be a movement in a definite direction which will, however, change from moment to moment. Fresnel probably included the ellipse and circle in his conception of the motion of ordinary light, though he does not mention them specifically. The elliptical vibration is the most general form, and we may, on Fresnel's hypothesis, consider natural light as an elliptical vibration, the form and orientation of which changes with great rapidity, passing through the circle and straight line as special forms.

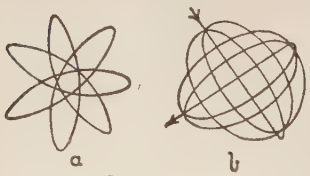


FIG. 253.

The change in the orientation of the ellipse cannot, for mechanical considerations, be conceived as taking place suddenly. If it takes place gradually, the curve ceases to be an ellipse and takes the complicated form shown in Fig. 253*a*. If the ratio of the axes changes as well as the orientation, we have a curve of the form shown in Fig. 253*b*, except that the change from the straight line to the circle is much more gradual, and the whole curve must be conceived of as constantly changing its orientation.

Curves of this nature occur in acoustics as the resultants of harmonic disturbances of different periods, and as early as 1831 Airy raised the objection that unless the ellipse changed its form and orientation suddenly, the light could not be strictly monochromatic, for a curve such as *b* can be considered as the resultant of two perpendicular plane-polarized components of different periods. Dove, in 1847, showed that light which had traversed a Nicol prism rotating at high speed showed all the properties of natural light, the

emergent beam being plane-polarized, the plane turning with the same speed as the prism. If a mica plate was added and made to revolve with the prism, the revolving elliptical vibration was found to have the properties of circularly polarized light. Airy pointed out, however, that the revolving plane-polarized vibration could be considered as the resultant of two oppositely polarized circular components of different periods. This case will be more fully dealt with presently.

Lippich[1] came to the conclusion that unpolarized vibrations are only possible with non-homogeneous light, and that only polarized vibrations are possible with strictly monochromatic light. His objections to the conception of unpolarized monochromatic vibrations were the same as those raised by Airy, but they are not serious, if we assume that the change in the orbit of the ether particle takes place very slowly in comparison to the time of revolution, *i.e.* if it executes several thousand revolutions in practically the same orbit, the departure from strict homogeneity of the light will be too slight to be detected.

Interference experiments under the condition of large difference of path, point out that the form of the vibration remains constant for many thousand periods, which makes the above assumption seem very probable. Michelson has obtained with unpolarized light, interference fringes, with a path-difference of 540,000 waves, which indicates that the light executes at least 540,000 vibrations before changing its state of polarization. A million vibrations take place in $2 \cdot 10^{-9}$ sec., and it is therefore impossible for the eye to detect traces of polarization in natural light even if it remains polarized for many million complete periods.

The lower limit for the duration of a constant condition of polarization is given by interference experiments, and is probably somewhere in the neighborhood of $1 \cdot 10^{-9}$ sec. The length of the path-difference in this case is about 32.4 cms.

The only conceivable way in which an upper limit for the constancy of the nature of the vibration might be obtained is by the employment of flashes of light of very brief duration. By a suitable arrangement of apparatus it is possible to obtain electric sparks the duration of which is as brief as $2 \cdot 10^{-9}$ sec.; the length of the wave-train from such a spark would be about 72 cms.

If no very great change of form of the vibration in one of these brief flashes took place, we might be able to discover traces of polarization in the light providing all the radiating centres of luminous energy in the spark vibrated in practically the same plane. In this case the light, of the spark would vary with the position of a Nicol prism through which it was viewed. It is quite conceivable that an electric discharge might start vibrations of similar form and orientation, but all experiments have failed to show evidences of polarization in the light of sparks and electric discharges in vacuum tubes, the few positive results that have been obtained having been subsequently shown to be due to the polariza-

[1] *Sitzungsber. d. Wien. Akad.*, Bd. xlviii., Abth. ii., page 146, 1863.

tion of the light by oblique transmission through the glass wall of the tube.

If the individual luminous centres in the spark were vibrating in different planes, the light would not appear polarized even if no change occurred in the states of vibration, for the smallest area of the spark which the eye could recognize, even under the microscope, would emit light from thousands of independent centres. The light from some would be cut off by the Nicol, while that from others would be transmitted; but the appearance would be the same as with natural light.

Light-Beats. — Airy's conception that revolving plane-polarized light was merely the resultant of two circularly polarized disturbances, of different periods and opposite directions of rotation, was tested experimentally by Righi.[1]

In the case of the interference between two sounds of nearly the same pitch we have the familiar phenomenon of beats. At a given point the intensity of the disturbance is a function of the time, the waves alternately reenforcing and destroying one another. The optical analogy would be a moving system of interference fringes, the illumination at a given point varying with the time.

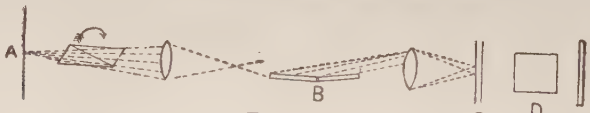


FIG. 254.

Righi arranged an optical system which effected interference between the two circularly polarized components of different periods, which were first transformed into plane-polarized vibrations by means of a $\frac{\lambda}{4}$ plate.

Light from a vertical slit A was passed through a Nicol prism which made n revolutions per second, and then brought to a focus by means of a lens. The revolving plane-polarized light was then reflected from a pair of Fresnel mirrors B, and again brought to a double focus by a second lens, the two images resulting from reflection from the inclined mirrors. By this device two similar sources are obtained side by side, the light from each being plane-polarized, the plane rotating with the same speed as the Nicol. Behind each image a $\frac{\lambda}{4}$ plate C was placed so oriented that the optic axes were mutually perpendicular, and at an angle of 45° with the vertical. We have seen that right and left-handed circular vibrations are transformed by a $\frac{\lambda}{4}$ plate into plane vibrations which make angles of $+45°$ and $-45°$ with the axes of the plate.

According to Airy's conception, if the light makes N vibrations per second and the plane of polarization n revolutions per second, it can be considered as the resultant of two circular vibrations, of

[1] *Journal de Physique* (2), p. 437, 1883.

periods $N+n$ and $N-n$ respectively. The two circular component from one source are resolved by the $\frac{\lambda}{4}$ plate into a vertical disturbance with a period $N+n$ and a horizontal disturbance with a period $N-n$. The light from the other source, since the $\frac{\lambda}{4}$ plate behind it is differently oriented, is resolved into vertical vibrations of period $N-n$ and horizontal ones of period $N+n$. The vertical vibrations of period $N+n$ from one source interfere with the vertical vibrations of period $N-n$ which come from the other, and since the number of beats per second is equal to the difference between the frequencies of the interfering disturbances, a given point will receive maximum illumination $2n$ times per second. The fringe system formed by the interference of the vertical vibrations was separated from that formed by the horizontal components by means of a doubly refracting prism D; and the fringes were found to be moving in opposite directions, passing a given point at the rate of $2n$ per second.

Righi has also employed an arrangement of apparatus in which the Nicol prism remains at rest while a mica plate revolves. Circularly polarized light, obtained by means of a Nicol and $\frac{\lambda}{4}$ plate, is brought to a double focus by means of a lens and bi-prism. A stationary mica plate is placed in front of one image and a revolving plate in front of the other, moving fringes being formed as in the previous experiment. These experiments are interesting chiefly as showing that the frequency of the vibration can be altered by allowing the disturbance to pass through revolving polarizing systems. It should be observed, however, that a statical treatment can be given as well as a kinematical, the fringes occupying positions corresponding to the position of the Nicol. The moving fringes seen in an interferometer are just as much an evidence of light beats, the change in wave-length in this case resulting from reflection from a moving mirror.

CHAPTER XII

METEOROLOGICAL OPTICS

The refraction and diffraction of light by water drops, ice crystals, dust particles, etc., in the earth's atmosphere, give rise to a great variety of optical phenomena, the study of which is of help in determining the atmospheric conditions producing them.

Of these the commonest example is the rainbow, the complete investigation of which is an extremely complicated problem. The elementary explanation given in 1637 by Descartes accounts for the bow by the laws of geometrical optics; it is wholly inadequate, however, for the explanation of the great diversity of color arrangement, and the fainter secondary bows which are so often seen. Other types of atmospheric halos, parhelia or mock suns, etc., are due to ice crystals, which may or may not have a definite orientation. If the air is quiet, *i.e.* free from turbulent motion, the ice spicules will lie for the most part with their long axes horizontal, the position assumed by a spindle-shaded body falling through a resisting medium. This orientation may give rise to concentrations of light at definite points, as in the case of mock suns. Mirage, scintillation, and related phenomena come under the head of meteorological optics, but they have been sufficiently treated in the Chapter on Refraction. A more comprehensive study of the whole subject will be found in Pernter's *Meteorologische Optik*. We will begin by a study of the rainbow.

The Rainbow. — The first theory of the rainbow was given by Descartes in 1637. He calculated the paths of various rays which together formed the parallel bundle incident upon the raindrop. These rays after suffering one internal reflection emerge from the underside of the drop in different directions. He made calculations, by the aid of Snell's law, of the paths of 10,000 rays incident on different parts of one side of the drop, and discovered that the angle between the extreme issuing rays, from the 8500th to the 8600th, is measured in minutes of arc. These are the so-called least deviated rays, and they leave the drop in the direction RT, Fig. 255, which is the direction followed by the ray figured in the drop, incident at P. Rays incident above and below P leave the drop in directions indicated by the dotted lines. There is thus a concentration of energy, or increased illumination along the direction RT. This light which suffers minimum deviation by the drop is the light by which the illuminated drops in the bow are seen. Its deviation is 137° 29′ and the angle which it makes with the incident rays is 42° 31′. All drops therefore which lie on a circle (more exactly a cone), the centre of which is opposite the sun, and of which the radius is 42° 31′, appear strongly illuminated.

This fact, which was laboriously ascertained by Descartes, can be easily found by the method of maxima and minima. We will make the calculation for any number k of internal reflections, which will give us the position of other bows.

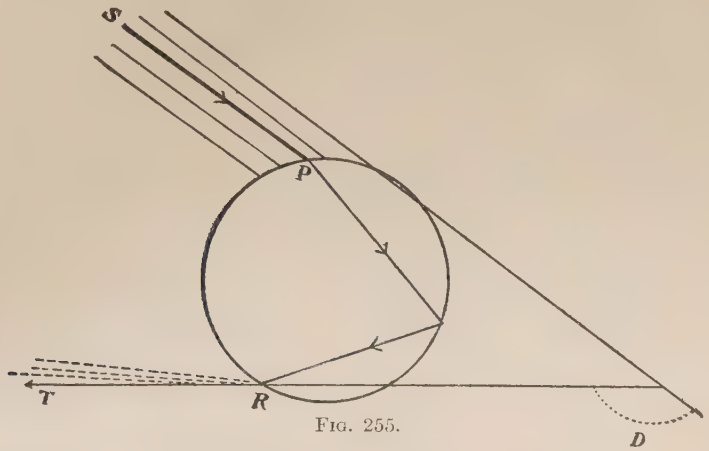


FIG. 255.

The deviation D is equal to $2\,(i-r)+\pi-2\,r$ for a single internal reflection, or for k internal reflections

$$D=2\,(i-r)+k(\pi-2\,r).$$

Differentiating,

$$dD=2\,di-2\,(k+1)dr,$$

and equating $\frac{dD}{di}$ to zero gives

$$\frac{di}{dr}=k+1.$$

If μ is the refractive index of water, we have

$$\sin i=\mu\,\sin r,$$

$$\cos i\,di=\mu\,\cos r dr,$$

$$\frac{di}{dr}=\mu\,\frac{\cos r}{\cos i};$$

or

$$\mu\,\frac{\cos r}{\cos i}=k+1.$$

Squaring,

$$\mu^2\cos^2 r=(k+1)^2\cos^2 i.$$

Adding

$$\mu^2\sin^2 r=\sin^2 i,$$

gives us

$$\mu^2=1+(k^2+2\,k)\cos^2 i$$

or

$$\cos i=\sqrt{\frac{\mu^2-1}{k^2+2\,k}}.$$

A second differentiation gives us a positive quantity which shows us that we are dealing with a minimum. Applying the last equation we find, since for water $\mu = \frac{4}{3}$ for $k = 1$,

$$i = 59° 23', \quad r = 40° 12'$$

$$D = \pi + 2i - 4r = 137° 58'.$$

The radius of the bow $\delta = 180° - 137° 58 = 42°$.

For $k = 2$, $D = 232° 30'$.

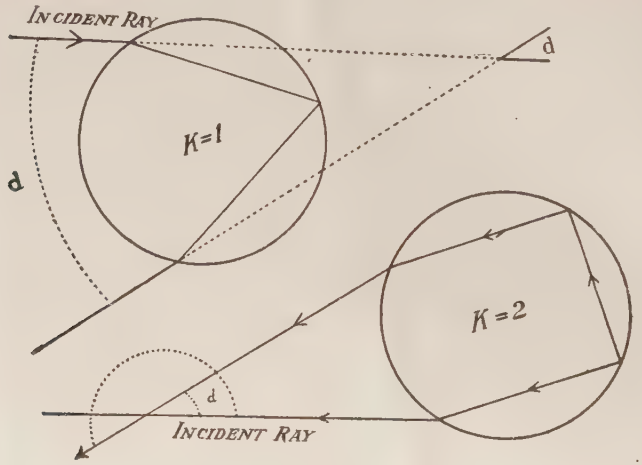


FIG. 256.

This gives us a bow radius $232° 30' - 180°$ or $\delta = 52° 30'$.

If we take dispersion into account it is clear from Fig. 256 that for one internal reflection δ will be larger for the red than for the violet, consequently the bow will be red on the outside and violet on the inside. Moreover the space between the bow and its centre will be more or less luminous, for the drops in this region will be seen by the feebler rays for which D is greater and δ consequently less. The bow due to the two internal reflections will appear outside of this, the drops being seen by rays which enter on the under side and leave on the upper side. The dispersion in this case will make D, and consequently δ, largest for the violet. This bow will accordingly be violet on the outside and red on the inside. The feebler rays will illuminate the region outside of this bow, the region between the two bows being absolutely dark, so far as light from the rain drops is concerned.

Three and four internal reflections give bows which are behind us as we face the primary bow, *i.e.* we should have to face the sun to see them. When the shower is between us and the sun, we see the drops powerfully illuminated by light which is refracted without undergoing internal reflection. This light overpowers the third and fourth bows; no bow corresponds to this directly refracted light, for the intensity falls off gradually as the angle between the drops and the sun increases. The fifth bow coincides

very nearly with the second, while the sixth falls inside the first. They are never seen, however, owing to the diminution in the intensity of the light by the refraction which accompanies each internal reflection.

The bow which is sometimes seen inside of the primary bow must not be mistaken for the sixth bow, as it is formed in quite another way, and cannot be accounted for by the elementary theory.

We will now take up the complete treatment which was first given by Airy.

According to the elementary treatment, the succession of colors in the rainbow should always be the same and the diameter and width constant. This is not the case. Rainbows are frequently seen which are not parts of true circles, and the succession of colors is frequently seen to be different in different parts of the same bow.

This was shown by Airy to depend upon the peculiar form of the wave-front which emerges from the drop. A plane or flat wave, incident upon a transparent sphere (rain drop), can be shown by very elementary methods to acquire, after two refractions and one reflection, the peculiar shape shown in *b*, Fig. 257, where the curved line 1, 2, 3 represents the wave-fronts, the elements of which are travelling in the direction of the arrows. The portion 1, 2 is convex in the direction of propagation, and will of course go on expanding; the part 2, 3 is concave, and converges to a focus. The curvature varies as we pass along the wave-front, being greatest at 3, zero at 2 (where the front is plane), and having a large value again at 1. The element at 3 comes to a focus first, passes through it and becomes convex instead of concave, forming a "cusp" on the wave as shown at *c*. Successive elements of the wave-front above 3 pass in turn through foci and build the rear front of the cusped wave. A full treatment of the propagation of cusped waves has been given in the Chapter on Reflection.

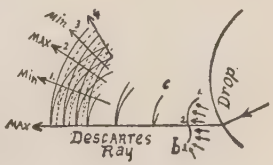


Fig. 257.

The caustic surfaces are always bordered by interference fringes, for we are dealing with two sets of wave-trains, formed by the front and rear surfaces of the cusped waves. If we draw the surfaces, as in Fig. 257, representing the crests of the waves by solid lines, and the troughs by dotted lines, we find that crests intersect troughs along the arrows 1 and 3, consequently these represent the positions of the interference minima. This method of looking at the phenomenon is more elementary than that given by Airy, who integrates the effect of the whole wave at a given point in front of it, as in diffraction problems. We thus see that in addition to the least deviated ray of Descartes, indicated by the arrow 0, there will be other directions (arrows 2, 4, etc.) in which there is considerable illumination. With a monochromatic sun we should, therefore, have a number of concentric bows, as can be shown experimentally with a spectrometer and glass cylinder or ball. The

distance between these bows will vary with the size of the rain drops, and since with white light we have, theoretically, an infinite number of bows, it is clear that the color of the actual rainbow at a given point can only be determined by computing the " mixed color " due to the superposed bows.

This is merely a qualitative explanation. For a complete solution we require the distance between the successive maxima and minima, and the position with reference to the least-deviated ray of Descartes. Airy solved the problem by integrating the effect of the wave-front at exterior points, and found that the first maximum, which gives the primary bow, fell a little inside of the Descartes ray, which makes the diameter of the bow slightly less than that called for on the geometrical theory. The discrepancy increases as the size of the drops diminishes. Very large drops give a bow of practically the same size as predicted by the elementary method; very small drops, however, give a bow which may have a diameter several degrees less. We thus see how it is possible for the curvature of a bow to change, if the drops which form one part of the bow are smaller than those which form another.

The successive maxima and minima give a series of so-called supernumerary bows within and concentric to the primary bow. With large drops they are closer together than with small. The actual rainbow is therefore a superposition of a number of bows, and the succession of colors will depend upon the spacing of the supernumerary bows.

The color distribution was first worked out by Pernter, who calculated the tints resulting from the mixtures of the primary colors of the superposed bows. If the red of the second bow falls upon the green of the first, we shall have a bow with an abnormally broad yellow band, for red and green lights, when mixed, form yellow. This is a very common type, a bow of red and yellow with green and blue nearly absent. If the drops are smaller, the red of the second may fall upon, or even within, the violet of the first. In this case we see a second bow just within the first, and perhaps a third and fourth.

It is not difficult to investigate these phenomena in the laboratory. Allow the light from an arc lamp to fall upon a vertical glass rod one or two millimetres in diameter, and view the refracted light with a telescope, placed close to the rod, covering the eye with a piece of red glass. A large number of maxima and minima will be seen; these are the fringes which border the caustic. Standing with our back to the arc, in a dark room, we see, if we throw the spray of an atomizer in front of us, not only the primary and secondary bow, but the first supernumerary bow, just inside of the primary.

By suspending a minute drop of water from the end of a very small glass thread (previously greased) and holding it very close to the eye, a little to one side, standing with our back to the arc, the sun reflected in a mirror, we can see a portion of both the primary and secondary bow most intensely colored, with the very dark region between them, and a set of supernumerary bows within the primary, and outside of the secondary. The best method of

getting the drop is to draw out a piece of small glass tubing into a fine capillary, breaking it off at a point where its diameter is about 0.3 mm. Grease the end slightly, introduce a little water, and blow a minute drop on the end of the capillary. This is the most instructive experiment of all, as we can vary the size of the drop and note the effect upon the spacing of the supernumerary bows.

The fact that we can see portions of a number of bows with a single drop presents no difficulty, for the drop is so close to the eye that most of the differently deviated rays enter the pupil. It is interesting to compare this case with that of the halos to be described presently, in which we see a bow resulting from the diffracted rays of light, the drop acting as an obstacle.

Halos, Mock Suns, and Related Phenomena. — The reflection and refraction of the sunlight by small ice crystals in the air give rise to a very complicated series of phenomena, which, unlike the rainbow, can be seen at all altitudes of the sun. They may be summed up briefly as follows :

A colored circle, surrounding the sun, red on the inside and white on the outside, having a radius of 22°, known as the *22° Halo.*

A larger circle, similar in appearance, but fainter, called the *46° Halo.*

A white horizontal circle passing through the sun, and having a constant height above the horizon, called the *Parhelic Circle.*

Concentrations of light on the parhelic circle, four colored ones at the points where it intersects the halos, are called *Parhelia* or mock suns. One white one at 180° from the sun is called the *Anthelion,* and two at 120° the *Paranthelia.*

Two oval arcs, sometimes joined together forming an ellipse which circumscribes the 22° halo, is known as the *Circumscribing Oval.*

The *Tangential Arcs* touching the 46° halo.

The *Oblique Arcs of Lorentz* which are sometimes seen under the parhelia when they extend beyond the 22° halo. These, the chief

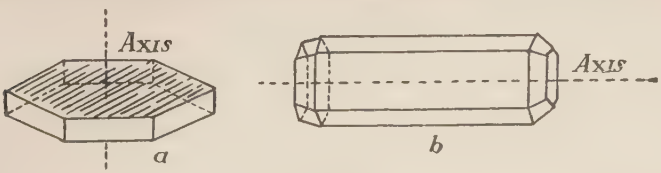


Fig. 258.

halo phenomena, can all be explained by the laws of reflection and refraction in the hexagonal ice crystals which have the forms shown in Fig. 258. In still air, these crystals will fall slowly in the position shown, since bodies moving through a resisting medium set themselves in the position of greatest resistance. A mistake has been made by several writers, Brevais and Pernter, for example, who state that the crystals fall in the position of least resistance. It must not be supposed, however, that this error invalidates in any way Pernter's admirable treatment of the optical problems involved,

for in each case we may substitute a horizontal hexagonal plate for a vertical hexagonal prism, and *vice versa*, without affecting the orientation of the angles. It is well to bear this in mind, as the formation of plates or spiculae of ice seems to depend upon the temperature, and it is possible that approximate estimates of the temperatures of the clouds could be made by considering whether the parhelia observed were of the type due to hexagonal crystals with the principal axis vertical (*i.e.* flat plates), or with the axis horizontal (*i.e.* spiculae). Erroneous conclusions would of course be drawn if the mistake here pointed out were not corrected.

This orientation of the crystals taken into account, we can explain all of the above-mentioned phenomena. The hexagonal prisms and plates can transmit light in various ways. The 120° angles are too large of course to act as prisms, but the alternate faces form 60° prisms, which transmit light at a minimum deviation of 22°. It is this transmission that is responsible for the 22° halo. The crystals are not considered as oriented in this case. A crystal situated 22° from the sun can send light at minimum deviation to the eye, and it can do this in six different positions. Crystals nearer the sun can send no light to the eye by this type of refraction and the inner edge of the halo is therefore sharply defined.

As we have seen in the Chapter on Refraction, the maximum quantity of light is transmitted by a prism at minimum deviation, hence the narrowness of the halo. The region outside the halo is slightly luminous, due to the light transmitted by prisms not in this position. The 46° halo is produced in a similar way by refraction through the 90° angles of the crystals. A glass prism of 90° will not transmit light, but ice has a much lower refractive index, and if the crystal is in just the right position, a small amount gets through, which accounts for the faintness of the ring.

These two halos are the only phenomena which can be explained by the action of crystals which have their axes directed in a fortuitous manner. All of the others require still air for their production.

The parhelic circle is white, and is due to reflection from the six sides of the flat plates, and the flat end faces of the elongated ones, the faces in question acting as vertical plane mirrors, which reflect the light of the sun to the eye. The crystals which are at the same elevation above the horizon as the sun are the ones which can send light to the eye in this way. The light of the parhelic circle can also come from rays which enter the upper surfaces of the flat plates, and suffering total reflection at one of the sides emerge from the under side. The two refractions in this case neutralize the dispersion, consequently no color is produced.

The parhelia are produced by refraction through the 60° and 90° prisms, which are oriented as described and for the moment situated at position of minimum deviation.

When the sun is at a considerable altitude, the rays no longer pass through these prisms parallel to their bases, and the effective angle of the prism is increased, and with it the deviation, which causes the parhelia to appear well outside of the halo, but always on the parhelic circle. Parhelia of 46° have been observed, but

only rarely. The 22° ones are frequently observed in winter, even in our latitude. The anthelion is a rarely observed phenomenon, only about thirty appearances of it having been recorded in the past 250 years. The paranthelia are still rarer. Both are formed by two internal reflections combined with two refractions. Pernter's treatment is not very satisfactory in this case, as he makes an erroneous assumption regarding the orientation of the crystals.

There are other phenomena too numerous to mention, such as the vertical pillars of light, seen above or below the sun, formed by reflection from the under surfaces of the horizontal flat plates. They are most frequently seen when the sun is near or below the horizon. Compare these with the parhelic circle. Crosses are sometimes seen, as when the ice cloud is located only at the point of intersection of a halo with the parhelic circle.

The reader is referred to Pernter's *Meteorological Optics* for a fuller account of the almost innumerable phenomena of this sort. There is another type of halo frequently seen surrounding the sun or moon, which is due to diffraction. These are usually of comparatively small diameter. The explanation of the manner in which they are produced has been given in the Chapter on Diffraction, and they can be distinguished from the halos just described from the circumstance that they always have the red light on the outside.

CHAPTER XIII

THEORY OF REFLECTION AND REFRACTION

A THEORY of reflection was worked out by Fresnel, based upon the elastic-solid hypothesis, and equations were obtained which represented the relations between the intensities of the reflected and refracted components, their states of polarization, etc.

This treatment, however, is only of historical interest, since it has been supplanted by one based on the electromagnetic theory, and we shall in the present chapter trace the derivation of the fundamental equations of the more modern theory of luminous disturbances, the foundations of which were laid down by Maxwell.

The luminous vibrations will be regarded as rapidly alternating displacement currents in the ether or in matter, as the case may be, these currents giving rise to magnetic forces similar to those brought into existence by currents flowing in conductors. On this theory not only are the optical and electrical properties of matter being rapidly harmonized, but predictions are being constantly made which are subsequently verified by experiment. In certain cases, however, the discussion from the elastic-solid standpoint is more intelligible, and we shall therefore make use of it from time to time, regarding the older theory more as a convenience, however, than as a true representation of what is actually going on. We will begin by the derivation of the fundamental equations of Maxwell.

Derivation of Maxwell's Equations. — The current may be defined either in electrostatic or electromagnetic units, and will be designated by i or i' accordingly. As we shall have occasion to pass from one system to the other frequently, it is well to fix firmly in the mind at the start that the accent is used to distinguish quantities measured in *electromagnetic* units from those measured in *electrostatic*.

The c.g.s. or electromagnetic system of units starts out with the definition of unit magnetic pole, which is a pole of such strength that, when placed at a distance of 1 cm. from an equal pole, it will exert a force of one dyne upon it. The unit of magnetic field strength is the "gauss," which is the strength of field which exerts a force of one dyne upon unit pole, *i.e.* it is the strength of the field at a distance of 1 cm. from unit pole.

The unit of current is the current which, when flowing in a wire, across a magnetic field of unit strength, will cause 1 cm. of the wire to be acted upon by a force of 1 dyne. It is roughly 10 amperes. The unit of quantity is the amount of such a current which flows across a given section of the wire in one second. The Faraday, or electrostatic, system of units starts out with unit charge, as the charge which, when placed at a distance of 1 cm. from an equal charge, exerts a force of one dyne upon it. The

electrostatic unit of current is the flow of one Faraday unit of charge per second, and the unit of magnetic field is a field which will exert a force of one dyne on a centimeter of wire carrying unit current.

We may also measure the current in the c.g.s. system by the work done in carrying unit magnetic pole once around the current against the lines of magnetic force which surround the current.

The force exerted by a current of intensity i' in a straight wire in a pole at distance r is $2\,i'/r$, and if the pole is carried once around a circle of radius r it moves a distance of $2\,\pi r$ against this force; the work done upon it is therefore $4\,\pi i'$.

If the current i' is of unit strength (c.g.s. system) the force will be 1 dyne if $r=1$.

The current i which flows through cross section q is defined as the number of electrostatic units which traverse q in unit time, so that if the quantity of electricity de flows through q in the element of time dt, we have

$$i=\frac{de}{dt}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and if q is equal to unit cross section, i is equal to j, the current density. The components of j along the x, y, z axes we will designate j_x, j_y, j_z. We will now derive an expression for the current in electromagnetic measure. The current is surrounded by a magnetic field, the lines of force being circles in the case of a current flowing along a cylindrical wire. An isolated magnetic pole will follow these lines of force, travelling around the wire as long as the current continues to flow.

If we carry the magnetic pole around the wire in the opposite direction, we are obliged to do a certain amount of work on it; and if it is allowed to move under the influence of the magnetic force, the current does work on it, developing a certain amount of kinetic energy. We shall define the current i' measured in electromagnetic units thus.

The work done will be proportional to the strength of the current, and for convenience we make use of the proportionality factor 4π. If A represents the work done by the current on unit magnetic pole in driving it around one complete turn, we write $A=4\,\pi i'$.

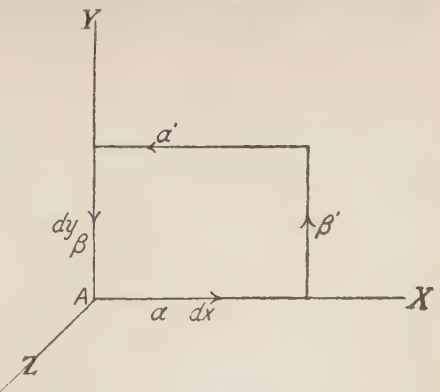


Fig. 259.

Now the work is represented by the force multiplied by the distance through which it acts. Assume that we have a rectangle dx, dy, which is traversed normally by a current $i'=j_z'dxdy$, j_z' being the z component measured in electro-magnetic units. If the current flows towards the observer (Fig. 259), a plus magnetic pole

will be carried around dx, dy in the direction indicated by the arrows. The total work done by the current in moving unit pole around the rectangle will be $A = \alpha dx + \beta' dy - \alpha' dx - \beta dy$, (2) in which α and β are the components of magnetic force along AB and AD, and α' and β' are the components along DC and BC. α may not be constant along dx, but if we regard it as variable, for example having the value α at A and $A + \partial\alpha$ at B, the average value will be $\alpha + \frac{\partial\alpha}{2}$, which, when multiplied by dx, gives us αdx + an infinitesimal of the second order. The minus signs occur for the obvious reason that the forces along DC and CB are oppositely directed from the forces along AB and AD. α' differs from α since it works along a line, the y coordinate of which is greater by an amount dy than that of AB. Under certain conditions of course α would be equal to α'.

If dy be taken sufficiently small $\frac{(\alpha' - \alpha)}{dy}$ may be regarded as the partial differential coefficient $\frac{\partial\alpha}{\partial y}$, and we have

$$\alpha' = \alpha + \frac{\partial\alpha}{\partial y} dy \text{ and } \beta' = \beta + \frac{\partial\beta}{\partial x} dx.$$

We now have for the work, substituting these values in (2) and cancelling,

$$A = \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx dy,$$

and since $A = 4\pi i' = 4\pi j_z' dx dy$,

$$4\pi j_x' = \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}, \text{ and similarly, } 4\pi j_z' = \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}, \quad 4\pi j_y' \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}, \qquad (3)$$

Maxwell's differential equations of the magnetic field.

If c represents the ratio of the two systems of units, *i.e.* $\frac{i}{i'} = c$ and $\frac{j_x}{j_x'} = c$, we can introduce j (defined electrostatically) into the equations, which now become

$$\frac{4\pi}{c} j_x = \frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}, \quad \frac{4\pi}{c} j_y = \frac{\partial\alpha}{\partial z} - \frac{\partial\gamma}{\partial x}, \quad \frac{4\pi}{c} j_z = \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}. \quad . \quad . \quad . \quad (4)$$

These equations hold for all media, for it can be shown that the work done in carrying the magnetic pole around the circuit is independent of the nature of the medium. The quantity c has the dimensions of velocity, and can be determined by observing the magnetic effect of discharging a quantity of electricity measured electrostatically through a circuit.

While the above equations, which connect the current with the magnetic force, hold for all media, we shall presently develop expressions connecting the current with the electric force, and these expressions take particular forms, depending upon the nature of the medium.

They will suffice for the study of reflection, absorption and dispersion, but when we come to consider the behavior of media when brought into a magnetic field, we shall require another set of similar equations, which connect a magnetic current with the lines of electric force which accompanies it.

The magnetic current or magnetic flux occurs when the strength of a magnetic field changes, and the lines of flow will be surrounded by lines of electric force just as the electric current is surrounded by lines of magnetic force. By determining the work done by the magnetic current in drawing unit charge once around the circuit, expressions are obtained which connect the strength of the flux with the accompanying electric field. The equations are similar to those which we have already deduced, and, like them, hold for all media:

$$\frac{4\pi}{c}s_x = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \quad \frac{4\pi}{c}s_y = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \quad \frac{4\pi}{c}s_z = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \quad (5)$$

Displacement Currents in Free Ether. — A displacement current will occur in the ether whenever the density of the lines of electric force changes, and the strength of the current will be proportional to the rate at which the change takes place. It is not easy to form a physical conception of the displacement current. We may perhaps think it of as a lateral shift of the ether, which takes place parallel to the lines of force; in this case our picture of the wave will not be unlike the conception of a wave in an elastic solid. A different way of looking at the matter is that adopted by J. J. Thomson in his *Recent Researches in Electricity and Magnetism*, the first chapter of which will be found of great assistance in forming a concrete picture of what may be taking place in the ether when it is traversed by waves. Thomson represents the phenomena of the electromagnetic field in terms of Faraday tubes (lines of electric force). The motion of one of these tubes gives rise to a magnetic force perpendicular to the direction of its motion, and an electromotive intensity (which we have spoken of as the electric force), which is perpendicular to both of the specified directions. The displacement current takes place in the direction of the electromotive intensity, which is not constant, unless the density of the movng tubes is constant.

Thomson showed that the equations which we have already examined, and those which we are about to consider, could be derived from the consideration of the motion of the Faraday tubes.

We will now derive expressions which connect the displacement current with the electromotive intensity (electric force). Since a charge e sends out $4\pi e$ lines of force, the product of the current density and 4π will be the change in the number of lines of force in unit time. It is obvious that in the case of steady currents there will be no change in the number of the lines, but in the case of displacement currents, where the current strength is changing with the time, the density of the lines of force changes. We can now

write

$$4\pi j_x = \frac{\partial N}{\partial t},\quad 4\pi j_y = \frac{\partial N_y}{\partial t},\quad 4\pi j_z = \frac{\partial N_z}{\partial t}. \quad \ldots \quad (6)$$

in which the expressions N_x, N_y, N_z represent the components of the density of the lines of electric force (polarization in free ether) parallel to the three axes. Similarly for the magnetic current we have

$$4\pi s_x = \frac{\partial M_x}{\partial t},\quad 4\pi s_y = \frac{\partial M_y}{\partial t},\quad 4\pi s_z = \frac{\partial M_z}{\partial t}. \quad \ldots \quad (6')$$

We can form an idea of a magnetic current such as we have in the case of light waves in the following way: Suppose that we have an iron wire with a coil of insulated wire around one end of it, which is traversed by an alternating current. The density of the magnetic lines of force in the iron wire varies periodically, rising from zero to a maximum, and then falling to zero during the first half period, and then rising again to a maximum, with a reversal in the direction of the force, however. The wire is thus traversed by a periodic magnetic current, which is surrounded by circular lines of electric force, which set up alternating induced currents in conductors which are brought into the field.

In the free ether the electric force is considered as numerically equal to the density of the lines of force, so that we may substitute for N_x, N_y, N_z their equivalents X, Y, Z. Our equations now become

$$\left.\begin{aligned} 4\pi j_x &= \frac{\partial X}{\partial t},\quad 4\pi j_y = \frac{\partial Y}{\partial t},\quad 4\pi j_z = \frac{\partial Z}{\partial t} \\ 4\pi s_x &= \frac{\partial \alpha}{\partial t},\quad 4\pi s_y = \frac{\partial \beta}{\partial t},\quad 4\pi s_z = \frac{\partial \gamma}{\partial t} \end{aligned}\right\} \quad \ldots \quad (7)$$

Substituting these values for the current in equations (4) gives us expressions which connect the variation of the electric force with the magnetic field which results from the displacement current.

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial X}{\partial t} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z},\quad \frac{1}{c}\frac{\partial Y}{\partial t} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x},\quad \frac{1}{c}\frac{\partial Z}{\partial t} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \\ \frac{1}{c}\frac{\partial \alpha}{\partial t} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y},\quad \frac{1}{c}\frac{\partial \beta}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z},\quad \frac{1}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \end{aligned}\right\} \quad \ldots \quad (8)$$

Isotropic Dielectrics. — The equations which we have just derived do not hold for dielectrics, for in media the force exerted by two charges $e'e''$ at distance r is less than the force which would be exerted in the free ether, being represented by $\frac{e'e''}{\epsilon r^2}$, in which ϵ is the dielectric constant of the medium. The dielectric constant is greater than unity for all media, and we shall see that the velocity with which the wave is propagated in the medium becomes less as the dielectric constant increases.

In the same way the force between two magnetic poles in a

medium is represented by $\frac{m'm''}{\mu r^2}$, μ being the magnetic permeability, a quantity which differs only slightly from unity except in the case of iron, and we shall see later on that we are justified in writing $\mu = 1$ in practically all optical problems. The change in the law of the force which occurs in ponderable media makes it necessary to modify our last equations, since with the same change in the current intensity the electric force is weaker in the proportion $\frac{1}{\epsilon}$, the current in dielectrics being represented by $4\pi j_x = \epsilon \frac{\partial X}{\partial t}$, etc., $4\pi s_x = \mu \frac{\partial \alpha}{\partial t}$, etc.

Equations (7) now become

$$\left.\begin{aligned} \frac{\epsilon}{c}\frac{\partial X}{\partial t} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, & \frac{\epsilon}{c}\frac{\partial Y}{\partial t} &= \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, & \frac{\epsilon}{c}\frac{\partial Z}{\partial t} &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \\ \frac{1}{c}\frac{\partial \alpha}{\partial t} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, & \frac{1}{c}\frac{\partial \beta}{\partial t} &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, & \frac{1}{c}\frac{\partial \gamma}{\partial t} &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \end{aligned}\right\}, \quad (9)$$

which expressions completely determine all properties of the magnetic field in an isotropic dielectric.

A comparison of equations (6) with the equations preceding (9) shows us that, if we consider the number of lines of force issuing from given charge independent of the surrounding medium, we have the condition within a medium of dielectric constant ϵ, and permeability μ,

$$N_x = \epsilon X, \; N_y = \epsilon Y, \; N_z = \epsilon Z,$$
$$M_x = \mu\alpha, \; M_y = \mu\beta, \; M_z = \mu\gamma;$$

in other words, the densities of the lines of magnetic and electric force are equal to the forces only in a vacuum, for which ϵ and μ both equal unity.

If a charge e is contained in the cube dx, dy, dz, $4\pi e$ lines of force issue from its surface. We can also reckon the number of lines issuing from the cube as the sum of the lines issuing from the six surfaces.

The two squares perpendicular to x contribute the part

$$-(N_x)_1 dydz + (N_x)_2 dydz.$$

By Taylor's Theorem $(N_x)_2 = (N_x)_1 + \frac{\partial N_x}{\partial x} dx$;

the two squares therefore contribute

$$\left((N_x)_1 + \frac{\partial N_x}{\partial x} dx\right) dydz - (N_x)_1 dydz = \frac{\partial N_x}{\partial x} dx \cdot dydz.$$

The total contribution of all six faces is

$$\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z}\right) dxdydz.$$

Equating this to $4\pi e$, and bearing in mind the expression for N^x, M^x, etc., gives us, if we write $\frac{e}{dxdydz} = \rho$ the charge in unit volume, an equation which we shall not make use of for the present.

$$4\pi\rho = \frac{\partial \epsilon X}{\partial x} + \frac{\partial \epsilon Y}{\partial y} + \frac{\partial \epsilon Z}{\partial z}, \quad \cdots \quad (9a)$$

Boundary Conditions. — Since in optical problems we are continually dealing with cases where the waves pass across the boundary which separates two media of different optical properties, it will be necessary to determine what changes, if any, occur in the components of the electric and magnetic forces at the surface of separation.

We will begin by considering that the transition is abrupt, *i.e.* that the dielectric constant changes suddenly in crossing a mathematical plane, which we will take parallel to the xy plane of our coordinate system. Let the dielectric constant of the upper medium be ϵ_1, and that of the lower ϵ_2, and let N_1 equal the density of the lines of force in the upper medium, that is, the number which pass in a normal direction through a plane of unit area. These lines of force are incident on the boundary at an angle θ_1 with the normal. We will first assume that the lines pass through into the second medium without change of direction. The electric force in the upper medium is $R_1 = \frac{4\pi N_1}{\epsilon_1}$, that in the lower medium $R_2 = \frac{4\pi N_2}{\epsilon_2}$, the force due to a given density of the lines of force decreasing as the dielectric constant increases. If $\epsilon_2 > \epsilon_1$, it is obvious that the electric force is less below the boundary, assuming as we have done that $N_1 = N_2$. Since R_2 is less than R_1 and the direction the same, it is obvious that the components of R_2, viz. X_2, Y_2, Z_2, will all be less in the second medium. We shall see that this condition is contrary to the principle of the conservation of energy, and that our original assumption that the lines passed through the boundary without change of direction was incorrect. If X_2 is less than X_1, and the boundary infinitely thin, we can derive an unlimited amount of work by carrying a charged particle along the boundary in the lower medium against the electric force, and then, carrying it across the boundary (which requires no work unless the force is infinite), allow it to move back in the upper medium through the same distance. It will obviously yield more work than has been spent upon it in the lower medium, since it is moving under the influence of a stronger force. We are thus forced to the conclusion that the x component has the same value on both sides of the boundary, otherwise a perpetual motion would be possible. The same is true for the y component, and we consequently have for the boundary condition of these two components $X_1 = X_2$ and $Y_1 = Y_2$. Let us now see how we can reconcile this condition with the smaller value of R_2 in the second medium. It is obvious that if we consider that if the lines of force bend away from the normal on crossing the boundary, we can reconcile the equality of the x and y components

on the two sides with the decrease in their resultant. But the bending of the lines of force in the direction specified results in a change in their density N_2, consequently this must be taken into account. Let the angle which the lines make with the normal in the upper medium be θ_1, and the angle which the refracted lines make with the normal be θ_2, as shown in Fig. 260. We will now determine the value of the Z component on the two sides of the boundary. In the upper medium the normal component of the electric force is $Z_1 = R_1 \cos \theta_1$; in the lower medium, $Z_2 = R_2 \cos \theta_2$. The density of the lines of force parallel to the z axis (normal polarization) for the two media is given by

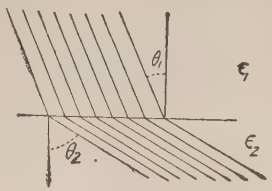


Fig. 260.

$$\frac{\epsilon_1 R_1 \cos \theta_1}{4\pi} \text{ and } \frac{\epsilon_2 R_2 \cos \theta_2}{4\pi}.$$

Now the normal polarization is the same in the two media, for the same number of lines pass through a plane of unit area which is perpendicular to the z axis, in whichever medium we consider the plane, consequently we can write

$$\frac{\epsilon_1 R_1 \cos \theta_1}{4\pi} = \frac{\epsilon_2 R_2 \cos \theta_2}{4\pi} \text{ or } \epsilon_1 Z_1 = \epsilon_2 Z_2,$$

which expression determines the boundary conditions of the z component.

In a similar way it may be shown that the boundary conditions for the components of the magnetic force are $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\mu_1\gamma_1 = \mu_2\gamma_2$. Since, however, $\mu = 1$ in practically all optical problems, we can write $\gamma_1 = \gamma_2$.

Velocity of the Wave. — To find the velocity of the wave we differentiate the first equation of (9) with respect to t,

$$\frac{\epsilon}{c}\frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \quad \frac{\epsilon}{c}\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma}{\partial t}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \beta}{\partial t}\right),$$

and substituting for $\frac{\partial \gamma}{\partial t}$ and $\frac{\partial \beta}{\partial t}$ the values given by the last two equations of (9),

$$\frac{\epsilon}{c^2}\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) - \frac{\partial}{\partial z}\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right)$$

$$= \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2} - \frac{\partial}{\partial x}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right). \quad \cdot (10)$$

We now differentiate the first three equations of (9) with respect to x, y, and z respectively,

$$\frac{\epsilon}{c}\frac{\partial^2 X}{\partial t \partial x} = \frac{\partial^2 \gamma}{\partial y \partial x} - \frac{\partial^2 \beta}{\partial z \partial x}, \quad \frac{\epsilon}{c}\frac{\partial^2 Y}{\partial t \partial y} = \frac{\partial^2 \alpha}{\partial z \partial y} - \frac{\partial^2 \gamma}{\partial x \partial y}, \quad \frac{\epsilon}{c}\frac{\partial^2 Z}{\partial t \partial y} = \frac{\partial^2 \beta}{\partial x \partial z} - \frac{\partial^2 \alpha}{\partial y \partial z}.$$

Addition of these three equations gives

$$\frac{\epsilon}{c}\left(\frac{\partial^2 X}{\partial t \partial x}+\frac{\partial^2 Y}{\partial t \partial y}+\frac{\partial^2 Z}{\partial t \partial y}\right)=0 \text{ or } \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)=0.$$

Since we are dealing with periodic changes of the electric force, the differential coefficient, with respect to the time, of the quantity in the parenthesis, can be considered as proportional to the quantity, with a phase increase of $\frac{\pi}{2}$ (since differentiating the sine gives the cos, the equivalent of a phase change of $\frac{\pi}{2}$).

This gives us

$$\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)=0, \quad \cdots \quad (11)$$

and we have as our final equation

$$\frac{\epsilon}{c^2}\frac{\partial^2 X}{\partial t^2}=\frac{\partial^2 X}{\partial x^2}+\frac{\partial^2 X}{\partial y^2}+\frac{\partial^2 X}{\partial z^2}=\Delta X. \quad \cdots \quad (12)$$

Similar equations hold for Y, Z, α, and β,

$$\frac{\epsilon}{c^2}\frac{\partial^2 X}{\partial t^2}=\Delta X, \quad \frac{\epsilon}{c^2}\frac{\partial^2 Y}{\partial t^2}=\Delta Y, \quad \frac{\epsilon}{c^2}\frac{\partial^2 Z}{\partial t^2}=\Delta Z,$$

$$\frac{\epsilon}{c^2}\frac{\partial^2 \alpha}{\partial t^2}=\Delta \alpha, \quad \frac{\epsilon}{c^2}\frac{\partial^2 \beta}{\partial t^2}=\Delta \beta, \quad \frac{\epsilon}{c^2}\frac{\partial^2 \gamma}{\partial t^2}=\Delta \gamma.$$

We have seen (page 7) that differential equations of this form represent waves travelling with a velocity $v=\frac{c}{\sqrt{\epsilon}}$.

Now the dielectric constant of the ether equals unity, consequently our equation shows us that the velocity of the wave in space is equal to c, the ratio of the two systems of electrical units. This has been confirmed by experiment, the velocity of light determined by optical methods being $2.9989.10^{10}$ cms. per sec., while the velocity of c determined by electrical methods is 3.10^{10} cm./sec.

In ponderable media the velocity must be smaller in the ratio $\frac{1}{\sqrt{\epsilon}}$. Now the refractive index of a medium is the ratio of the velocity of light in free space to the velocity in the medium, consequently we have $n=\sqrt{\epsilon}$, or the square of the refractive index equals the dielectric constant. As a matter of fact this result is not confirmed by experiment except in a few cases, for the reason that the dielectric constant is in reality a function of the period of the vibration, or in other words of the wave-length, and the dielectric constant determined by all electrical methods is the value of n^2 for infinitely long waves, consequently our expression does not hold for the very short waves of light. In the case of some of the gases, in which the dispersion is small, we have fair agreement, as is shown by the following table:

	n	$\sqrt{\epsilon}$
Air,	1.000294	1.000295
CO_2,	1.000449	1.000473
H_2,	1.000138	1.000132

As examples of substances which show especially marked deviations, we may take water and ethyl alcohol:

	n	$\sqrt{\epsilon}$
Water,	1.33	9.0
Alcohol,	1.36	5.0

The reason of these discrepancies we shall see when we come to the study of dispersion.

Deduction of the Laws of Reflection and Refraction for Transparent Media. — If a ray of light is incident upon the boundary separating two media of different optical densities, the percentage reflected and refracted will depend, not only upon the relative refractive indices of the two media, and the angle of incidence, but also upon the nature of the incident vibration as regards its state of polarization.

A plane-wave is represented by the equations

$$\left.\begin{aligned} X &= A_x \cos \frac{2\pi}{T}\left(t - \frac{mx+ny+pz}{V}\right) \\ Y &= A_y \quad \text{" } \quad \text{" } \quad \text{" } \\ Z &= A_z \quad \text{" } \quad \text{" } \quad \text{" } \end{aligned}\right\}, \quad \cdots \quad (13)$$

in which X, Y, and Z are the values of the components of the electric force at any time and any point in space, A_x, A_y, and A_z the maximum values of the components, T the periodic time of the disturbance, V the velocity of the wave, and m, n, and p the direction cosines of the normal to the wave-front.

These equations also represent transverse waves in an elastic solid, if we call X, Y, Z the components of the displacement and A_x, A_y, A_z the components of the amplitude.

We will now deduce the laws which govern the directions of the reflected and refracted rays, and their relative intensities and states of polarization.

Suppose a ray of light coming from A (behind the plane of the paper) to be incident at angle Φ, on the surface of a block of glass (Fig. 261).

Resolve the incident electric force into two components, E_s perpendicular to the plane of incidence and E_p parallel to it. We therefore write for the y component of the electric force, since

$$m = \sin \Phi, \; n = 0, \text{ and } p = \cos \Phi,$$

$$Y_e = E \cos \frac{2\pi}{T}\left(t - \frac{x \sin \Phi + z \cos \Phi}{V_1}\right), \quad \cdots \quad (14)$$

in which V_1 is the velocity of the light in air.

The other component E_p, which is also perpendicular to the ray, has x and z components given by $A_x = E_p \cos \Phi$ and $A_z = -E_p \sin \Phi$.

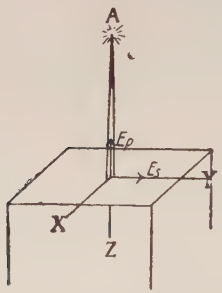


FIG. 261.

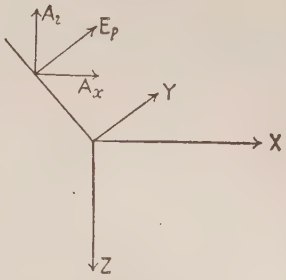


FIG. 262.

The positive direction of E_p is upwards from the boundary which makes the z component negative, as will be seen from Fig. 262, which gives us the side view of the block.

While the component E_s, perpendicular to the plane of incidence, is parallel to the y axis, and therefore enters the expression as the amplitude term, the component E_p, parallel to the incidence plane, must be resolved into its components along x and z, and these components form the amplitude terms in our expressions for X_e and Z_e.

The x and z components of the electric force are therefore given by

$$\left.\begin{aligned} X_e &= E_p \cos \Phi \,.\, \cos \frac{2\pi}{T}\left(t - \frac{x \sin \Phi + z \cos \Phi}{V_1}\right) \\ Z_e &= -E_p \sin \Phi \qquad \text{,,} \qquad\quad \text{,,} \qquad\quad \text{,,} \end{aligned}\right\} \quad \cdot \cdot \cdot \quad (14)$$

We use X_e, Y_e, and Z_e instead of X_1, Y_1, and Z_1, since the forces in the upper medium are not alone those in the incident wave. We have forces X_r, Y_r, and Z_r contributed by the reflected wave, and X_1 is the sum of X_e and X_r.

The magnetic forces associated with these are obtained at once by differentiating the above and substituting in equations (9). For example, the x component α is given by

$$\frac{1}{c}\frac{\partial \alpha}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} = E_s \sin \frac{2\pi}{T}\left(t - \frac{x \sin \Phi + z \cos \Phi}{V_1}\right) \frac{2\pi}{t} \frac{\cos \Phi}{V_1},$$

$$\frac{\partial \alpha}{\partial t} = cE_s \frac{2\pi}{T} \frac{\cos \Phi}{V_1} \sin \frac{2\pi}{T} \qquad \text{,,} \qquad \text{,,}$$

$$\alpha_e = -cE_s \frac{\cos \Phi}{V_1} \cos \frac{2\pi}{T}\left(t - \frac{x \sin \Phi + z \cos \Phi}{V_1}\right),$$

and since $V_1 = \frac{c}{\sqrt{\epsilon_1}}$

$$\alpha_e = -E_s \cos\Phi\sqrt{\epsilon_1}\cos\frac{2\pi}{T}\left(t - \frac{x\sin\Phi + z\cos\Phi}{V_1}\right)$$

The component of magnetic force along x is due to the displacement current resulting from the component of electric force E_s along y. The magnetic force is perpendicular to E_s, *i.e.* parallel to E_p, and must be resolved into components along x and y in the same way. We thus see the significance of the cos Φ in the amplitude term of the above expression.

Similarly, $\beta_e = E_p\sqrt{\epsilon_1}\cos\frac{2\pi}{T}\left(t - \frac{x\sin\Phi + z\cos\Phi}{V_1}\right)$

$\gamma_e = E_s\sqrt{\epsilon_1}\cos\frac{2\pi}{T}$ " "

. . . (15)

Writing for the refracted wave,

$$X_2 = D_p\cos\chi\cos\frac{2\pi}{T}\left(t - \frac{x\sin\chi + z\cos\chi}{V_2}\right)$$

$Y_2 = D_s\cos\frac{2\pi}{T}$ " "

$Z_2 = -D_p\sin\chi\cos\frac{2\pi}{T}$ " "

. . (16)

in which D_p and D_s are the components of amplitude parallel and perpendicular to the plane of incidence. If now the boundary conditions are to be complied with, there will be a reflected wave except when $\sqrt{\epsilon_1} = \sqrt{\epsilon_2}$. Let us take the simplest possible case of a plane-polarized vibration parallel to the x axis at normal incidence. The boundary conditions are $X_1 = X_2$, $\beta_1 = \beta_2$.

The magnetic components of the refracted wave are obtained in the same way as those of the incident.

They are given by

$$\alpha_2 = -D_s\cos\chi\sqrt{\epsilon_2}\cos\frac{2\pi}{T}\left(t - \frac{x\sin\chi + z\cos\chi}{V_2}\right)$$

$\beta_2 = D_p\sqrt{\epsilon_2}$ " " "

$\gamma_2 = D_s\sin\chi\sqrt{\epsilon_2}$ " " "

. . (17)

At the boundary we have

$X_1 = X_2$, that is, $E_p = D_p$, since $\Phi = \chi = 0$

$\beta_1 = \beta_2$ or $E_p\sqrt{\epsilon_1} = D_p\sqrt{\epsilon_2}$

$\sqrt{\epsilon_1} = \sqrt{\epsilon_2}$

. . . (18)

The boundary conditions will hold for the incident and refracted wave only under the above condition, that is, when the refractive indices of the two media are the same. If ϵ_1 differs from ϵ_2, we shall

have a reflected wave, and the sum of the forces of the incident and reflected wave constitute the force at the boundary in the upper medium, which is to be equated to the force in the lower medium. The direction of the force in the reflected wave is opposite to that in the incident, for as the reflecting power increases, the force in the lower medium must diminish.

We now write for the electric and magnetic components of the reflected wave,

$$\left.\begin{aligned} X_r &= R_p \cos\Phi' \cos\frac{2\pi}{T}\left(t - \frac{x\sin\Phi' + z\cos\Phi'}{V_1}\right) \\ Y_r &= R_s \cos\frac{2\pi}{T} \qquad \text{,,} \qquad \text{,,} \\ Z_r &= -R_p \sin\Phi' \cos\frac{2\pi}{T} \qquad \text{,,} \qquad \text{,,} \end{aligned}\right\} \quad . \quad . \quad (19)$$

and

$$\left.\begin{aligned} \alpha_r &= -R_s \cos\Phi'\sqrt{\epsilon_1}\cos\frac{2\pi}{T}\left(t - \frac{x\sin\Phi' + z\cos\Phi'}{V_1}\right) \\ \beta_r &= R_p\sqrt{\epsilon_1} \qquad \cos\frac{2\pi}{T} \qquad \text{,,} \qquad \text{,,} \\ \gamma_r &= R_s \sin\Phi'\sqrt{\epsilon_1} \qquad \cos\frac{2\pi}{T} \qquad \text{,,} \qquad \text{,,} \end{aligned}\right\} \quad . \quad . \quad (20)$$

From these equations we can deduce the laws of reflection and refraction, as well as the relation between the intensities of the reflected and refracted rays for various states of polarization.

The relations between the angles of incidence, reflection, and refraction follow at once from the boundary conditions, which are only fulfilled when for $z=0$ we put all of the forces proportional to the same function of t, x, and y.

This gives us $$\frac{\sin\Phi}{V_1} = \frac{\sin\Phi'}{V_1} = \frac{\sin\chi}{V_2},$$

or $$\frac{\sin\Phi}{\sin\chi} = \frac{V_1}{V_2} = n, \text{ the refractive index.} \quad . \quad (21)$$

We will now deduce expressions for the intensities of the reflected and refracted components, the relations between the phases of the vibrations, their dependence upon the angle of incidence and the state of polarization. The force X_1, on the upper side of the boundary, is equal to the sum of the forces in the incident and reflected waves, $X_e + X_r$, which is to be equated to the force on the lower side of the boundary.

$$\left.\begin{aligned} &(1)\ X_e + X_r = X_2 \text{ or } && (E_p - R_p)\cos\Phi = D_p\cos\chi \\ &(2)\ Y_e + Y_r = Y_2 \text{ or } && E_s + R_s = D_s \\ &(3)\ \alpha_e + \alpha_r = \alpha_2 \text{ or } && (E_s - R_s)\sqrt{\epsilon_1}\cos\phi = D_s\sqrt{\epsilon_2}\cos\chi \\ &(4)\ \beta_e + \beta_r = \beta_2 \text{ or } && (E_p + R_p)\sqrt{\epsilon_1} = D_p\sqrt{\epsilon_2} \end{aligned}\right\} \quad . \quad . \quad (22)$$

The positive directions of the components R_p and D_p are shown in Fig. 263.

$$\left.\begin{aligned}
&\text{Add (2) and (3)}, && 2E_s = D_s\left(1+\frac{\sqrt{\epsilon_2}\cos\chi}{\sqrt{\epsilon_1}\cos\Phi}\right)\\
&\text{Elim. } D_s \text{ from (2) and (3)},\\
& && E_s\left(\frac{\sqrt{\epsilon_1}\cos\Phi}{\sqrt{\epsilon_2}\cos\chi}-1\right) = R_s\left(\frac{\sqrt{\epsilon_1}\cos\Phi}{\sqrt{\epsilon_2}\cos\chi}+1\right)\\
&\text{Add (1) and (4)}, && 2E_p = D_p\left(\frac{\cos\chi}{\cos\Phi}+\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}\right)\\
&\text{Elim. } D_p \text{ from (1) and (4)},\\
& && E_p\left(\frac{\cos\Phi}{\cos\chi}-\frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\right) = R_p\left(\frac{\cos\Phi}{\cos\chi}+\frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\right)
\end{aligned}\right\}. \quad (23)$$

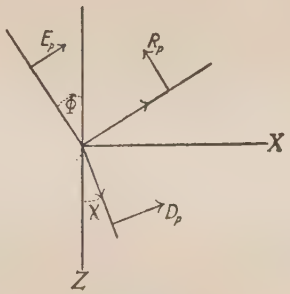


Fig. 263.

Substitute for $\frac{\sqrt{\epsilon^2}}{\sqrt{\epsilon_1}} = n$ its equivalent $\frac{\sin\Phi}{\sin\chi}$ and we obtain equations identical with the formulæ of Fresnel, from which the phases and intensities of the reflected and refracted waves can be calculated.

$$\left.\begin{aligned}
&\text{Reflected amplitudes:}\\
&R_s = -E_s\frac{\sin(\Phi-\chi)}{\sin(\Phi+\chi)}, \quad R_p = E_p\frac{\tan(\Phi-\chi)}{\tan(\Phi+\chi)}\\
&\text{Refracted amplitudes:}\\
&D_s = E_s\frac{2\sin\chi\cos\Phi}{\sin(\Phi+\chi)}, \quad D_p = E_p\frac{2\sin\chi\cos\Phi}{\sin(\Phi+\chi)\cos(\Phi-\chi)}
\end{aligned}\right\}. \quad (24)$$

It should be noticed that these formulæ are unsuitable for perpendicular incidence, for when $\Phi=0$, $\chi=0$, and the expression becomes indeterminate.

We will now examine these formulæ in detail.

It is evident that the component perpendicular to the plane of incidence in the reflected light never vanishes, whatever be the values of Φ and χ in the formula for R_s.

It is different, however, in the case of the formula for R_p, the parallel component. As we increase the angle of incidence from 0, it is evident that we shall eventually reach a point at which the reflected and refracted rays are at right angles, for the angle between them is greater than 90° near perpendicular incidence, and less than 90° at grazing incidence. At the angle in question it is evident that $(\Phi + \chi) = 90°$ and $\tan(\Phi + \chi) = \infty$, that is, $R_p = 0$, or the component parallel to the plane of incidence is wholly absent. This means simply that the reflected light is plane-polarized, and the angle in question is known as the angle of polarization or the Brewsterian angle.

The refractive index $n = \dfrac{\sin \Phi}{\sin \chi}$, consequently if Φ' be the angle of polarization, we have

$$\sin \chi' = \sin\left(\frac{\pi}{2} - \Phi'\right) = \cos \Phi' \text{ and } \frac{\sin \Phi'}{\cos \Phi} = \tan \Phi' = n,$$

a relation which has been fully discussed in the Chapter on Polarization.

Polarizing Power of Oblique Plates. — A method has been given, in the Chapter on Polarization, for the determination of the percentage of polarization, by compensation with inclined plates. It is important, therefore, to develop the formula for determining the polarization produced by one or more glass plates at any given angle. For the ratio of the amplitudes transmitted through a single surface we have from (24)

$$\frac{D_s}{D_p} = \frac{E_s}{E_p} \cos(\Phi - \chi),$$

or for two surfaces (*i.e.* a plate),

$$\frac{D_s}{D_p} = \frac{E_s}{E_p} \cos^2(\Phi - \chi).$$

If $E_s = E_p$, the condition equivalent to natural or unpolarized light, we have $D_s < D_p$, that is, the transmitted light is partially polarized, the plane of the vibration which is in excess being in the plane of incidence. If the incident vibration makes an angle of 45° with the plane of incidence, $E_s = E_p$ and the transmitted light is plane-polarized, the plane, however, being more or less rotated towards the plane of incidence, on account of the fact that the component D_p is larger than the component D_s. The percentage of polarization for a single plate increases with the angle of incidence. At the polarizing angle, $\tan \Phi = n$ and $\Phi + \chi = 90°$; therefore, if $E_s = E_p$, we have

$$\frac{D_s}{D_p} = \sin^2 2\Phi = \frac{4n^2}{(1+n^2)^2}.$$

The polarization is not a maximum at this angle, however, but goes on increasing as the angle of incidence increases, in marked contrast to polarization by reflection. This holds only for one plate,

as we shall see presently. As an illustration of the use of this formula, let us calculate the percentage of polarization in a source of light, when it is completely compensated by transmission through a glass plate at an angle of 59°. (This was the case for the polarized fluorescence of sodium vapor observed by the author.)

To do this we have only to calculate the percentage of polarization in ordinary unpolarized light after transmission through a glass plate at an incidence of 59°. For glass $n=1.52$, $\Phi=59°$. By Snell's law we calculate $\chi=34°\,20'$,

$$\Phi-\chi=24°\,40'.$$

For unpolarized light we take $E_s=E_p$ and find for the amplitude ratio

$$\frac{D_s}{D_p}=\cos^2(\Phi-\chi)=.826.$$

The intensity ratio we find by squaring this,

$$\frac{I_s}{I_p}=.682=\frac{68}{100}.$$

Now the total intensity of the light is $I_s+I_p=168$.

The amount of polarized light is $I_p-I_s=100-68=32$, and the percentage of polarization is 19.

If we have more than one plate, we raise the intensity ratio to the power equivalent to the number of plates. If in the above experiment we had used four plates, we should have

$$\frac{I_s}{I_p}=.682^4=.21$$

or 65% of the light polarized.

Stokes has given us a very complete paper on the transmission and reflection of light by a pile of plates, considered as perfectly transparent and as partially absorbing.

The table which accompanies his paper is so often of use in optical investigations that it is reproduced in full. The incident light is supposed to have an intensity 1000, and the proportions transmitted (χ) and reflected (Φ) are given for m plates. The loss by absorption for a transit through a single plate is represented by $\delta=1-e^{-qT}$. For perfect transparency we have $\delta=0$. If 2% is absorbed, $\delta=\frac{1}{50}$, and if 10%, $\delta=\frac{1}{10}$. In the table Φ_s and Φ_p represent the components of vibration perpendicular to and parallel to the plane of incidence. The case of normal incidence, $i=0$, incidence at the polarizing angle Φ', and at an angle two degrees greater than the polarizing angle are considered. Quoting from the original paper in regard to this table:

"The intensity of the light reflected from a pile consisting of an infinite number of similar plates falls off rapidly with the transparency of the material of which the plates are composed, especially at small incidence. Thus at perpendicular incidence we see from

Table for Reflection and Transmission by Transparent and Partially Absorbing Plates

No. of Plates m	$\delta = 0$								$\delta = \frac{1}{50}$						$\delta = \frac{1}{10}$					
	$i = 0$		$i = \Phi'$		$i = \Phi' + 2°$				$i = 0$		$i = \Phi'$				$i = 0$		$i = \Phi'$			
	Φ	χ	Φ_s	χ_s	Φ_s	χ_p	Φ_p	χ_p	$\frac{\chi_s}{\chi_p}$	Φ	χ	Φ_s	χ_s	χ_p	$\frac{\chi_s}{\chi_p}$	Φ	χ	Φ_s	χ_s	χ_p	$\frac{\chi_s}{\chi_p}$
1	82	918	271	729	300	700	1	999	.701	80	900	265	711	976	.728	74	826	245	639	881	.725
2	151	849	426	574	459	541	2	998	.542	145	815	410	544	953	.571	125	686	351	435	777	.559
4	262	738	598	402	628	372	4	996	.373	244	679	555	355	908	.391	185	479	427	215	604	.357
8	416	584	749	251	771	229	8	992	.231	364	490	656	182	824	.221	229	237	451	57	365	.156
16	587	413	856	144	870	130	16	984	.132	464	276	695	58	679	.086	243	59	453	4	133	.030
32	740	260	922	78	931	69	32	968	.071	509	97	699	7	461	.014	244	4	453	0	18	.001
∞	1000	0	1000	0	1000	0	1000	0	.000	516	0	699	0	0	.000	244	0	453	0	0	.000

the above table that the reflected light is reduced to little more than one-half when 2% is absorbed in a single transit, and to less than a quarter when 10% is absorbed.

"With imperfectly transparent plates little is gained by multiplying the plates beyond a very limited number, if the object be to obtain light, as bright as may be, polarized by reflection. Thus the table shows that four plates of the less defective kind reflect 79%, and four plates of the more defective kind as much as 94%, of the light that could be reflected by a greater number, whereas four perfectly transparent plates reflect only 60%. The table shows that while the amount of light transmitted at the polarizing angle by a pile of a considerable number of plates is materially reduced by a defect of transparency, its state of polarization is somewhat improved. This result might be seen without calculation. For while no part of the transmitted light which is polarized perpendicularly (vibration *in*) to the plane of incidence underwent reflection, a large part of the transmitted light polarized the other way was reflected an even number of times, and since the length of the path of the light within the absorbing medium is necessarily increased by reflection, it follows that a defect of transparency must operate more powerfully in reducing the intensity of light polarized in (vibration $\perp$ to) than of light polarized perpendicularly to the plane of polarization.

"But the table shows that a far better result can be obtained as to the perfection of the polarization of the transmitted light, without any greater loss of illumination, by employing a larger number of plates of a more transparent kind.

"With a single plate the polarization of the transmitted light continually improves up to grazing incidence, but with a pile of plates the polarization attains a maximum at an angle of incidence which approaches indefinitely to the polarizing angle as the number of plates is indefinitely increased.

"For a given number of plates the angle of maximum polarization may be readily found by the method of trial and error. The following table gives for assumed angles of incidence, decreasing by 5° from 90°, the number of plates required to make these angles the angles of maximum polarization of the transmitted light, and the value Ψ, which determines the defect of polarization."

$i = 90°$	85°	80°	75°	70°	65°	60°	56° 40′ ($=\Phi'$)
$m = 1$	1.33	1.94	2.91	4.92	9.77	30.37	∞
$\Psi = .433$	.422	.390	.337	.265	.177	.075	0

In the larger table the column $\frac{\chi_s}{\chi_p}$ gives the measure of the defect of polarization of the transmitted light. No such column was required for $\delta = 0$ and $i = \Phi'$ because in this case $\chi_p = 1000$.

The table will be found interesting in connection with the thin plates formed by roasting thin laminæ of mica described in the Chapter on Radiation. (Temperature radiation of solids.)

Perpendicular Incidence. — As has been stated, our formulæ do not hold for perpendicular incidence, for then $\Phi = \chi = 0$.

Substituting in equations (23) for $\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$ the refractive index n, we have, since $\frac{\cos\Phi}{\cos\chi} = \frac{1}{1}$,

$$E_r\left(\frac{1+n}{n}\right) = E_i\left(\frac{1-n}{n}\right),$$

$$E_r = E_i\left(\frac{1-n}{1+n}\right) = -E_i\left(\frac{n-1}{n+1}\right),$$

$$E_r = E_i\left(\frac{n-1}{n+1}\right).$$

From the first equation it is clear that if $n > 1$, the reflected electric amplitude is oppositely directed from the incident, since the direction of the vector depends on its sign. The second equation shows the same condition, for when $\Phi = 0$, similar signs mean opposite directions, as will be seen by referring to Fig. 263. The presence of the reflected wave will thus reduce the amplitude of the incident wave *at the reflecting boundary*, and, if the intensity of the reflected wave is equal to that of the incident, the amplitude will be reduced to zero. This was found to be the case by Wiener in his experiments upon stationary light waves, the node occurring at the reflecting surface. These experiments were fully described in the Chapter on Interference. The opposite condition will be found for the magnetic vectors, which are similarly directed in the incident and reflected waves. They will therefore add their effects at the boundary.

It must be remembered that the above formulæ express the *amplitudes* of the reflected vibrations; the *intensity* of the reflected light, if the intensity of the incident light is 1, is given by

$$R_r^2 = I = \frac{(n-1)^2}{(n+1)^2}.$$

This formula has been verified for water by Lord Rayleigh, who found that the observed value agreed with the calculated within 1.5 per cent.

Change of Phase by Passage through the Polarizing Angle. — The formula for the reflected amplitude, parallel to the plane of incidence, shows us that the phase changes suddenly by 180° on passage through the angle of polarization; for $(\Phi + \chi)$ is obtuse or acute according as the angle of incidence is less or greater than the polarizing angle. Suppose now that the incident light is plane-polarized at an azimuth of 45°. At the polarizing angle the component of the vibration, which is parallel to the surface, will be the only one reflected, and it can be completely quenched by means of a Nicol prism held with its short diagonal vertical. On either side of the polarizing angle we shall have a reflected component

perpendicular to the other, but the directions of the vectors will be opposite on opposite sides of the angle. The resultant will be in each case a plane-polarized vibration, which will, however, be turned slightly towards the plane of incidence, the direction of the rotation from the plane parallel to the surface being opposite in the two cases. This will be readily understood by drawing the horizontal component, and compounding it first with a small vertical component directed upwards, and then with one directed downwards, the change of direction corresponding to the phase change of 180°.

It was found by Jamin and others that in the majority of cases the light was not completely polarized by reflection at the Brewsterian angle. Moreover, if the incident light was polarized, and reflection occurred in the neighborhood of this angle, the reflected light, instead of being plane-polarized, as the formulæ indicate, showed traces of elliptical polarization. This indicates that the phase change, instead of occurring abruptly at the polarizing angle, enters by degrees; Drude observed in 1889 that the elliptical polarization produced by a freshly split surface of rock salt was very small, but that it increased rapidly on the exposure of the surface to the air. Shortly afterward Lord Rayleigh found that the ellipticity produced by reflection from water could be completely eliminated by removing the surface film of grease, which is always present unless special precautions are taken.

These experiments indicate that the disagreement with the formulæ is caused by surface films having optical properties different from those of the body of the substance. We will now take up the investigation of the effects of these films, and the calculation of their probable thickness.

Elliptical Polarization. Surface Films or Strains? — The theory of reflection applied to boundaries between media of different optical densities has led us to the conclusion that plane-polarized light should always be reflected as plane-polarized light. As a matter of fact, we find that this is seldom the case. If the incident light is polarized at an angle of 45° with the plane of incidence, almost no change in intensity is seen when the reflected light is examined through a revolving Nicol, if the reflecting surface is a metal, while even in the case of transparent substances it is seldom possible to completely extinguish the reflected light with the analyzing Nicol. The cause of this has been supposed to be the almost universal presence of a so-called "surface-layer" within which the optical density changes gradually from that of the upper medium to that of the lower. In the previous treatment we considered that the change at the boundary was abrupt, and deduced our boundary conditions on this assumption.

Drude has developed the equations which represent the reflection of the two components, from a surface at which the change in the refractive index is not abrupt; that is, he considers a surface film present with a refractive index less or greater than that of the reflecting material. His treatment will be found in his text-book on Optics, or in the earlier edition of this work.

It has seemed best to omit it, as doubt has been cast upon its

validity by the recent work of Lummer and Sorge,[1] who have shown that the elliptical polarization can be altered by subjecting the medium to pressure.

If ϵ_1 is the dielectric constant of the first medium, and ϵ_2 that of the second (or reflecting substance), and ρ is the amplitude ratio, Drude's treatment shows us that ρ has a positive sign when $\epsilon_2 > \epsilon_1$, and the dielectric constant ϵ of the film has an intermediate value. If the incident light is considered as coming towards us and the plane of vibration makes an angle of 45° with the vertical (the rotation away from the vertical being clockwise), the direction of the elliptical vibration will be clockwise for positive values of $\bar{\rho}$ and counter-clockwise for negative values, a change of sign of the amplitude ratio, amounting to the same thing as a phase-difference of 180° between the components. We can easily determine the value of $\bar{\rho}$ experimentally by either of the methods given in the Chapter on Elliptical Polarization. In the case of reflection at a glass surface in air it has a value not far from .007, though for flint glass with a high refractive index the value may be as high as .03. Negative ellipticity, which occurs when $\epsilon_1 > \epsilon_2$, has been observed in the case of water and certain solids with very low refractive indices.

In the case of water Lord Rayleigh has shown that the ellipticity is due to a very thin film of grease, which naturally has a higher refractive index than water. In the case of solids, a higher refractive index of the surface film may perhaps be explained by some action of the polishing material upon the surface.

By carefully cleaning the surface of the water, Lord Rayleigh was able to almost completely destroy all traces of elliptical polarization, the value of ρ being not more than .00035. Quite recently he has found that the positive ellipticity of glass can be changed into negative by repolishing the surface.

Drude found that freshly cleaned crystal surfaces showed no traces of elliptical polarization, but that it appeared after the surfaces had been exposed to the air for some time, owing to the formation of surface films.

Assuming that the dielectric constant is uniform throughout the film, Drude calculated the thickness of the film necessary to produce a given axis ratio $\bar{\rho}$.

For glass of refractive index $n = 1.5$ and $\bar{\rho} = .007$ he found that the thickness of the film necessary to account for the ellipticity is less than $\frac{1}{200}$ of the wave-length of the light.

Lummer and Sorge found, in repeating and extending the work of Lord Rayleigh, that the elliptical polarization of the light reflected from the surface of a glass prism was altered by polishing or rubbing one of the other faces, or even by subjecting the prism to pressure applied to the bases.

The prism was a right-angled one, and they found that by rubbing the two base surfaces the ellipticity was decreased for all three of the other surfaces, that for the hypothenuse passing through 0 (the

[1] *Annalen der Physik*, Band 31, page 325 (1910).

condition for the ideal case of sudden transition) and becoming slightly negative. It is impossible to conceive how the refractive index of a surface film could be altered, without touching the surface, much less how it could be changed from a value less than that of the medium to one of greater, as we should be obliged to assume for the conditions found in the case of the hypothenuse surface of Lummer's prism.

In view of these experiments it appears more probable that the elliptical polarization is caused in some way by surface strains. At all events further work is required before we can feel justified in establishing a theory. Some unpublished experiments by the author have pointed towards a change in the reflecting power resulting from strains. A very thin film of collodion on glass has been found to more than double its reflecting power under certain conditions. The refractive index of collodion is, however, not very different from that of glass, and we should expect but little reflection from the transition surface. A layer of thick glue spread over a glass surface reduces its reflecting power nearly to zero. When the glue dries, the surface is subjected to a strain so great that glass chips are often torn off the surface. The reflecting power of the surface separating the dry glue and the glass was a little higher than that of a surface between glass and damp glue. The change in the refractive index could hardly account for this, as it would be in the wrong direction.

Total Reflection. — We have seen in the Chapter on Refraction that when a ray of light is incident at the boundary separating an optically dense from a rarer medium, the refracted ray vanishes for incidence angles greater than a certain value, the energy being totally reflected. We will now apply our equations to this phenomenon. In this cases in χ turn out to be greater than unity, *i.e.* χ is no longer real. We can study the nature of the reflected light, however, by substituting $\frac{\sin \Phi}{n}$ for $\sin \chi$ in the equations (23).

$$\therefore \cos \chi = \sqrt{1 - \frac{\sin^2 \Phi}{n^2}},$$

which quantity is imaginary if $\sin \Phi > n$.

We can write this in the form

$$\cos \chi = -i \sqrt{\frac{\sin^2 \Phi}{n^2} - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Substitution of the above in equation (23) gives us reflected light with a complex amplitude, which as we have seen can be interpreted as a change of phase which results at the moment of reflection. This change of phase, if it is different for the two components of the incident vibration, will result in the transformation of a linear vibration into an elliptical one, and as total reflection is one of the methods commonly employed to produce elliptically and circularly polarized light, the subject is of some importance. To calculate this phase change we write as before

$R_p e^{i\delta_p}$ and $R_s e^{i\delta_s}$ for the components of the reflected amplitude, and obtain, since $\sqrt{\epsilon_2}/\sqrt{\epsilon_1} = n$, by substitution in (23)

$$E_s\left(\frac{i\cos\Phi}{\sqrt{\sin^2\Phi - n^2}} - 1\right) = R_s e^{i\delta_s}\left(\frac{i\cos\Phi}{\sqrt{\sin^2\Phi - n^2}} + 1\right),$$

$$E_p\left(\frac{i\cos\Phi\cdot n}{\sqrt{\sin^2\Phi - n^2}} - \frac{1}{n}\right) = R_p e^{i\delta_p}\left(\frac{i\cos\Phi\cdot n}{\sqrt{\sin^2\Phi - n^2}} + \frac{1}{n}\right).$$

If we multiply these equations by their complex conjugates, obtained by writing in them $-i$ for i, we find that $E_s^2 = R_s^2$ and $E_p^2 = R_p^2$, *i.e.* the intensities of the reflected components are equal to those of the incident.

Suppose now that our incident light is plane-polarized vibrating in azimuth 45°. In this case $E_s = E_p$ and $R_s = R_p$, and if we substitute these values in the above equations and divide, we get

$$\frac{i\cos\Phi - \sqrt{\sin^2\Phi - n^2}}{i\cos\Phi\cdot n - \frac{1}{n}\sqrt{\sin^2\Phi - n^2}} = e^{i(\delta_s - \delta_p)}\,\frac{i\cos\Phi + \sqrt{\sin^2\Phi - n^2}}{i\cos\Phi\cdot n + \frac{1}{n}\sqrt{\sin^2\Phi - n^2}},$$

or

$$e^{i(\delta_p - \delta_s)} = e^{i\Delta} = \frac{\sin^2\Phi + i\cos\Phi\sqrt{\sin^2\Phi - n^2}}{\sin^2\Phi - i\cos\Phi\sqrt{\sin^2\Phi - n^2}},$$

and

$$\frac{1 - e^{i\Delta}}{1 + e^{i\Delta}} = \frac{-i\cos\Phi\sqrt{\sin^2\Phi - n^2}}{\sin^2\Phi}.$$

Multiplying this by its complex conjugate gives

$$\frac{1 - \cos\Delta}{1 + \cos\Delta} = \left\{\frac{\cos\Phi\sqrt{\sin^2\Phi - n^2}}{\sin^2\Phi}\right\}^2,$$

since

$$e^{i\Delta} + e^{-i\Delta} = 2\cos\Delta;$$

therefore

$$\tan\tfrac{1}{2}\Delta = \frac{\cos\Phi\sqrt{\sin^2\Phi - n^2}}{\sin^2\Phi}. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This expression shows us that the relative phase-difference Δ is zero for grazing incidence $\left(\Phi = \frac{\pi}{2}\right)$, and also at the critical angle $(\sin\Phi = n)$, in which n is the relative refractive index. If the denser medium has a refractive index 1.51 and the reflection occurs at an air surface, n in our equations will be $\frac{1}{1.51}$.

To find the value of Φ which will give Δ its maximum value, we differentiate the last equation with respect to Φ, and obtain

$$\frac{1}{2\cos^2\frac{1}{2}\Delta}\frac{\partial\Delta}{\partial\Phi} = \frac{2n^2 - \sin^2\Phi(1 + n^2)}{\sin^3\Phi\sqrt{\sin^2\Phi - n^2}}$$

and the maximum value of Δ is obtained at an incidence angle Φ' determined by

$$\sin^2 \Phi' = \frac{2n^2}{1+n^2}.$$

The corresponding value of Δ is given by

$$\tan \tfrac{1}{2} \Delta' = \frac{1-n^2}{2n}.$$

For glass of refractive index 1.51, $\Phi' = 51° 20'$ and $\Delta' = 45° 36'$. A value of 45° occurs at incidence angles 48° 37′ and 54° 37′. Two reflections at this angle will give us $\Delta = 90°$, and circularly polarized light will result if the incident light was polarized in azimuth 45°. This is accomplished by the Fresnel rhomb described in the Chapter on Elliptical Polarization.

Penetration of the Disturbance into the Second Medium. — If we apply the equations (23), which express the relation between the incident and the refracted amplitudes, to the case of total reflection, we reach the somewhat astonishing conclusion that the refracted amplitude is not zero, which appears to be inconsistent with the *total* reflection of the energy. The case is a peculiar one, for although D may have a large value close to the boundary, it becomes zero at a distance of a few wave-lengths, the energy being entirely thrown back into the first medium. This decrease in the amplitude, as we advance from the boundary in the direction of the z axis, can be seen from equations (15) (16), which show that electric and magnetic forces in the second medium,

$$e^{i\frac{2\pi}{T}\left(\frac{t - x \sin \chi - z \cos \chi}{V_2}\right)},$$

are proportional to the real part of the complex quantity which, if we substitute for $\cos \chi$ the value given by equation (25), takes the form

$$e^{-\frac{2\pi}{TV_2}\sqrt{\frac{\sin^2 \Phi}{n^2} - 1} \cdot z} \cdot e^{i\frac{2\pi}{T}\left(t - \frac{x \sin \Phi}{nV_2}\right)}. \quad . \quad . \quad . \quad . \quad (27)$$

This formula represents a wave disturbance moving parallel to the x axis, which implies that the energy stream is along the boundary, and not down into the second medium. The amplitude, which is represented by the underscored part of (27), decreases as z increases, becoming sensibly zero when z is large in comparison to the wave-length $\lambda_2 = TV_2$. These boundary waves possess another peculiarity, in that they are not transverse, for in a transverse disturbance moving along the x axis in the second medium, X_2 must equal zero, which is not the case.

The existence of these waves can be shown experimentally by bringing a convex surface of glass of large radius of curvature into contact with the surface at which total reflection is taking place. The light will be found to enter the lens in an annular region surrounding the point of contact. This is due to the fact that the two glass surfaces come into optical contact, *i.e.* we may regard the

air film as completely squeezed out. This circular patch over which total reflection fails, and complete transmission obtains, is surrounded by a ring which *transmits* light of a reddish, and *reflects* light of a bluish, tinge. The glass surfaces are not in contact here, but the air film is too thin for total reflection to take place. Transmission will obviously occur for red light first, since the thickness of the film necessary to reflect light is measured in comparison with the wave-length. This experiment dates back to the time of Newton and Fresnel, and shows us that if the rarer medium is of extreme thinness, total reflection no longer occurs. Voigt [1] has described an experiment designed to show the possibility of separating this surface wave from the incident and reflected waves, and allowing it to spread out into space. A prism of the form shown in Fig. 264 was constructed, and light passed into it in the direction indicated by the arrows. The angle of incidence exceeded the critical angle at both of the surfaces, and must therefore have exceeded it along the edge *a*, which can be regarded as a cylinder of very minute radius of curvature. Voigt observed that the edge *a* was luminous, and that the intensity was greatest when viewed from a point *b*, decreasing steadily as the eye was moved towards *c*. He regarded this emission of light by the edge as due to the breaking away of the surface wave from the wave disturbances in the glass.

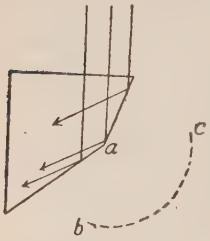


FIG. 264.

Ketteler [2] criticised this experiment, and claimed that it was a physical impossibility to obtain a separation of the two wave disturbances in the manner imagined by Voigt. An attempt was made by the author some years ago to repeat Voigt's experiment with a prism of very nearly identical form, but no distinct evidence of the phenomenon was observed, and it seems possible that the effect observed might have been due to the fact that the edge was not in fact a perfectly polished cylinder. It is difficult, however, if not impossible, to pass judgment on an experiment of this nature without actually witnessing it.

Another method of showing the presence of a luminous disturbance in the rarer medium, which is to be recommended on account of its simplicity, is to scatter minute particles on the reflecting surface; for example, smoke the hypothenuse surface of a right-angled prism *very lightly* with a flame. On sending a strong beam of light into the prism the smoked patch will be illuminated, and if viewed under a powerful microscope, each individual carbon particle will be seen to scatter light in all directions. This method was used by the author in establishing the granular nature of certain metallic films, which will be described in the Chapter on The Scattering of Light. The method was subsequently and independently originated by Cotton as a means of rendering visible ultra-microscopic particles.

[1] *Wied. Ann.*, 67, page 185, 1899.

[2] *Wied. Ann.*, 67, page 879.

CHAPTER XIV

THE THEORY OF DISPERSION

Previous to the discovery of anomalous dispersion, all that was required of a dispersion theory was a satisfactory explanation of a steady increase in the refractive index with decreasing wave-length, and the development of a mathematical relation between the two which should conform to the dispersion curves determined by experiment. The expression for the velocity of transverse waves $c = C\sqrt{\frac{\epsilon}{d}}$, where ϵ is the elasticity and d the density of the medium, is developed on the assumption that the wave-length is large in comparison to the distance between the vibrating particles. If this is not the case, the velocity of propagation is a function of the wave-length, as was shown by Cauchy, who assumed that in refracting media it was not allowable to assume the wave-length large in comparison to the distance between the particles, on account of the shortening of the waves by retardation. Cauchy deduced the expression

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4},$$

which gives the refractive index in terms of the wave-length and three constants, for the determination of which we require determinations of n for three different wave-lengths.

This formula was found to represent the dispersion of many transparent substances with considerable accuracy, and the fact that the dispersion increases as we pass down the spectrum into the region of the short waves, makes it appear at first sight as if his assumption was justifiable. It was pointed out by Biot, however, that the shortening of the waves was not sufficient to account for the phenomenon, since if, in dispersing media, the wave-length has a value comparable to the distance between the particles, the same must be true in free ether, which should, therefore, show evidences of dispersion; in other words, there is not sufficient change in the wave-length. It is evident that this must be the case, for the wave-length of red light in glass is greater than that of violet light in free ether. That the formula expresses the relation between n and λ in certain cases is purely accidental, and, as we shall see presently, it is a special case of a much more elaborate formula, developed from quite different fundamental assumptions.

The discovery of anomalous dispersion, and the relation existing between absorption and dispersion, put the matter in a new light. The refractive index of a medium, which exercised strong selective absorption, was found to increase rapidly as the absorption band

was approached from the region of longer wave-lengths. This made it seem extremely probable that the dispersion of so-called transparent media was due to absorption bands in the ultra-violet; in other words, that there was no essential difference between normal and anomalous dispersion, the former being only a special case of the latter, the observations being restricted to a range of wave-lengths too narrow to show any anomalies.

As has been recently pointed out by Lord Rayleigh, the foundation of the modern theory of dispersion was in reality laid by Maxwell in the form of a question propounded in an examination paper (*Camb. Calendar*, 1869, Math. Tripos Exam.). The same idea subsequently occurred to Sellmeier, who has always been regarded as the founder of the theory. Sellmeier sought for the cause of dispersion in the vibrations of the atoms of the molecule caused by the repeated impacts of the light-waves. These atoms would naturally have free-periods of vibrations of their own, and would be set in motion by the light-waves exactly as a tuning-fork is set in vibration by waves of sound.

Sellmeier deduced a formula which is practically identical with a special case of the more recent electro-magnetic dispersion formula, and which represents the dispersion for those wave-lengths for which the medium is comparatively transparent, *i.e.* on both sides of the absorption band. Within the region of absorption it breaks down, for reasons which will appear presently. Sellmeier's formula is as follows:

$$n^2 = 1 + \frac{DT^2}{T^2 - T_m^2},$$

where n is the refractive index for light of periodic time T, and T_m is the periodic time of the vibrating atom. Substituting wave-lengths for periodic times, the formula becomes

$$n^2 = 1 + \frac{D\lambda^2}{\lambda^2 - \lambda_m^2},$$

where λ is the wave-length in ether of light of the same vibration frequency as the absorbing atom. If more than one absorption band exists, the formula takes the form

$$n^2 = 1 + \sum \frac{D\lambda^2}{\lambda^2 - \lambda_m^2},$$

the summation being taken for as many terms as there are atoms of different periods. We will now compare this formula with the Cauchy formula:

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}.$$

As the wave-length increases, the refractive index, as expressed by this formula, becomes less, approaching asymptotically the limiting value A. This was formerly supposed to agree with experiment, but more recent investigations in the infra-red have shown that the

dispersion curve after running nearly horizontal for a certain distance may again descend.

In Fig. 265 $XABC$ is an experimentally determined curve. The portion AB follows the Cauchy formula, which, if applied to values

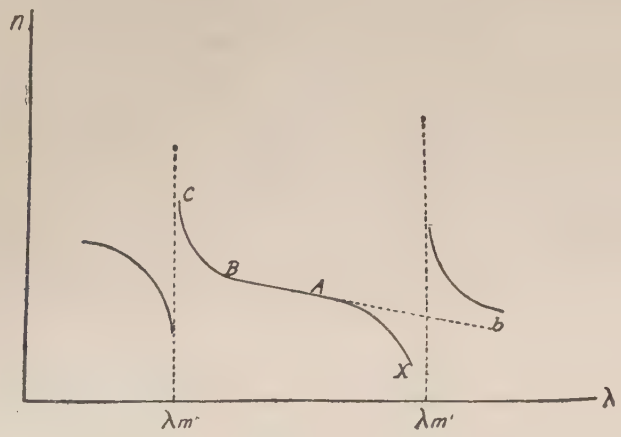


FIG. 265.

of λ greater than the value at A would give the dotted curve Ab instead of the actual curve AX. For infinitely short waves the refractive index would be infinite.

Let us now examine the formula of Sellmeier,

$$n^2 = 1 + \frac{D\lambda^2}{\lambda^2 - \lambda_m{}^2}.$$

For very long waves the denominator becomes approximately equal to λ^2, and we have

$$n = \sqrt{1 + D},$$

which, if D is small, as is sometimes the case, does not differ much from unity. As λ decreases the value of the fraction $\frac{\lambda^2}{\lambda^2 - \lambda_m{}^2}$ increases, becoming infinite when $\lambda = \lambda_m$. For values of λ less than λ_m the sign of the term changes, and we have values of n, which are less than unity, the lowest values being for wave-lengths close to the absorption band; as λ decreases n increases, becoming unity for infinitely short waves. The form of the curve is shown in Fig. 266. As we shall see later on, the Sellmeier formula represents most perfectly the dispersion of sodium vapor, in which the refraction and dispersion are due almost entirely to the influence of a single absorption band (in reality a close double band).

If we confine our attention to the region AB of the curve, we see that the dispersion here is normal, the refractive index increasing with decreasing λ, and the curve convex towards the axis of abscissae, as is the case with all ordinary transparent media for visible

radiations. The decrease in refractive index occurs when we pass across the absorption band, in the above case the drop being very sudden. A further decrease in the value of λ causes an increase in

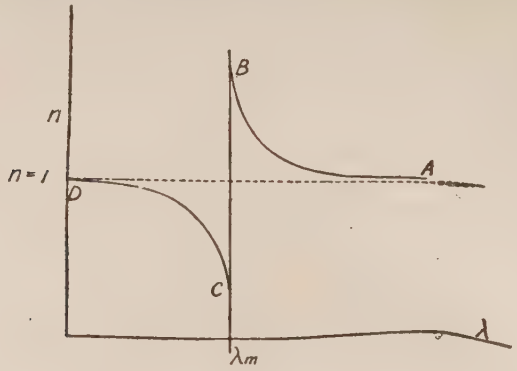


FIG. 266.

n, the curve now being *concave* towards the axis of abscissæ. The dispersion here is normal in that n increases as λ decreases, but it differs from the dispersion along the branch AB in that the rate of change of n with λ becomes less as λ becomes less, while along AB the rate of change of n with λ increases as λ decreases. This is due to the fact that in the former case we are receding from the absorption band, while in the latter we are approaching it.

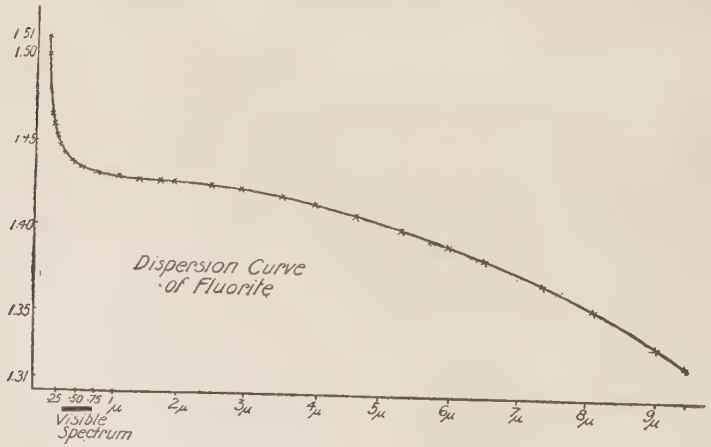


FIG. 267.

The dispersion of glass or water, for example, is represented by a curve of form similar to AB, consequently we may infer that an absorption band in the ultra-violet is responsible for the dispersion. Curves of the form CD are never found for ordinary transparent media in the visible spectrum. If, however, the investigations are

carried into the infra-red region, we frequently find that the curve, which in the visible region of the spectrum was convex towards the axis of abscissae, eventually becomes concave in this direction, indicating that an absorption band is being approached. The curve for fluorite (Fig. 267) is an example.

If we apply Sellmeier's formula to the dispersion of some substance such as glass, where the refractive index appears to approach a limiting value of say 1.5 in the extreme red, instead of unity as in the first case considered, we find that the only difference lies in the larger value of the constant D, which must be equal to 1.25 if $n=\sqrt{1+D}=1.5$.

A fuller discussion of the matter can be undertaken to better advantage after we have actually developed the dispersion formula, for we are then in a position to recognize the physical significance of the constants.

Thus far all that has been attempted is to show that normal dispersion is only a special case of the so-called anomalous dispersion, and that the general form of the curve can be represented by a formula, in which the difference between the squares of the observed wave-length, and the wave-length at the centre of the absorption band occurs in the denominator.

Schuster has proposed the term "Selective Dispersion" for the phenomenon which we observe in the vicinity of an absorption band, as there is in reality nothing anomalous about it, and the suggestion is an excellent one, and will be followed from now on.

The formula of Sellmeier was incomplete in that it was inapplicable to values of λ very near the value of λ_m. The curve which the formula represents runs to infinity on one side of the centre of the absorption band, and to zero on the other. On the fundamental assumptions made by Sellmeier there would in fact be no absorption, for he introduced no term which provided for a transformation of the radiant energy into energy of some other form, which must occur if light is actually absorbed by the medium.

The conception of something akin to friction accompanying the vibration of the atom was introduced by Helmholtz, who formed separate differential equations for the vibration of the ether and that of the ponderable atom, introducing a term representing a frictional loss into the latter. The absorption of the light is here accounted for as a frictional transformation of the radiant energy into heat, and the final formula which expresses the variation of n with λ shows that the dispersion curve is continuous through the absorption band. The factor expressing friction enters into the formula in such a way that its tendency is to *decrease* the value of n as the absorption band is approached from the region of longer waves. This factor becomes larger as we near the centre of the band, consequently the curve, instead of running off to infinity, turns as we enter the band, and running down through it meets the other branch, the whole curve being continuous.

We will now proceed with the development of the complete dispersion formula, first on the mechanical theory and then on the electro-magnetic theory. The former treatment will be made in-

dependent of complex quantities, and though longer, is perhaps easier to follow; the latter will involve the use of imaginaries, and though the final equation is not very different from the other, it will help us to understand the physical significance of the constants, and the relations existing between the optical and electrical properties of various media.

Helmholtz's Mechanical Theory of Dispersion. — In the following treatment we shall regard the ether in the nature of an elastic solid, *i.e.* made up of small particles, which when displaced are urged back into their original position by forces of restitution. The refracting medium we assume to be made up of molecules, between which the ether penetrates freely. The atoms of these molecules are capable of vibrating in periods of their own like pendulums, and any displacement of an ether particle is assumed to cause a displacement of one or more atoms; in other words, forces exist between the ether particles and atoms, similar to those existing between the ether particles themselves. When a wave enters the medium, we consider it propagated through the agency of the ether alone; that is, there is no direct propagation of a disturbance from molecule to molecule or from atom to atom. Helmholtz considered that the molecule remained at rest, but that the atom could be displaced from its position of equilibrium by the vibration of the ether, and when so displaced was drawn back by a force of restitution proportional to the displacement. Though the atoms are independent of each other, and each is free to vibrate by itself, they will, when disturbed in succession by a passing wave, have displacements which collectively form a wave curve, just as chips floating on water-waves, though not transmitting the waves, will be arranged in the form of the wave-curve, the difference in this case being an absence of any force of restitution tending to draw back the displaced chips.

Thus far the conception is not very different from that of Sellmeier. Helmholtz next assumes the vibration of the atom to be accompanied by friction, for if this were not the case the energy taken from the ether would be given back to the ether by the vibrating atom and no absorption would take place. The velocity with which waves are propagated through a medium can be determined in terms of the elasticity and density of the medium. The measure of the elasticity is the force of restitution exerted upon a displaced particle by the neighboring particle. In the case which we are about to consider, the displaced ether particle is urged back into its position of equilibrium not only by the forces exerted by neighboring ether particles, but also by the forces existing between the ether and the atoms. The atoms are, however, not stationary, but are set in motion to a greater or less degree by the waves. If the waves have the same period as the atom, the latter will be violently agitated, in a manner analogous to that of a tuning-fork when subjected to sound-waves of the same period as its own. If the period of the light-waves be different from the free period of the atom, the latter will be forced to vibrate with a period different from its natural period, and the amplitude of the vibration will be

less than before. We have to determine the velocity of waves of different length, as influenced by the reaction upon the ether of the vibrating atoms, taking into account the damping which they experience as the result of friction.

Let μ = the mass of the ether particle and η its displacement at time t (Fig. 268). Let m = the mass of the atom and y its displacement at time t. If ϵ is the elasticity of the ether, the equation of motion of the ether alone, the atom considered absent, would be

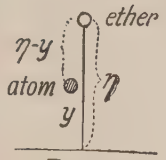


Fig. 268.

$$\frac{d^2\eta}{dt^2} = a\frac{\epsilon}{\mu}\frac{d^2\eta}{dx^2}.$$

To this acceleration must be added that due to the force exerted on the ether particle by the atom. If β is the force for unit displacement, the force in this case is $-\beta(\eta-y)$, the negative sign being given since the force acts in a direction opposite to the displacement. The expression for the acceleration of the ether particle now becomes

$$(1) \qquad \frac{d^2\eta}{dt^2} = a\frac{\epsilon}{m}\frac{d^2\eta}{dx^2} - \beta(\eta-y).$$

This expression is not sufficient to determine the motion of the ether particle; we require in addition an expression for the motion of the atom.

The ether particle exerts on the atom, in the direction of the displacement of the force $\beta(\eta-y)$. The atom is drawn in the other direction by a force $-\gamma y$, where γ is the force exerted when the displacement is unity.

From this is to be subtracted a term representing the effect of friction, which is proportional to the velocity. The equation for the motion of the atom of mass m is

$$(2) \qquad \frac{d^2y}{dt^2} = \frac{\beta}{m}(\eta - y) - \frac{\gamma y}{m} - \frac{\delta}{m}\frac{dy}{dt},$$

where δ is the frictional resistance for unit velocity.

These two equations are sufficient to determine the motion of the ether and the atom.

We shall now get values for $\frac{d^2\eta}{dt^2}$ and $\frac{d^2y}{dx^2}$ by twice differentiating the usual expression for the displacement in the case of wave-motion, and substituting the values thus found in the above equations.

Let t be the time which has elapsed since the disturbance reached the boundary of the medium, T the period of the vibration of the wave, x, the distance of the particle in question from the boundary, l the wave-length in the medium, and a the amplitude of the particle.

Then $$\eta = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{l}\right),$$

in which η is the displacement of a particle at distance x from the boundary at time t.

As we are dealing with a medium in which absorption may take place, we must modify this expression somewhat. The amplitude will decrease as the disturbance penetrates into the medium, owing to the friction of the vibrating atoms, and it will decrease by the same fractional amount for equal increments of x.

Let A be the amplitude at the boundary. At distance x from the boundary the amplitude α will be

$$\alpha = Ae^{-kx}, \text{ where } k \text{ is a constant.}$$

Substituting this in our last equation we have

$$\eta = Ae^{-kx} \sin 2\pi\left(\frac{t}{T} - \frac{x}{l}\right). \tag{3}$$

In the same way, if B is the amplitude of the atom at the boundary, we have

$$y = Be^{-kx} \sin 2\pi\left(\frac{t}{T} - \frac{x}{l} - \Delta\right), \tag{4}$$

where Δ is a possible difference of phase between the ether particle and the atom.

We now twice differentiate η and y with respect to t, regarding x as constant, and twice with respect to x, regarding t as constant, and substitute these values in equations (1) and (2).

$$\frac{d^2\eta}{dt^2} = -\frac{4\pi^2}{T^2} Ae^{-kx} \sin 2\pi\left(\frac{t}{T} - \frac{x}{t}\right),$$

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2}{T^2} Be^{-kx} \sin 2\pi\left(\frac{t}{T} - \frac{x}{l} - \Delta\right),$$

$$\frac{d^2\eta}{dx^2} = k^2 Ae^{-kx} \sin \Phi + \frac{2\pi}{l} k(Ae^{-kx} \cos \Phi) + k\frac{2\pi}{l} Ae^{-kx} \cos \Phi$$

$$- \frac{4\pi^2}{l^2} Ae^{-kx} \sin \Phi,$$

in which

$$\Phi = 2\pi\left(\frac{t}{T} - \frac{x}{l}\right).$$

The expression for $\frac{d^2\eta}{dt^2}$ is substituted as it stands, while in the case of the expression for $\frac{d^2\eta}{dx^2}$ we group the sin Φ terms and cos Φ terms thus:

$$\left(k^2 Ae^{-kx} - 4\frac{\pi^2}{l^2} Ae^{-kx}\right) \sin \Phi = \left(k^2 - \frac{4\pi^2}{l^2}\right) Ae^{-kx} \sin \Phi,$$

which, when multiplied by $a\frac{\epsilon}{\mu}$, gives

$$-a\frac{\epsilon}{\mu}\left(\frac{4\pi^2}{l^2} - k^2\right) Ae^{-kx} \sin \Phi.$$

Grouping the cos Φ terms in the same way gives

$$2a\frac{\epsilon}{\mu}\left(\frac{4\pi^2}{l^2}-k^2\right)Ae^{-kz}\sin\Phi.$$

We require also an expression for $(\eta-y)$,

$$\frac{\beta}{\mu}(\eta-y)=\frac{\beta}{\mu}\left[Ae^{-kz}\sin\Phi-Be^{-kz}\sin(\Phi-\Psi)\right]$$

$$=\frac{\beta}{\mu}Ae^{-kz}\sin\Phi-\frac{B}{A}\sin(\Phi-\Psi)$$

in which $\Psi=2\pi\Delta$.

Substituting in equation (1) for $\frac{d^2\eta}{dt^2}$, $\frac{d^2\eta}{dx^2}$ and $(\eta-y)$ the values obtained from the equations (3) and (4) we have

$$-\frac{4\pi^2}{T^2}Ae^{kz}\sin\Phi=-a\frac{\epsilon}{\mu}\left(\frac{4\pi^2}{l^2}-k^2\right)Ae^{kz}\sin\Phi+2a\frac{\epsilon}{\mu}\frac{2\mu}{l}kAe^{-kz}\cos\Phi$$

$$-\frac{\beta}{\mu}Ae^{-kz}\left[\sin\Phi-\frac{B}{A}\sin(\Phi-\Psi)\right].$$

If n is the refractive index of the medium for a disturbance of period T, the wave-length of which in free ether is λ, then $l=\frac{\lambda}{n}$; we next introduce a new constant κ such that $k=\kappa\frac{2\pi}{\lambda}$, and put $\zeta^2=a\frac{\epsilon}{\mu}$.

We have seen that $a\frac{\epsilon}{\mu}$ is the square of the velocity of propagation of the disturbance; $\therefore\ \zeta=\frac{\lambda}{T}$.

Dividing each term in the expression by $Ae^{-kz}\zeta^2$ or its equivalent, we get

$$-\frac{4\pi^2}{\lambda^2}\sin\Phi=-\left(\frac{4\pi^2n^2}{\lambda^2}-\kappa^2\frac{4\pi^2}{\lambda}\right)\sin\Phi+\frac{4\pi n\kappa}{\lambda}\frac{2\pi}{\lambda}\cos\Phi-\frac{4\pi^2}{\lambda^2}\sin\Phi$$

$$-\frac{\beta}{\mu\zeta^2}\left[\sin\Phi-\frac{B}{A}\sin(\Phi-\Psi)\right]-\frac{4\pi^2}{\lambda^2}(n^2-\kappa^2)\sin\Phi$$

$$+\frac{4\pi^2}{\lambda^2}2n\kappa\cos\Phi-\frac{\beta}{\mu\zeta^2}\left(\sin\Phi-\frac{B}{A}\sin(\Phi-\Psi)\right).$$

Writing now for $\sin\Phi(-\Psi)$ the equivalent $\sin\Phi\cos\Psi-\cos\Phi\sin\Psi$, we get

$$\left\{(n^2-\kappa^2-1)\frac{4\pi^2}{\lambda^2}+\frac{\beta}{\mu\zeta^2}\left(1-\frac{B}{A}\cos\Psi\right)\right\}\sin\Phi$$

$$\left(2n\kappa\frac{4\pi^2}{\lambda^2}-\frac{\beta}{\mu\zeta^2}\frac{B}{A}\sin\Psi\right)\cos\Phi=0.$$

This equation must hold for every value of Φ, which is only possible when the coefficients of $\sin\Phi$ and $\cos\Phi$ both equal zero; otherwise, with increasing Φ in the first quadrant, the first member would increase and the second diminish and the equation no longer hold.

The equation therefore falls into two, and dividing through by $\frac{4\pi}{\lambda^2}$,

we get (5) $$n^2-\kappa^2-1=-\frac{\beta\lambda^2}{4\pi^2\mu\zeta^2}+\frac{\beta\lambda^2}{4\pi^2\mu\zeta^2}\frac{B}{A}\cos\Psi,$$

and (6) $$2n\kappa=\frac{\beta\lambda^2}{4\pi^2\mu\zeta^2}\frac{B}{A}\sin\Psi.$$

These equations give us the refractive index and extinction coefficient κ in terms of the ratio of the vibration amplitudes of the atom and ether, and the phase between them. We have next to determine these two quantities. Substituting in equation (2) the values found for the first and second derivatives, and for $(\eta-y)$ and y, gives

$$-\frac{4\pi^2}{T^2}Be^{-kx}\sin(\Phi-\Psi)=\frac{\beta}{m}Be^{-kx}\left(\frac{A}{B}\sin\Phi-\sin(\Phi-\Psi)\right)$$

$$-\frac{\gamma}{m}Be^{-kx}\sin(\Phi-\Psi)-\frac{\delta}{m}Be^{-kx}\frac{2\pi}{T}\cos(\Phi-\Psi).$$

Divide through by Be^{-kx}, and writing as before for $\sin(\Phi-\Psi)$ its equivalent, we find that this equation also breaks up into two:

$$-\frac{4\pi^2}{T^2}\cos\Psi=\frac{\beta}{m}\frac{A}{B}-\frac{\beta+\gamma}{m}\cos\Psi-\frac{\delta}{m}\frac{2\pi}{T}\sin\Psi$$

and $$\frac{4\pi^2}{T^2}\sin\Psi=\frac{\beta+\gamma}{m}\sin\Psi-\frac{\delta}{m}\frac{2\pi}{T}\cos\Psi,$$

which gives us the amplitude ratio and the phase-difference in terms of the constants of equation (2).

Multiply the first equation by $\cos\Psi$, and the second by $\sin\Psi$, and substract the first from the second, and we have

$$\frac{4\pi^2}{T^2}=-\frac{\beta}{m}\frac{A}{B}\cos\Psi+\frac{\beta+\gamma}{m}.$$

Multiply the first by $\sin\Psi$ and the second by $\cos\Psi$ and add, and we have

$$0=\frac{\beta}{m}\frac{A}{B}\sin\Psi-\frac{\delta}{m}\frac{2\pi}{T}.$$

The first equation gives $$\frac{B}{A}=\frac{\frac{\beta}{m}\cos\Psi}{\frac{\beta+\gamma}{m}-\frac{4\pi^2}{T^2}}.$$

The second $$\frac{B}{A}=\frac{\frac{\beta}{m}\sin\Psi}{\frac{\delta}{m}\frac{2\pi}{T}}.$$

The quantity $B+\gamma$ in the first of these two equations is the sum of the forces acting on the atom when displaced unit distance, the ether being regarded as at rest (or $\eta=0$). The equation of motion of the atom under this condition is, disregarding friction, $\frac{d^2y}{dt^2}=-\frac{\beta+\gamma}{m}y$, and its period of oscillation we will designate by T_m.

$\frac{\beta+\gamma}{m}=\frac{4\pi^2}{T_m{}^2}=\frac{4\pi^2\zeta^2}{\lambda_m{}^2}$, where λ_m is the wave-length in ether of a disturbance of periodic time T_m, ζ being the velocity of propagation. (See last equations in section on Periodic Motion, chap. i.)

Substituting this value in the equations for the amplitude ratio gives us

(9) $$\frac{B}{A}=\frac{\beta}{4\pi^2\zeta^2m}\frac{\cos\Psi}{\frac{1}{\lambda_m{}^2}-\frac{1}{\lambda^2}}=\frac{\beta}{4\pi^2\zeta^2m}\frac{\lambda_m{}^2\lambda^2}{\lambda^2-\lambda_m{}^2}\cos\Psi.$$

(10) $$\frac{B}{A}=\frac{\beta}{4\pi^2\zeta^2m}\frac{\sin\Psi}{\frac{\delta}{2\pi m\zeta}\cdot\frac{1}{\lambda}},$$

in which we recognize the term $\frac{\gamma^2}{\lambda^2-\lambda_m{}^2}$, which we have seen accounts for anomalous dispersion.

We will now write α for $\frac{\delta}{2\pi m\zeta}\lambda_m{}^2$ (all the quantities being constants), and (10) becomes $\frac{B}{A}=\frac{\beta}{4\pi^2\zeta^2m}\frac{\lambda_m{}^2\lambda}{\alpha}\sin\Psi$, and dividing this by (9) we have $\frac{1}{\tan\Psi}=\frac{\lambda_m{}^2\lambda}{\alpha}\cdot\frac{\lambda^2-\lambda_m{}^2}{\lambda_m{}^2\lambda^2}$.

(11) $$\tan\Psi=\frac{\alpha\lambda}{\lambda^2-\lambda_m{}^2},$$

an expression for the phase-difference between the ether and the atom in terms of the constants and the wave-length of the light.

We shall presently require an expression for $\sin^2\Psi$ and $\cos^2\Psi$, which we can get from

$$\tan^2\Psi=\frac{\sin^2\Psi}{\cos^2\Psi}\text{ and }\sin^2\Psi=1-\cos^2\Psi;$$

$$\therefore\ \frac{1-\cos^2\Psi}{\cos^2\Psi}=\frac{\alpha^2\lambda^2}{(\lambda^2-\lambda_m{}^2)^2}$$

$$\alpha^2\lambda^2\cos^2\Psi=(\lambda^2-\lambda_m{}^2)^2-(\lambda^2-\lambda_m{}^2)^2\cos^2\Psi.$$

$$\cos^2\Psi = \frac{(\lambda^2-\lambda_m^2)^2}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2}, \tag{12}$$

$$\sin^2\Psi = \frac{\alpha^2\lambda^2}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2}, \text{ since } \sin^2\Psi+\cos^2\Psi=1, \tag{13}$$

expressions which we shall presently substitute in (9). We now multiply equation (9) by $\cos\Psi$ and (10) by $\sin\Psi$, and substitute for the $\cos^2\Psi$ and $\sin^2\Psi$ the expressions given by (12) and (13),

$$\frac{B}{A}\cos\Psi = \frac{\beta}{4\pi^2\zeta^2 m}\cdot\frac{\lambda_m^2\lambda^2(\lambda^2-\lambda_m^2)}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2},$$

$$\frac{B}{A}\sin\Psi = \frac{\beta}{4\pi^2\zeta^2 m}\cdot\frac{\alpha\lambda_m^2\lambda^3}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2}.$$

Substitute these values in equations (5) and (6),

$$n^2-\kappa^2-1 = -\frac{\beta}{4\pi^2\zeta^2\mu}\cdot\lambda^2+\frac{\beta^2}{(4\pi^2\zeta^2)^2\mu m}\cdot\frac{\lambda_m^2\lambda^4(\lambda^2-\lambda_m^2)}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2},$$

$$2n\kappa = \frac{\beta^2}{(4\pi^2\zeta^2)^2\mu m}\cdot\frac{\alpha\lambda_m^2\lambda^5}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2}.$$

Let $$P=\frac{\beta}{4\pi^2\zeta^2\mu} \text{ and } Q=\frac{\beta^2\lambda_m^2}{(4\pi^2\zeta^2)^2 m\mu},$$

which we may do since only constants are involved, and we have the final equations,

$$n^2-\kappa^2 = 1-P\lambda^2+Q\frac{\lambda^4(\lambda^2-\lambda_m^2)}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2}, \quad 2n\kappa = Q\frac{\alpha\lambda^5}{(\lambda^2-\lambda_m^2)^2+\alpha\lambda^2}.$$

In these equations n is the refractive index of the medium for wave-length λ, $2\pi\kappa$ is the fraction of the light lost by absorption in distance λ within the medium $\left(\text{since } k=\kappa\frac{2\pi}{\lambda}\right)$. (See eq. (2*a*).)

The dispersion formula gives us not only the relation between the refractive index and the wave-length, but also the relation between the absorption and the wave-length. If we have atoms with different free periods, we must place a summation sign before each of the terms in the right-hand member of the equation. They are then applicable to media with more than one absorption band.

$$n^2-\kappa^2-1 = -\Sigma P\lambda^2+\Sigma Q\frac{\lambda^4(\lambda^2-\lambda_m^2)}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2},$$

$$2n\kappa = \Sigma Q\frac{\alpha\lambda^5}{(\lambda^2-\lambda_m^2)^2+\alpha^2\lambda^2}.$$

We will now proceed to examine the equations in some detail, applying them to transparent and then to absorbing media.

Application to Transparent Media. — As we have seen, we must define transparent media as media with absorption bands outside of the visible spectrum. For such media, if we confine our attention to

values of λ in the freely transparent region, we can put $\kappa=0$ in the equation, since there is no absorption. If $\kappa=0$, then α must also equal zero, since if there is no absorption there is no friction.

The first equation now reduces to

$$n^2=1-P\lambda^2+Q\frac{\lambda^4}{\lambda^2-\lambda_m{}^2}.$$

This expression differs somewhat from the Sellmeier formula,

$$n^2=1+D\frac{\lambda^2}{\lambda^2-\lambda_m{}^2},$$

and from the electro-magnetic dispersion formula, which we shall presently develop, and which, for transparent media, reduces to a form identical with the Sellmeier equation. The term $-P$ reminds us of a similar term in the complete dispersion formula of Ketteler which we shall take up presently, which represents the effect of an absorption band in the infra-red. There is, however, no connection between the two, as is at once evident when we remember that the expression was developed for a *single* absorption band. The λ^4 in the numerator makes it seem at first sight as if the two formulæ must be quite different. This apparent difference disappears as soon as we calculate the values of the constants P and Q.

These constants cannot be determined from theory, but they can be calculated if we determine three values of the refractive index n_1, n_2, n_3 for wave-lengths λ_1, λ_2, λ_3.

Writing the equation $n^2-1=-P\lambda^2+Q\frac{\lambda^4}{\lambda^2-\lambda_m{}^2}$ in the form

$$\left(\frac{n^2-1}{\lambda^2}+P\right)\left(1-\frac{\lambda_m{}^2}{\lambda^2}\right)-Q=0,$$

and letting $\frac{n_1{}^2-1}{\lambda_1{}^2}=r$, $\frac{n_2{}^2-1}{\lambda_2{}^2}=s$, and $\frac{n_3{}^2-1}{\lambda_3{}^2}=t$,

we have three equations,

$$(r+P)\left(1-\frac{\lambda_m{}^2}{\lambda_1{}^2}\right)-Q=0,\quad (s+P)\left(1-\frac{\lambda_m{}^2}{\lambda_2{}^2}\right)-Q=0,$$

$$(t+P)\left(1-\frac{\lambda_m{}^2}{\lambda_3{}^2}\right)-Q=0,$$

from which the three constants P, Q, and λ_m are determined.

The constants of the Cauchy formula can be determined from two observed values of n and λ; thus

$$n_1=A+\frac{B}{\lambda_1{}^2},\qquad n_2=A+\frac{B}{\lambda_2{}^2}$$

$$B=\frac{n_2-n_1}{\frac{1}{\lambda_2{}^2}-\frac{1}{\lambda_1{}^2}},\qquad A=n_1\frac{\frac{1}{\lambda_2{}^2}-n_2\frac{1}{\lambda_1{}^2}}{\frac{1}{\lambda_2{}^2}-\frac{1}{\lambda_1{}^2}}$$

For water at 19°.5 C. the constants are as follows:

Helmholtz formula	Cauchy formula
$\lambda_m^2 = .87979$	$A = 1.324137$
$P = .865895$	$B = .30531$
$Q = .865767$	

These constants once determined, we can test the formula by calculating the values of n for other wave-lengths, and compare them with observed values. In the following table are given the values for water obtained by Wüllner, the values of λ being designated by the Fraunhofer lines. The differences between the observed and calculated values are given in the last two columns:

		Calculated values differ by	
λ	n observed	(Helmholtz)	(Cauchy)
B	1.33048	.00000	+.00012
C	1.33122	−.00005	−.00001
D	1.33307	.00000	−.00012
E	1.33527	+.00005	−.00015
F	1.33720	−.00000	−.00003
G	1.34063	+.00001	−.00002
H	1.34350	+.00004	+.00014

This shows the accuracy with which the two formulæ represent the dispersion.

The constants P and Q of the Helmholtz formula are seen to be very nearly equal, a fact which is true for other substances than water.

If we put $P=Q$, the formula $n^2 = 1 - P\lambda^2 + Q\frac{\lambda^4}{\lambda^2 - \lambda_m^2}$ reduces at once to $n^2 = 1 + Q\frac{\lambda^2\lambda_m^2}{\lambda^2 - \lambda_m^2}$, which is identical with the formula of Sellmeier if we write D for $Q\lambda_m$. We can, moreover, by a different transformation cause the Helmholtz formula to take the form of the Cauchy series.

If $\lambda_m < \lambda$, which must be the case, otherwise we should have values of n less than 1, we can write $Q\frac{\lambda^4}{\lambda^2 - \lambda_m^2} = \frac{Q\lambda^2}{\left(1 - \frac{\lambda_m^2}{\lambda^2}\right)}$.

By the binomial theorem, $\frac{1}{1 - \frac{\lambda_m^2}{\lambda^2}} = 1 + \frac{\lambda_m^2}{\lambda^2} + \frac{\lambda_m^4}{\lambda^4} + \ldots$.

Substituting this series in our dispersion formula gives us

$$n^2 = 1 - P\lambda^2 + Q\lambda^2 + Q\lambda_m^2 + Q\lambda_m^2\frac{\lambda_m^2}{\lambda^2} + Q\lambda_m^2\frac{\lambda_m^4}{\lambda^4},$$

or

$$n^2 = 1 + Q\lambda_m^2 - (P - Q)\lambda^2 + Q\lambda_m^2\frac{\lambda_m^2}{\lambda^2} + Q\lambda_m^2\frac{\lambda_m^4}{\lambda^4}.$$

Since $P=Q$, the term $-(P-Q)\lambda^2$ falls out, and we have the Cauchy formula, writing $A=1+Q\lambda_m{}^2$, $B=Q\lambda_m{}^4$, etc. This explains why the Cauchy formula is capable of representing a dispersion curve as well as it does, its agreement with the Helmholtz formula being accidental, of course.

Calculation of the Position of the Absorption Bands of Transparent Media. — The Helmholtz equation was modified by Ketteler, who obtained a formula containing a term, the square root of which represented the refractive index of the medium for infinitely long waves. This formula, being essentially identical with the electromagnetic dispersion formula which we shall develop later on, will for the present be assumed. It enables us to push our investigations over a wider range of wave-lengths than was possible with the other equation, and is commonly spoken of as the Ketteler-Helmholtz dispersion formula.

The two equations are

$$n^2-\kappa^2-n_\infty{}^2=\sum\frac{M(\lambda^2-\lambda_m{}^2)}{(\lambda^2-\lambda^2)+\alpha\lambda^2},$$

$$2n\kappa=\sum\frac{M_a\lambda}{(\lambda^2-\lambda_m{}^2)+\alpha^2\lambda^2}.$$

This formula has been verified over a wide range by the investigations of Paschen, Rubens, and others, who have measured the dispersion of various substances in the infra-red, visible, and ultra-violet regions, and compared the observed values with those calculated from the formula, finding most perfect agreement.

By measuring the dispersion in the visible spectrum and determining the constants, it is possible to calculate the positions of the absorption bands in the infra-red and ultra-violet, even if we cannot observe them.

In this way bands of absorption have been definitely located in the infra-red region of various media, and subsequently found by experiment.

We will now examine in some detail the application of the Ketteler-Helmholtz formula to the study of the optical properties of quartz.

For transparent media the equation becomes

$$n^2=n_\infty{}^2+\sum\frac{M_m}{\lambda^2-\lambda_m{}^2}.$$

This formula must hold over the entire spectrum with the exception of the small gaps where the absorption bands lie.

It was found sufficient to take but two terms of the member $\sum\frac{M_m}{\lambda^2-\lambda_m{}^2}$ in all cases where in one term $\lambda_m{}^2$ was considerably smaller than λ^2, and in the other considerably larger; in other words, to consider but two absorption bands, one in the ultra-violet, the other in the infra-red.

The expression $$n^2 = n_\infty{}^2 + \overset{\text{Ultra-violet}}{\frac{M_1}{\lambda^2 - \lambda_1{}^2}} + \overset{\text{Infra-red}}{\frac{M_2}{\lambda^2 - \lambda_2{}^2}}$$

can be written in the form

$$n^2 = n_\infty{}^2 + \frac{M_1}{\lambda^2 - \lambda_1{}^2} - \left(\frac{M_2}{\lambda_2{}^2} + \frac{M_2\lambda^2}{\lambda_2{}^4} + \frac{M_2\lambda^4}{\lambda_2{}^6}\right),$$

the expansion being made by change of sign and division. The members in the parenthesis form a rapidly convergent series, and for very diathermous substances, in which the absorption band is far out in the infra-red (*i.e.* λ_2 very large in comparison to λ), a sufficiently good approximation is obtained when we write

$$n^2 = \alpha^2 + \frac{M_1}{\lambda^2 - \lambda_1{}^2} - k\lambda^2, \text{ in which } \alpha = n_\infty{}^2 + \frac{M_2}{\lambda_2{}^2},\ k = \frac{M_2}{\lambda_2{}^4}.$$

If the diathermancy is less, or if we are working nearer to the absorption band, we add a second member of the series and get

$$n^2 = \alpha^2 + \frac{M_1}{\lambda^2 - \lambda_1{}^2} - k\lambda^2 - k_1\lambda^4.$$

Now $k = \frac{M_2}{\lambda_2{}^4}$ and $k_1 = \frac{M_2}{\lambda_2{}^6}$; $\therefore \frac{k}{k^1} = \lambda_2{}^2$, and by determining the constants k and k_1, we have at once determined the position of the infra-red absorption band.

Quartz is a substance exceedingly transparent to the ultra-violet, visible, and infra-red, and is consequently well adapted for a verification of the dispersion formula over a wide range of wave-lengths.

The dispersion is measured in the visible spectrum with the spectrometer, in the ultra-violet by means of photography, and in the infra-red either with the thermopile, bolometer, or radiometer. In these instruments the heating effect of the rays is the means by which they are detected, the thermopile furnishing a current, and deflecting a galvanometer needle when the rays strike it, while the bolometer gives evidence of the rays by the change in the resistance of a fine strip of blackened platinum due to the heating. The radiometer, which was brought to a high degree of perfection by E. F. Nichols, has the advantage of being uninfluenced by magnetic disturbances, but is not quite as convenient to work with. It consists of a double vane of mica suspended in a vacuum by a quartz fibre. The rays, falling upon one of the vanes, cause a deflection, which is measured by a mirror and scale. A fuller description of this instrument will be found in the chapter on Radiation.

Rubens employed the bolometer in his earlier investigations. The arrangement of his apparatus is shown in Fig. 269. The radiation from a zirconia burner Z was concentrated by a rock-salt lens, which is very transparent to the long waves, on the slit of a reflecting spectrometer, dispersed by a wire diffraction grating, and focussed on the slit of a second reflecting spectrometer, on the table of which the quartz prism was mounted. By means of the grating

spectrometer, heat radiation of approximately a single wave-length was thrown on the slit of the spectrometer, and the deviation of these rays by the quartz prism was determined by means of the bolometer, which took the place of the cross-hairs in the eye-piece of the ordinary spectrometer. With this apparatus Rubens investigated the dispersion of fluorite to 8.9 μ, and of quartz to 4.26 μ. Taking Sarasin's measurements in the ultra-violet and visible spectrum, and his own in the infra-red, Rubens compiled the following table. The differences between the observed and calculated values are given in the δ columns. The values in column 1 were calculated from the formula, making use of one member of the convergent series. The calculated values agree with the observed, from the extreme ultra-violet up to nearly 2 μ, beyond which a rapidly increasing difference is found, due to the fact that we are getting so near the absorption band that it is no longer sufficient to take two members of the series. Three members of the series were used in compiling the values of column 2, and it is seen that the agreement is perfect out to 4.26 μ, which was as far as it was possible to obtain measurements at the time. The constants of the equation were determined as follows:

$$a^2 = 2.35681, \quad k = .01113, \quad k_1 = .0001023,$$
$$M_1 = .010654, \quad \lambda_1^2 = .010627.$$

		I			II	
	λ	m. obs.	m. cal.	δ I	m. cal.	δ II
	.198	1.65070	1.65077	+.7	1.65077	+.7
	.274	1.5875	1.58757	+.7	1.68757	+.7
	.358	1.56400	1.56395	−.5	1.56395	−.5
Visible	.434	1.553869	1.5539	+.3	1.5539	+.3
Spectrum	.534	1.54663	1.5466	+.3	1.5466	+.3
	.656	1.541807	1.5419	+.9	1.5419	+.9
	1.160	1.5329	1.5330	+.1	1.5329	0
	1.617	1.5271	1.5271	0	1.5269	− 2
	1.969	1.5216	1.5221	+ 5	1.5216	0
	2.32	1.5156	1.5162	+ 6	1.5152	− 4
	2.60	1.5099	1.5111	+ 12	1.5096	− 3
	2.86	1.5039	1.5057	+ 18	1.5034	− 5
	3.06	1.4985	1.5013	+ 28	1.4983	− 2
	3.21	1.4942	1.4980	38	1.4944	+ 2
	3.42	1.4877	1.4926	49	1.4879	+ 2
	3.67	1.4790	1.4861	63	1.4799	+ 1
	3.84	1.4739	1.4812	73	1.4738	− 1
	4.01	1.4678	1.4761	83	1.4683	+ 5
	4.15	1.4619	1.4717	98	1.4616	− 3
	4.26	1.4567	1.4682	115	1.4570	+ 3

Calculation of the position of the infra-red absorption band from k and k_1 showed it to be at 10.4 μ.

The approximate position of the ultra-violet band is at .103 μ, the

two being $6\frac{1}{2}$ octaves apart. Of this region $4\frac{1}{2}$ octaves could be measured, and the agreement between the observed and calculated values within this range was found to be perfect.

Radiometric Observations on Quartz. — The behavior of quartz with respect to very long heat waves was investigated by Nichols with a radiometer.

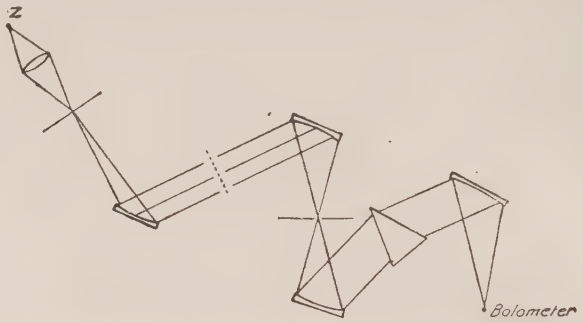


Fig. 269.

He found a strong absorption band between 8 and 9 μ which agreed fairly well with the position predicted by Rubens from the constants of the formula. If, however, we use this experimentally determined value of λ_m for calculating the refractive indices, we no longer find agreement between the two sets.

The reason of this appeared upon the completion of a subsequent piece of experimental work by Rubens and Nichols.

Nichols had found that for the wave-length corresponding to the centre of the absorption band (8.5 μ), quartz reflected almost as powerfully as a metal. This, as we shall see in the chapter on Absorption, is universally true of substances showing powerful selective absorption. The phenomenon is commonly spoken of as "Surface color," the aniline dyes exhibiting it to a very marked degree. Quartz was found to reflect about 80 % of the incident radiation of wave-length 8.5 μ, and only about 2 % of the radiation at 4 μ. The transmission and reflection curves are shown in Fig. 270.

On this property of quartz Rubens and Nichols based the very beautiful method of isolating heat waves of great wave-length. In brief, the method consisted in reflecting the radiant energy, coming from a zirconia button heated in the oxyhydrogen flame, in succession from several polished surfaces of quartz. It is apparent that, if the quartz reflects like a metal for wave-length 8.5 μ and like a transparent medium for all other wave-lengths, the radiant energy after a sufficient number of reflections will contain practically nothing but the metallically reflected waves. The energy after five reflections was examined with a wire diffraction grating, and found to consist principally of waves of length 8.5 μ. In addition to these waves, the grating showed that wave-length 20 μ was also present in the reflected energy, conclusive proof that there was a second absorption band at 20 μ. Taking 8.5 μ and 20 μ as values

for λ_2 and λ_3 in the formula, adding a term $\frac{M_3}{\lambda^2 - \lambda_3^2}$ for the new absorption band, it was found that perfect agreement was again obtained between the observed and calculated values.

The error in the calculated position of the band (10.4 against 8.5 μ) was due to the fact that the second band had been neglected. This shows us how the presence of an absorption band beyond the reach of observation may be detected. More will be said of this method further on in the chapter.

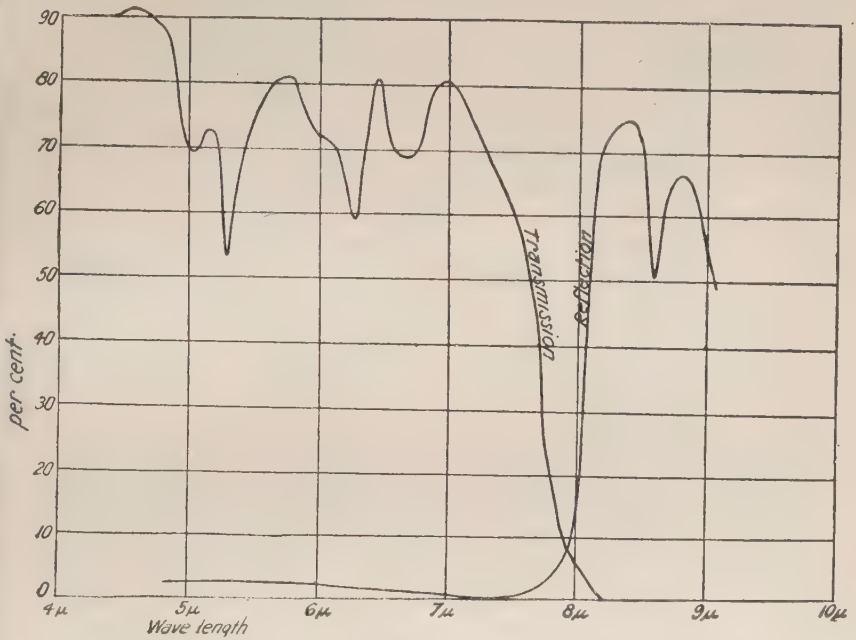


FIG. 270.

A further remarkable verification of the formula has been found by Rubens and Aschkinass (*Ann. der Physik*, 67, p. 459, 1899), who have measured the refractive index of quartz for heat-waves of length 56 μ. These waves are isolated by means of repeated reflections from surfaces of fluorite. Inasmuch as they lie on the side of the infra-red bands towards the region of longer wave-length, we should expect a higher value of the refractive index than in the visible spectrum. Calculating the refractive index from the dispersion formula, we find the extraordinary value 2.20, higher even than for the ultra-violet. The value found experimentally by Rubens was 2.18, a remarkably close agreement.

Dispersion and Absorption of Various Substances in the Infra-red. — A knowledge of the dispersion and absorption of transparent substances in the infra-red region is of the greatest importance in the design of apparatus for investigations in this region.

Figure 271 represents according to Rubens and Trowbridge the behavior of fluorite, rock-salt, and sylvite, the substances chiefly used for the manufacture of prisms and plates. I have added the quartz dispersion curve for the sake of comparison, but have omitted the transmission curve. This curve has been determined

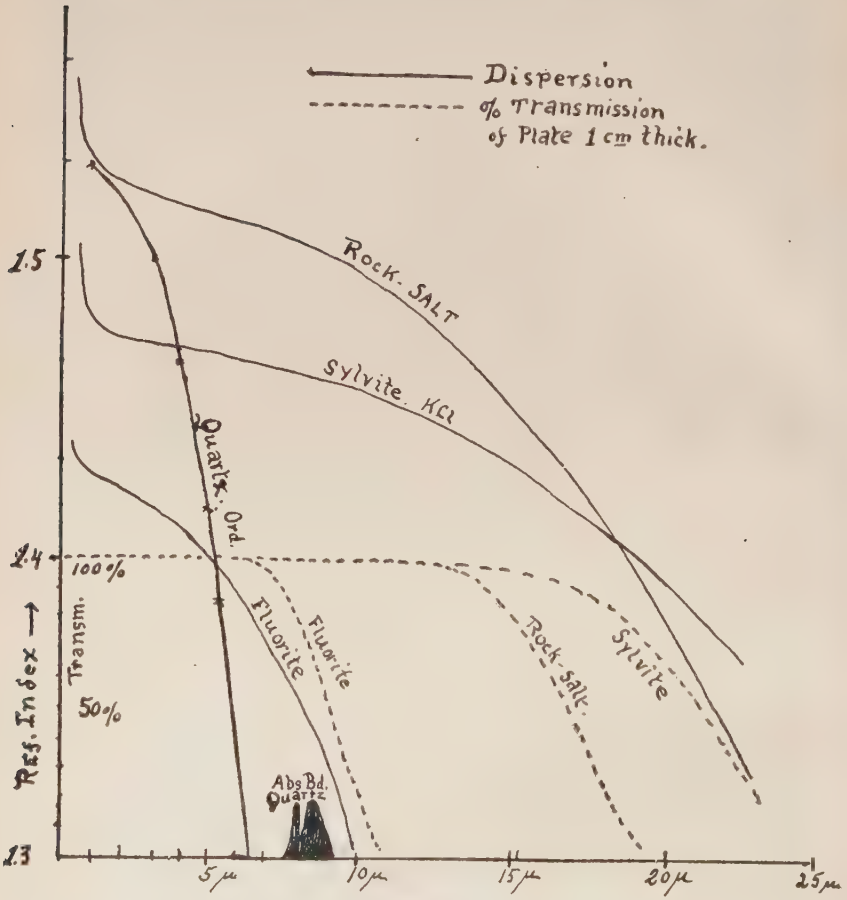


Fig. 271.

by the prism method only as far as 4.5 μ. The rest of the curve is calculated from the reflecting powers measured by Nichols. The steepness of the curve is due to the presence of the band of metallic absorption at 8.5 μ, which is indicated in the figure. Trowbridge observed that powdered quartz showed a minimum of reflecting power at 4.5 μ, indicating a weak absorption at this point, which makes itself manifest as a "body-color" effect (see next chapter). This band is not strong enough to have any influence on the dispersion. There is probably a partial though slight return of trans-

parency between 5 μ and 8.5 μ, though it has never been detected in the case of thin plates.

It is very instructive to compare these dispersion curves with the transmission curves and note the change of curvature as the region of strong absorption is approached. As can be seen from the figure, fluorite makes much the best material for prisms, if we only wish to work out as far as 10 μ, on account of its greater dispersing power. Rock-salt comes next, and sylvite last, though with it we can go out a little further.

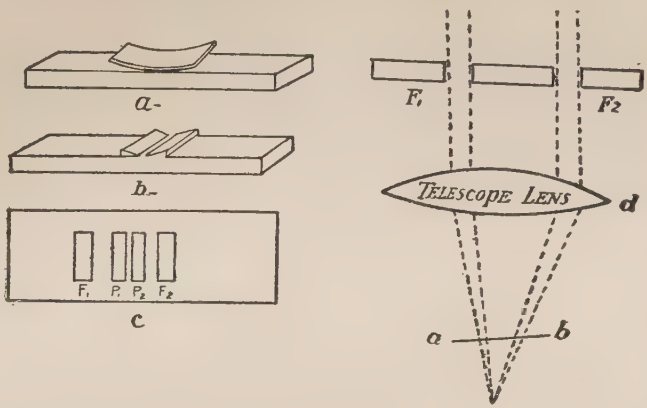


Fig. 272.

Application to Absorbing Media and Selective Dispersion. — Pflüger (*Ann. der Phys.*, 65, p. 113, 1898) was the first to undertake a careful series of observations of n and κ in the case of strongly absorbing substances, with a view of testing the dispersion formula near to and within the absorption band. He employed solid bi-prisms of small angle, obtained by evaporating an alcoholic solution of an aniline dye between a glass plate and a curved segment of a glass tube, as shown in Fig. 272*a*. On removing the segment of the tube, two prisms of small angle remained on the plate, the refracting edges being turned towards the centre, as shown at *b*. The refractive indices could be obtained by means of these prisms even at the centre of the absorption band, though in this case the image of the slit of the spectrometer was greatly broadened by diffraction, since only the extreme edges of the prism transmitted the light. The values of the constant κ for the different values of λ were determined by means of the spectro-photometer, thin films of different thicknesses being used. The general form of the curves found in the case of cyanine is shown in Fig. 273.

In determining the deviations produced by these prisms of small angle, it was necessary to take certain precautions, brief mention of which will be of use to any engaged in similar observations. The glass plate, with its bi-prism, was covered by a black paper screen provided with four apertures, F_1 and F_2 exposing clear glass, P_1 and P_2 the two opposed prisms, Fig. 272*c*. In measuring the deviations,

the plate is mounted on the table of a spectrometer and the slit illuminated with intense monochromatic light, furnished by a spectroscope. The windows F_1F_2 and P_1 are covered, and the broadened image of the slit brought into coincidence with the cross-hair. It is first necessary, however, to be sure that the eye-piece is focussed exactly, otherwise false readings will be obtained, as can be

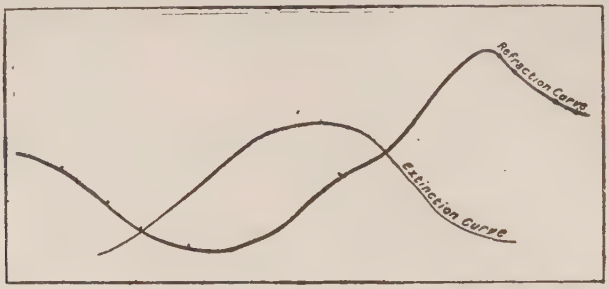


FIG. 273.

seen by referring to Fig. 272*d*. Suppose the eye-piece to be focussed on the plane a–b instead of on the focal point. On account of the narrowness of the beams coming through F_1 and F_2 the images will be about as sharp here as at the true focus, but we shall see *two* of them, since the ray bundles are separated, and if we screened the windows in succession, we should obtain two positions of the image, though no deviation has resulted from passage of the rays through the windows. It is clear that the same error will be introduced, though in a less degree, for the windows P_1 and P_2, which cover the prisms. We should therefore place the eye-piece in such a position that no change in the position of the image can be detected when the apertures F_1 and F_2 are covered in succession. This will only be the case when it is correctly focussed.

The prism angles are determined by reflecting the light from their surfaces.

To determine the extinction coefficient, two films of different thickness are deposited on glass, and the difference in thickness determined by laying a second glass plate over them, and determining the thickness of the air films above them by an interference method. The absorption of each film for light of different wavelengths is then determined with a spectro-photometer. The difference of absorption obtained in this way is the true absorption of a film of thickness equal to the difference of thickness: the loss by reflection, being the same in each case, falls out. If we are working with absorbing liquids or solutions, the difference in thickness can be obtained by putting a small plate of clear glass in the cell containing the liquid.

The dispersion and absorption were found to be well represented by the Ketteler-Helmholtz formulæ in the forms

$$n^2-\kappa^2-1=\sum\frac{D\lambda^2(\lambda^2-\lambda_m{}^2)}{(\lambda^2-\lambda_m{}^2)^2+g^2\lambda^2},\qquad 2n\kappa=\sum\frac{Dg\lambda^3}{(\lambda^2-\lambda_m{}^2)^2+g^2\lambda^2}.$$

The dispersion of cyanine was also investigated by Wood and Magnusson (*Phil. Mag.*, January, 1901) by means of cyanine prisms of larger angles than the ones employed by Pflüger, as well as by means of observations of the shift of the fringes formed by the Michelson interferometer, caused by the introduction of a thin film of solid cyanine into one of the optical paths. The continuity of the dispersion curve through the absorption band was well brought out in the photographs obtained with the interferometer.

The most complete investigation of the dispersion and absorption of solutions of aniline dyes is that made by Stöckl.[1] The refractive indices were determined by measuring with the spectrometer the deviations produced by fluid prisms of small angle. In work of this kind, when observations are required near and within the absorption band, prisms of extremely small angle are necessary, and even then light is only transmitted by a narrow strip along the refracting edge. This limitation of the transmitted beam interferes seriously with the resolving power of the instrument, the image of the slit being broadened by diffraction. Stöckl used a prism made of a pair of glass plates, the angle between which could be varied to suit the conditions. When observing at a distance from the absorption band, larger angles can be used which naturally give more accurate results.

Determination of the Extinction Coefficient. — The determination of κ is not as easy as the determination of n. By our definition of the constant, the amplitude of the light-wave of length λ decreases in the ratio $1 : e^{-2\pi\kappa}$ in traversing a layer of thickness λ. If the thickness of the layer is d, the ratio expressing the decrease of amplitude is $1 : e^{-2\pi\kappa\frac{d}{\lambda}}$.

Now the intensity of the light is measured by the square of the amplitude, and the intensity therefore decreases in the ratio $1 : e^{-4\pi\kappa\frac{d}{\lambda}}$. To avoid the error due to reflection from the surfaces of the layer, or the glass plates between which it is held, it is customary to employ layers of different thicknesses. Let these thicknesses be d_1 and d_2, and the intensity of the incident light be J_0. The intensities after traversing the two layers will be $J_1 = J_0 e^{-4\pi\kappa\frac{d_1}{\lambda}}$ and $J_2 = J_0 e^{-4\kappa\pi\frac{d_1}{\lambda}}$.

$$\log J_1 = \log J_0 - 4\pi\kappa\frac{d_1}{\lambda}\log e,$$

$$\log J_2 = \log J_0 - 4\pi\kappa\frac{d_2}{\lambda}\log e,$$

$$\log J_1 - \log J_2 = \frac{4\pi\kappa}{\lambda}\log e(d_2 - d_1).$$

$$\log\frac{J_1}{J_2} = \frac{4\kappa\pi}{\lambda}(d_2 - d_1)\log e.$$

From this equation we can calculate the extinction coefficient κ, by measuring the intensities of the transmitted beams with a

[1] Stöckl, Dissertation, Munich, 1900.

spectro-photometer. The layers of different thickness are best obtained by pouring the liquid into a glass cell containing a glass plate which reduces the thickness of the layer along the bottom of the cell. The spectro-photometer best adapted to the purpose is the instrument designed by Vierordt. It is provided with a double slit, before which the cell is placed in such a position that the dividing line between the two layers coincides with the junction of the two slits. The intensities of the two spectra, which lie one above the other, can be made equal for any value of λ by altering the widths of the slits. Equal illumination is obtained when the slit widths b_1 and b_2 are inversely proportional to the intensities of the illuminating beams, that is, when

$$\frac{J_1}{J_2} = \frac{b_2}{b_1}.$$

Our equation now takes the form

$$\log \frac{b_2}{b_1} = \frac{4\pi\kappa}{\lambda}(d_2 - d_1) \log e,$$

$$\kappa = \frac{1}{4\pi(d_2 - d_1) \log e} \lambda \log \frac{b_2}{b_1}.$$

The absorption coefficient which we have called k is given by

$$k = \frac{1}{2(d_2 - d_1) \log e} \log \frac{b_2}{b_1}, \text{ since } \kappa = \frac{k\lambda}{2\pi}.$$

Further particulars regarding the measurements will be found in Stöckl's paper. His measurements were made with solutions of varying concentrations, and the results plotted as curves. The curve for cyanine in alcohol is shown in Fig. 274. The dispersion curve for pure alcohol is represented by an unbroken line, the solution dispersion curves by dotted lines. It will be seen that on the red side of the absorption band, the cyanine increases the refractivity of the alcohol, while on the blue side it decreases it up to a certain point, and then increases it again. The family of curves pass through two common points, which are the intersections of the curve for alcohol with that of solid cyanine, the refractive indices of which have been measured by Pflüger and by Wood.

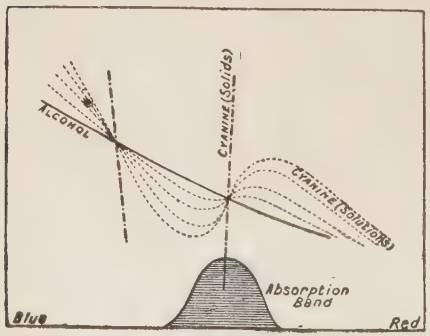


FIG. 274.

The curves for κ are given below the dispersion curves. The wavelength for which the maximum absorption occurs lies further towards the red end of the spectrum than in the case of the solid dye.

Stöckl calculated the position of the maximum of the curve for solid cyanine from these observations of n and κ by employing the formula for $2n\kappa$, and obtained a value which agreed closely with the value observed by Pflüger.

Dispersion of Nitroso-dimethyl Aniline. — The remarkable optical properties of this substance have been investigated by the author (*Phil. Mag.*, 1903). It is of especial interest in that, while fairly transparent for wave-lengths comprised between the red and blue, it has its band of metallic absorption in the violet. This circumstance gives it an enormous dispersive power in the yellow and green, a prism of the substance yielding a spectrum about fifteen times as long as the spectrum given by a glass prism of the same angle.

The substance melts at 85° C. and can be formed into prisms between small strips of thin plate glass. The strips should be about two centimeters long, and are best fastened together with one of the small clamps used with rubber tubing. It is best to melt the material on the end of one of the strips, the other being warmed over the same flame, and then clamp the two together with a piece of a match between the ends, to give the required prismatic form. A candle flame viewed through the prism is spread out into a most remarkable spectrum. It is instructive to have a prism of the same angle made of Canada balsam or some such substance pressed out between two similar glass strips.

The refractive indices were measured with a spectrometer, the slit of which was illuminated with approximately monochromatic light obtained from a spectroscope furnished with a narrow slit in place of the eye-piece. In the more transparent region a prism of 8° 7′ was used, while in the vicinity of the absorption band it was necessary to employ one of less than one degree, on account of the opacity of the substance. The values found are given in the following table:

Prism Angle 8° 7′				Prism Angle 53′	
λ	n	λ	n	λ	n
508	2.025	636	1.647	497	2.140
516	1.985	647	1.758	500	2.114
525	1.945	659	1.750	506	2.074
536	1.909	669	1.743	513	2.020
546	1.879			577	1.826
557	1.857			647	1.754
569	1.834			669	1.743
584	1.815			696	1.723
602	1.796			713	1.718
611	1.783			730	1.713
620	1.778			749	1.709
627	1.769			763	1.697

The curve for carbon bisulphide, which has the strongest dispersion of any transparent substance in common use, is given for the sake of comparison (Fig. 275). The extraordinary dispersion

of the nitroso in the visible region is at once apparent. Carbon bisulphide absorbs strongly below wave-length 36, and its dispersion can only be measured in the ultra-violet by employing very acute prisms.

Fig. 275.

The substance was found to become transparent again on the ultra-violet side of the absorption band, and measurements were made in this region by means of photography. A small quartz spectrograph was used, the nitroso prism being mounted with its refracting edge horizontal, immediately behind the quartz prism of the instrument. This device will be at once recognized as the method of crossed prisms.

The undeviated spectrum was photographed by the rays which passed below the edge of the small prism, and by measuring the distances between it and the deviated portion, it was possible to calculate the refractive index. One of these photographs is reproduced in Fig. 275. It will be seen that the deviation is a maximum in the green at the edge of the absorption band, while on the other side of the band the deviation is zero, *i.e.* the refractive index equals unity for this wave-length. The continuity of the dispersion curve can be traced through the absorption band, though the deviated spectrum is so broadened by diffraction that accurate measurements cannot be made in this region. The vertical lines are the bright spectrum lines of the cadmium spark which served as a source of light. The complete dispersion curve is shown graphically in Fig. 276, the position of the absorption bands being recorded as well.

Optical Properties of Stibnite. — Stibnite, or natural sulphide of antimony, is a substance which exhibits very remarkable optical properties. It has a metallic lustre resembling that of freshly cut lead, and is opaque to visible light. Nevertheless, as Coblentz has shown, it is extremely transparent to the infra-red radiations, a slab 0.4 mm. in thickness transmitting 46 % of the energy out to 12 μ, while one 4.9 mm. thick allowed 43 % to pass. This shows us that the absorption is very nearly nil, and that the rest of the energy is reflected at the surfaces. The reflecting power of the substance is about 37 %, and results from the very high value of the refractive index, which is 5.53 for blue light and 4.69 for red, according to the calculations of E. C. Müller, made from observations of the constants of elliptical polarization.

Measurements made by the author with a Michelson interferometer, and by observing Newton's rings, with a film of stibnite

obtained by cathode deposition, give the value 3 for yellow light, the lower value doubtless resulting from a slightly spongy condition of the deposit, which exhibited a lower reflecting power than the cleavage surfaces of the mineral.

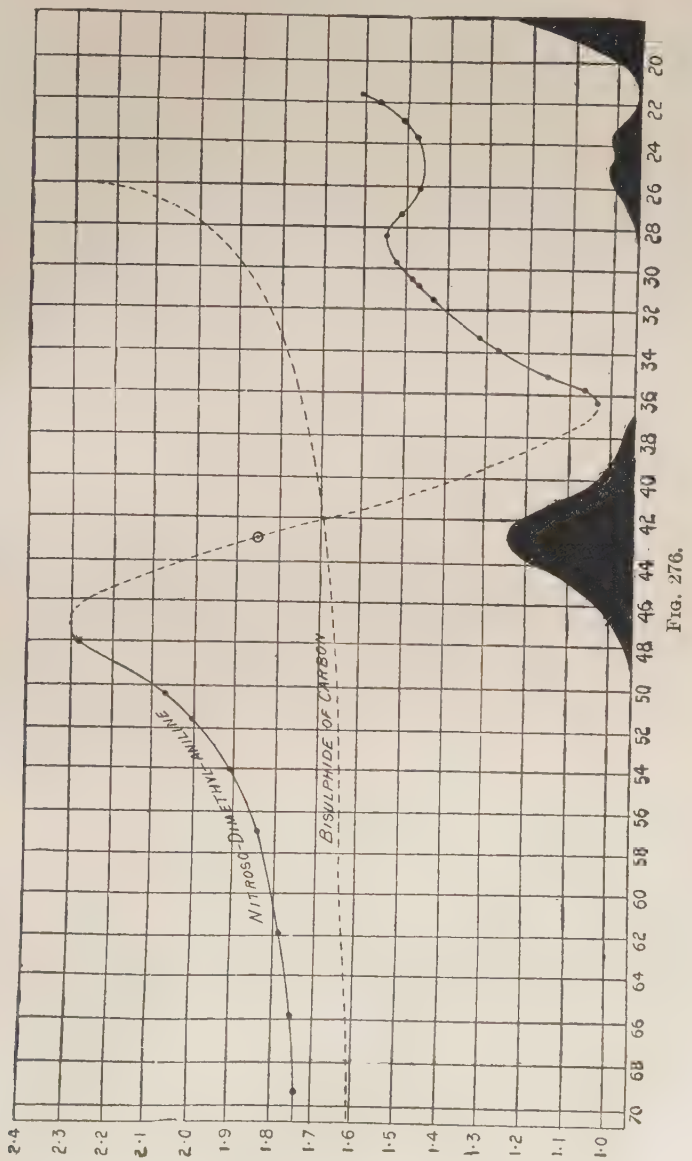


Fig. 276.

Electro-magnetic Theory of Dispersion. — In the previous treatment we have considered the atoms set in motion by waves in a

medium having the properties of an elastic solid. We will now take up the development of the electro-magnetic dispersion formula, which is not unlike the one which we have already examined. Instead of assuming the atoms to be the oscillating particles, we will consider that the medium contains charged electrons. These may or may not be identical with the electrons which we consider as the carriers of electricity in metallic conduction. They are at all events not the same as the ions of electrolysis. It is necessary to consider them charged either positively or negatively, if they are to be set in motion by the rapidly reversing electric force which constitutes light. We must also think of them as having fixed positions of equilibrium with reference to the atoms to which they belong. The application of a steady electric force will displace them, but this displacement once produced, there will be no further movement until the force ceases, when they will resume their former positions. It is clear that these electrons are incapable of conduction, *i.e.* there will be no current produced by the application of a constant difference of potential, for they are bound to the atoms, and their displacement by an electric force is accompanied with something that corresponds to a force of restitution, which we shall consider as proportional to the displacement, as in the Helmholtz treatment.

As an introduction to the present treatment, the reader is advised to glance over the derivation of the expression for the dielectric constant in absorbing media, in the chapter on the Optical Properties of Metals. The electrons behave differently, of course, in the case of metals, but we shall require one or two of the expressions representing their effects in the present chapter.

If the electron is displaced by a force which instantly ceases, it will be drawn back by the elastic force of restitution, and vibrate with a definite period, depending on its mass, charge, and the force of restitution. Forces akin to friction may damp this vibration and eventually bring the electron to rest. A damping due to radiation may also occur; though this is comparatively small, and will not be considered in the present chapter. The electron is analogous in every respect to the Sellmeier vibrator, and, as we shall see, will cause the medium to absorb radiant energy of a period similar to its own. This period depends on the chemical constitution of the medium; in other words, upon the arrangement of the atoms, which makes it appear doubtful if we can consider the ion, at least in some cases, as a minute part of the atom. For example, the aniline dyes, complicated organic compounds, with powerful selective absorption, are made up of atoms which, when existing as elements, or when entering into the composition of other compounds, do not show this absorption at all. A certain knowledge of the nature of the electron is not necessary for the construction of a satisfactory theory of dispersion. We have merely to assume that an electrical vibration of some sort can be set up within the molecule, it being quite immaterial whether this consists of the to and fro excursion of a group of electrons, or of a single electron within an atom.

Calling m the mass of the electron, e its charge, and ξ its displace-

ment along the x axis, we have its motion represented by an equation similar to the fundamental equation of the Helmholtz treatment,

$$m\frac{\partial^2\xi}{\partial t^2}=eX-\frac{4\pi e^2}{\theta}\xi-re^2\frac{\partial\xi}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation eX is the force applied by the wave, θ may be defined as the reciprocal of the elastic force which urges the electron back, when displaced unit distance. If the electron were in equilibrium under the action of steady force X, we should have $e\xi=\frac{\theta}{4\pi}X$. In the case of metals, where the electrons are free to move continuously under the action of a steady force, we have $\theta=\infty$.

The last term of the equation represents the action of some damping factor analogous to friction, which is proportional to the velocity $\frac{\partial\xi}{\partial t}$ and a constant r. In the two last terms e^2 is written to show that the direction of the force is independent of the sign of the charge.

The current along the x axis will consist of two parts, a displacement current in the ether

$$(j_x)_0=\frac{1}{4\pi}\frac{\partial X}{\partial t},$$

and a convection current due to the motion of the electrons, proportional to the number in unit volume and their velocity,

$$(j_x)_1=eN\frac{\partial\xi}{\partial t},$$

in which N is the number of electrons in unit volume and $\frac{\partial\xi}{\partial t}$ their velocity. The total current will be

$$j_x=(j_x)_0+(j_x)_1=\frac{1}{4\pi}\frac{\partial X}{\partial t}+\frac{\partial}{\partial t}(eN\xi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For periodic disturbances we have $\xi=Ae^{\frac{t}{i\tau}}$, in which $\tau=\frac{T}{2\pi}$, T being the periodic time of the disturbance which enters the medium, and ξ the real part of the complex quantity to which it is equated. The calculations can be much simplified by the introduction of complex quantities, and we can return at the end to the physical conception, *i.e.* the real part of the complex. Differentiating we have

$$\frac{\partial\xi}{\partial t}=\frac{i}{\tau}\xi, \quad \frac{\partial^2\xi}{\partial t^2}=-\frac{1}{\tau^2}\xi.$$

Multiplying (1) through by $\frac{\theta}{e4\pi}$, and substituting the above

derivatives, gives

$$e\ddot{\xi}+\frac{re\dot{i}\dot{\xi}\theta}{\tau 4\pi}-\frac{m\ddot{\xi}\theta}{e4\pi\tau^2}=\frac{\theta}{4\pi}X,$$

$$e\ddot{\xi}\left(1+\frac{i}{\tau}\frac{r\theta}{4\pi}-\frac{1}{\tau^2}\frac{m\theta}{4\pi e^2}\right)=\frac{\theta}{4\pi}X.$$

Writing $$a=\frac{r\theta}{4\pi},\quad b=\frac{m\theta}{4\pi e^2},$$

gives us $$e\ddot{\xi}=\frac{1}{4\pi}X\frac{\theta}{1+\frac{i}{\tau}a-\frac{b}{\tau^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we have a number of ions with different values of r and θ, the expression for the current (eq. (2)) takes the form, by substituting for $\frac{Xi}{\tau}$ its equivalent $\frac{\partial X}{\partial t}$ (see Chapter on Optics of Metals), eq. (2),

$$j_x=\frac{1}{4\pi}\frac{\partial X}{\partial t}\left(1+\sum\frac{\theta N}{1+\frac{i}{\tau}a-\frac{b}{\tau^2}}\right), \quad . \quad . \quad . \quad . \quad (4)$$

an equation similar to the one which we derived for an insulator except that the dielectric constant ϵ is, in this case, represented by the complex quantity in the parenthesis.

It will be seen that the complex dielectric constant is dependent on the period $T=2\pi\tau$ of the light. The relation which this value of the dielectric constant bears to the value ϵ, determined electrically, can be found as follows.

In the electrical determinations we make use either of very long periods (method of electrical waves) or static charges, for either of which we can write $\tau=\infty$. Substituting this value of τ in the expression for the complex dielectric constant, which we may call the optical dielectric constant ϵ', gives us

$$\epsilon=\epsilon'_\infty=1+\Sigma\theta N.$$

We can define θN as the dielectric constant of one of the groups of electrons.

The constant b is identified with the natural free-period of the electron, the friction coefficient a being neglected. For this case $X=0$ and $a=r=0$, under which conditions equation (1) becomes

$$m\frac{\partial^2\xi}{\delta t^2}=-\frac{4\pi e^2}{\theta}\xi$$

$$-\frac{m\theta}{4\pi e^2}\frac{1}{\tau_m{}^2}\xi=-\xi,\ \text{since}\ \frac{\partial^2\xi}{\partial t^2}=-\frac{1}{\tau_m{}^2}\xi,$$

$$b=\tau_m{}^2\ \text{and}\ \tau_m=\frac{T_m}{2\pi},$$

T_m being the period of the electron.

In the Chapter on the Optical Properties of Metals we shall see that a complex dielectric constant means absorption, and if we substitute the value which we have obtained for ϵ', namely, $1+\sum\frac{\theta N}{1+\frac{ia}{\tau}-\frac{b}{\tau^2}}$ we obtain an expression connecting the refractive index, and the index of absorption, with the period of the light vibration and the natural free period.

The equation of wave-motion can be applied here if we substitute for ϵ the complex value ϵ',

$$\frac{\epsilon'}{c^2}\frac{\partial^2 X}{\partial t^2}=\frac{\partial^2 X}{\partial z^2}. \quad \ldots\ldots\ldots \quad (5)$$

To integrate this, we write

$$X=Ae^{i\frac{2\pi}{T}(t-mz)}. \quad \ldots\ldots\ldots \quad (6)$$

X has here the significance of the real part of the imaginary. The equation gives us the value of the electric force parallel to the x axis of a plane-polarized disturbance travelling along the z axis, at any time t, and at any point on z, m being of course the reciprocal of the velocity with which the disturbance travels in the medium.

Differentiating (6), $\frac{\partial^2 X}{\partial t^2}=-\frac{4\pi^2}{T}X$, $\frac{\partial^2 X}{\partial z^2}=-\frac{4\pi^2 m^2}{T^2}X$,

and substituting in (5),

$$\frac{\epsilon'}{c^2}\frac{4\pi^2}{T^2}X=\frac{4\pi^2 m^2}{T^2}X,\quad \frac{\epsilon'}{c^2}=m^2.$$

Since ϵ' is complex, m must also be complex, and we can write $m=\frac{1-i\kappa}{V}$, in which V is the velocity of propagation and κ a constant.

Substituting this value in (6),

$$X=Ae^{\frac{2\pi i}{T}\left(t-\frac{(1-i\kappa)z}{V}\right)}=Ae^{\frac{2\pi it}{T}-\frac{2\pi iz}{TV}-\frac{2\pi\kappa z}{TV}},$$

in which $TV=\lambda$,

$$X=Ae^{-2\pi\kappa\frac{z}{\lambda}}e^{2\pi i\left(\frac{t}{T}-\frac{z}{\lambda}\right)}. \quad \ldots\ldots\ldots \quad (7)$$

In this expression $Ae^{-2\pi\kappa\frac{z}{\lambda}}$, which represents the maximum value which X has during a complete reversal, is seen to diminish as z increases. This means that the intensity falls off as we proceed along the z axis, or that absorption takes place.

The light, after traversing a thickness equal to the wave-length λ, is decreased in amplitude by an amount $e^{-2\pi\kappa}$. The constant κ is the measure of absorption, and is called the absorption index.

We are now in a position to get the equations which connect the refractive index n and the absorption index κ, with the constants of the medium.

$$\frac{\epsilon'}{c^2}=\left(\frac{1-i\kappa}{V}\right)^2, \text{ and, since } n=\frac{c}{V},\ \epsilon'=\frac{c^2}{V^2}(1-\kappa^2-2i\kappa);$$

$$\therefore \epsilon'=n^2(1-\kappa^2-2i\kappa). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By equations (4) and (8) we have

$$n^2(1-i\kappa)^2=1+\sum\frac{\theta N}{1+\frac{i}{\tau}a-\frac{\tau_m^2}{\tau^2}}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

By separating the real and imaginary parts of this equation we can derive two relations, from which n and κ can be determined.

Normal Dispersion. — In the case of normal dispersion we are dealing with a range of frequencies which does not include the free-period of the electron. The term $\frac{i}{\tau}a$ can in this case be neglected, since it represents friction, and friction is not brought into play, since the electron is not thrown into vibration. This makes the right-hand member of the equation real, and $\kappa=0$. The expression for the refractive index reduces to

$$n^2=1+\sum\frac{\theta_h'}{1-\left(\frac{\tau_m}{\tau}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which $\theta_h'=\theta N$.

For a medium with two absorption bands, one in the infra-red, the other in the ultra-violet, the formula may be written

$$n^2=1+\frac{\theta_v'}{1-\left(\frac{\tau_v}{\tau}\right)^2}+\frac{\theta_r'}{1-\left(\frac{\tau_r}{\tau}\right)^2},$$

in which θ_v' is the value of θN for the electrons vibrating with ultra-violet periods, and θ_r, the value of θN for the infra-red electrons, $\tau_v=\frac{T_v}{2\pi}$, $\tau_r=\frac{T_r}{2\pi}$, T_v and T_r being the free-periods of the two sets of electrons.

This formula can be shown to be the equivalent of a Cauchy series, with four constants,

$$n^2=-A'T^2+A+\frac{B}{T^2}+\frac{C}{T^4},$$

in which T is the period of the light.

If T differs considerably from T_v and T_r, as is the case when the radiations belong to the visible spectrum, and the medium is transparent, $\frac{\tau_v}{\tau}$ will be a small fraction, and we have, by division,

$$\frac{1}{1-\left(\frac{\tau_v}{\tau}\right)^2}=1+\left(\frac{\tau_v}{\tau}\right)^2+\left(\frac{\tau_v}{\tau}\right)^4+\cdots$$

For the infra-red electrons $\frac{\tau}{\tau_r}$ is a small fraction, and

$$\frac{1}{1-\left(\frac{\tau_r}{\tau}\right)^2}=\frac{\tau^2}{\tau^2-\tau_r^2}=\frac{\tau^2}{\tau_r^2}\frac{1}{\frac{\tau^2}{\tau_r^2}-1}=-\frac{\tau^2}{\tau_r^2}\frac{1}{1-\left(\frac{\tau}{\tau_r}\right)^2}.$$

Developing the fraction into a series as before, the expression becomes

$$-\frac{\tau^2}{\tau_r^2}\left\{1+\left(\frac{\tau}{\tau_r}\right)^2+\left(\frac{\tau}{\tau_r}\right)^4\right\}+\ldots.$$

The dispersion formula now takes the form (writing in place of τ the actual period T)

$$n^2=1+\theta_v'+\frac{\theta_v'T_v^2}{T^2}+\frac{\theta_v'T_v^4}{T^4}-\frac{\theta_r'T^2}{T_r^2}-\frac{\theta_r'T^4}{T_r^4}.$$

which is identical with the four-constant Cauchy formula just given. The term A of this formula, which is independent of T, has the physical significance

$$A=1+\theta_v'.$$

The dielectric constant $\epsilon=1+\Sigma\theta'$; $\therefore\ \epsilon-A=\theta_r$,

or the difference between the dielectric constant and the term of the dispersion formula which is free from T, represents the dielectric constant of the group of electrons with periods corresponding to those of infra-red radiations.

The coefficient A' in the formula represents the effect of the electrons with infra-red periods, and in the case of substances with dispersions represented by the three-constant formula $n^2=A+\frac{B}{\tau^2}+\frac{C}{\tau^4}$, though we cannot be sure that there are no absorption bands in the infra-red region, we know that the dielectric constant of the electrons is small. For such substances A should represent the dielectric constant. Drude has applied his treatment to the dispersion of water.

The coefficient A' has a larger value for water than for any other transparent substance, which is what we should expect from the circumstance that water is the least diathermanous of all the transparent media. If we assume that but one absorption band exists in the infra-red, we can calculate its position from the experimentally determined value of the constants A' and $\epsilon-A$. Referring to the formula we find that A' corresponds to $\frac{\theta_r'}{T_r^2}$ and $\epsilon-A=\theta^{4\prime}$, from which we have for the period of the infra-red electron

$$T_r=\sqrt{\frac{\epsilon-A}{A'}}.$$

A' and A are calculated from observations of the dispersion, while ϵ is the electrically determined dielectric constant.

The absorption band determined in this way is situated at a point in the spectrum corresponding to wave-length .078 mm. Rubens and Aschkinass found (*Wied. Ann.*, page 65, 252, 1898) that the long heat-waves obtained by multiple reflections from sylvite ($\lambda = 61\,\mu$) were more strongly absorbed by water-vapor than the rays from rock-salt ($\lambda = 51\,\mu$). This makes it seem probable that the infra-red band is somewhere beyond 61 μ, which would be in agreement with the value 78 μ calculated by Drude.

Recent work by Rubens with rays reflected from bromide of potassium ($\lambda = 80\,\mu$) has shown, however, that water only reflects 9 % of the incident energy in this region, and has calculated that the refractive index is only 1.41. Moreover, a column of water-vapor at a temperature of 150° and atmospheric pressure 40 cms. in length, transmitted 33 % of the energy.

Still more recently the reflecting power of ice was very carefully investigated in a room at a low temperature by A. Trowbridge, in a search for possible selective reflection, but no trace of the phenomenon was found. These results vitiate the calculations made by Drude completely. The dispersion of water does not appear to be well represented by any formula at present at our disposal.

Equation (10) can be written

$$n^2 = 1 + \sum \frac{\theta_h' \tau^2}{\tau^2 - \tau_h^2} = 1 + \Sigma\theta_h' + \sum \frac{\theta_h' \tau_h^2}{\tau^2 - \tau_h^2}$$

or

$$n^2 = b^2 + \sum \frac{M_h}{\lambda^2 - \lambda_h^2}, \quad \cdots \quad (11)$$

in which b^2 represents the dielectric constant. In this expression we have substituted wave-lengths for periods, as they are more convenient to work with, λ_h being the wave-length *in ether* of a disturbance of the same peroid as that of the electron and $M_h = \theta_h' \lambda_h^2$. The dispersion of the ordinary ray in quartz is well represented if we take the summation for three terms, *i.e.* for three absorption bands.

The constants have been determined as follows:

$$\begin{array}{lll} M_1 = .0106, & \lambda_1^2 = .0106, & \\ M_2 = 44.224, & \lambda_2^2 = 78.22, & b^2 = 4.58. \\ M_3 = 713.55, & \lambda_3^2 = 430.56. & \end{array}$$

From equation (11) we see that these constants must fulfil the relation

$$b^2 - 1 = \Sigma\theta_h' = \frac{M_1}{\lambda_1^2} + \frac{M_2}{\lambda_2^2} + \frac{M_3}{\lambda_3^2}.$$

The sum of the fractions is 3.2, while $b^2 - 1 = 3.58$.

The discrepancy can be accounted for by assuming the existence of one or more other absorption bands in the remote ultra-violet. The wave-lengths corresponding to the positions of these bands are so small that we can consider them equal to zero.

If the sum of the dielectric constants of these groups of electrons

is θ_o', we have $b^2=1+\theta_o'+\Sigma\theta_k'$, $M_k=\theta_k'\lambda_k^2$,

$$\theta_o'=b^2-1-\sum\frac{M_k}{\lambda_k^2}.$$

In the present case $\theta_o'=.38$.

The dielectric constant for quartz has been found to be in the neighborhood of 4.6, which agrees well with the value of b^2 determined optically.

Selective Reflection in Absorption Region: Residual Rays. — The method of "Rest-Strahlen" or Residual Rays, originated by Rubens and Nichols, has been extensively used by other investigators.

Not all absorption bands exhibit the phenomenon of quasi-metallic reflection, and it is of interest to inquire into the conditions governing its appearance. Absorption results from resonance vibration of the electrons, but in some cases they throw practically all of the energy back, and there is little or no true absorption. When we come to the Chapter on the Optical Properties of Metals, we shall see that there is a type of absorption resulting from the presence of free electrons which have no definite period of vibration. If these electrons are free to move without friction under the periodic forces of the light-waves, the electric intensity within the medium will be neutralized within the medium and we shall have complete reflection. This would be the case for a perfect conductor. If the electrons experience a resistance to their motion, the neutralization will not be complete and a portion of the energy will enter the medium and be absorbed or transformed into heat (ohmic heating). It is the surface layer only which contributes to the reflecting power; in other words, the resonators must be so densely packed that the wave is practically stopped within less than a wave-length of the surface. That this is so follows from the principle of interference. Suppose each layer of resonators to reflect a small percentage of the incident energy. The phases of the disturbances would not be in agreement, and they would destroy each other. Now we find that ultra-violet absorption bands, say those of glass or quartz, do not give rise to selective reflection. From this we should infer that the disturbance penetrates a short distance into the medium and is absorbed. Thin films of glass, only a few wave-lengths in thickness, are quite transparent, even to the extreme ultra-violet. They would, without doubt, be found to be absolutely opaque to the infra-red at 8.5 μ. It is possible to obtain scales of mica less than a wave-length thick, and it would be interesting to compare their transmission at 8.5 μ and in the remote ultra-violet, say at 0.2 μ. The absorption bands of many substances, both in the infra-red and visible region, are of the same nature as these ultra-violet bands, and cannot therefore be located by the method of residual rays. Selemite and alum belong to this class, as was found by Aschkinass. In some cases traces of selective reflection can be found by employing only a single surface. The residual rays are in this case much diluted with continuous spectrum, but they can be located if due precautions are taken. This method was adopted

by Coblentz, who investigated the reflecting power of a large number of substances.

Curves showing the percentage reflected by a single surface of a number of substances examined by Coblentz are reproduced in Fig. 277. Obviously in cases where the reflecting power is below 70 %, most of the energy would be lost after four or five reflections. For 50 % reflecting power, four surfaces would only give us $\frac{1}{16}$ of the total intensity of a band of residual rays.

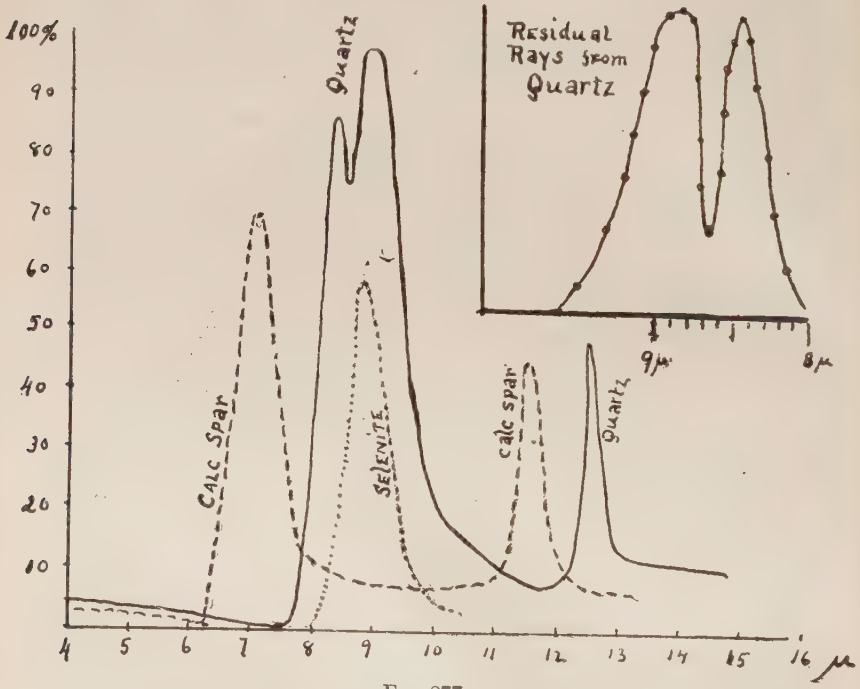


Fig. 277.

The double maximum of quartz at 8.4–9.8 μ has recently been investigated by A. Trowbridge and R. W. Wood, with the echelette gratings described previously. Its true form is shown in the smaller figure. The minimum is very much deeper than it is usually figured, and the maxima are of nearly equal height. A curve of this type, obtained from observations made with too small resolving powers, would come out as figured by previous investigators.

Interesting observations have been made by Pfund on the rays reflected from solutions. Especially interesting are mixtures of fuming sulphuric acid and water. The pure acid gives bands at 7.2, 10.3, and 8.6 μ. The addition of water causes the first two to disappear and a fourth band at 9.6 μ to appear, which are ascribed to ionization produced by the water.

Residual Rays from Powders. — The selective reflection of powdered quartz in the infra-red has been investigated by A. Trowbridge. He found a *minimum* of reflecting power in a region

of the spectrum corresponding to the absorption band at 2.95 μ, and a *maximum* at the absorption band at 8.5 μ. This is very interesting, and shows us the difference in the behavior of the two types of absorption bands. The one at 2.95 μ is weak, and only manifests itself when a considerable thickness is traversed. The quartz therefore shows "Body color" by reflection, *i.e.* a color resulting from absorption, the energy penetrating deeply into the powdered mass, and finally emerging as the result of repeated reflection and refractions among the particles, robbed of energy of wave-length 2.95 μ. The band at 8.5 μ is metallic in character, and the energy is selectively reflected from the upper surfaces of the particles. If the particles were very small and the surface flat, specular or regular reflection would appear. Trowbridge used a rather coarse powder, and studied the diffuse reflection. The distinction between the two types of bands must be taken into account in all investigations made with surfaces which permit of the formation of "Body color," otherwise the results will appear to be very anomalous in character.

Interferometer Investigation of Long Heat-Waves. — An investigation of remarkable interest has been made recently by Rubens and Hollnagel[1] with an interferometer designed especially for work in the infra-red. This investigation has resulted in the discovery of heat-waves 96 μ in length, the longest Rest-Strahlen observed. The arrangement of the apparatus was as follows. Light from a Welsbach lamp, without a chimney, rendered parallel by a mirror, was passed through the interferometer. The plates of this instrument were of quartz, cut perpendicular to the axis, .6 mm. in thickness. Thin plates of quartz are fairly transparent for the very long waves, and on account of the high value of the refractive index in this region, possess a very high reflecting power. Previous investigations by Rubens and Aschkinass had shown that the refractive index for the waves reflected from sylvite ($\lambda = 56\,\mu$) is as high as 2.18, which gives a reflecting power of nearly 14 %. After passage through the interferometer the rays suffer a fourfold reflection from the reflecting surfaces (which are made of crystal plates, or slabs of fused salts), and are then focussed upon the radiomicrometer. The quartz plates were first placed in contact, and the scale deflection observed. They were then separated by constantly increasing amounts, readings being taken for each position. With properly adjusted plates, an interference maximum or minimum could be made to cover completely the sensitive surface of the recording instrument, and by plotting the scale deflections against the readings of the interferometer wheel an intensity curve was obtained precisely similar in every respect to the visibility curves obtained by Michelson with his instrument. The substances investigated were rock-salt, sylvite, and the bromide and iodide of potassium. The curves obtained from some of these substances are shown in Fig. 278. From the similarity to the visibility curves obtained with sodium light we recognize the presence of two wave-lengths.

[1] *Sitz. Ber der Preus. Akad.*, Jan. 20, 1910.

From the known separation of the quartz plates for each maximum and minimum, the intensity curve of the radiation can be calculated. The longest waves were obtained from plates of fused iodide of potassium, which gives us Rest-Strahlen 95.6 μ in length. The wave-length of the selectively reflected bands from the substances selected are given in the following table:

Rock-salt	53.6 μ and	46.9 μ
Sylvite	69.9	61.6
Bromide of potassium	86.8	
Iodide of potassium	96.5 μ	

Intensity curves of the radiation reflected from rock-salt, sylvite, and potassium bromide are reproduced in Fig. 279.

An investigation also made of the absorption and reflection of the rays from potassium bromide and iodide is recorded in the following table:

Material	Thickness	Percentage Transmitted for	
		KBr Rays	KI Rays
	mm.		
Quartz	.60	64.9	—
Quartz	2.00	47.6	59.2
Quartz	3.03	39.2	—
Quartz	4.03	31.4	50.4
Paraffine	2.90	47.6	54.5
Mica	.02	51.7	25
Hard rubber	.40	30.3	43
Water	.0026	77.3	—
Fluorite	3.48	0	0
Rock-salt	3.00	0	0
Glass	2.2	0	0
Sylvite	4.1	0	0
Water-vapor	400	38.7	33
Carbonic acid	400	100	—

As will be seen, quartz, paraffine, and hard rubber are the only substances which show any considerable transparency to the radiation. Mica, however, can be split into such thin sheets that they exercise no appreciable influence either by absorption or reflection. The water-film investigated was a soap-bubble film containing 10% of glycerine and 1% of oleate of soda. The water-vapor was contained in a tube 40 cms. long and 9 cms. in diameter, with open ends kept at a temperature of 150° by electrical heating. Steam at a temperature of 100° and atmospheric pressure was blown into this tube from the side, the powerful superheating preventing

condensation at the open ends of the tube. It was found that the rays from rock-salt and sylvite were completely absorbed by the

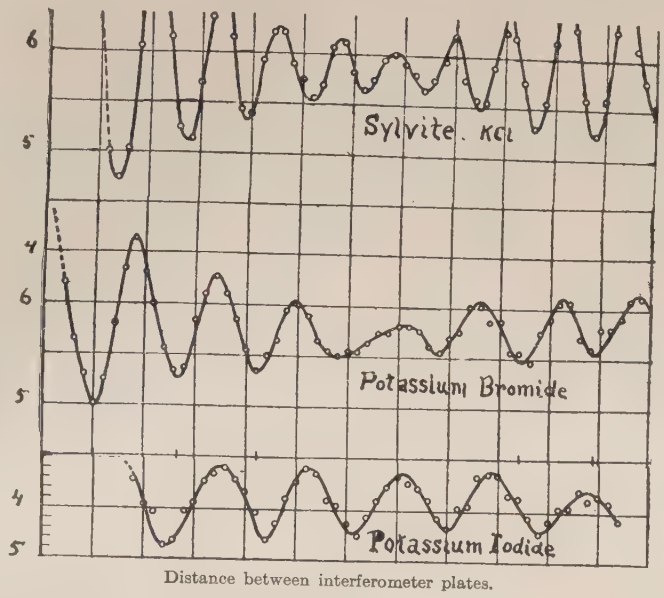


FIG. 278.

water-vapor, while the radiations from bromide and iodide of potassium were partially transmitted. The transmissions of quartz

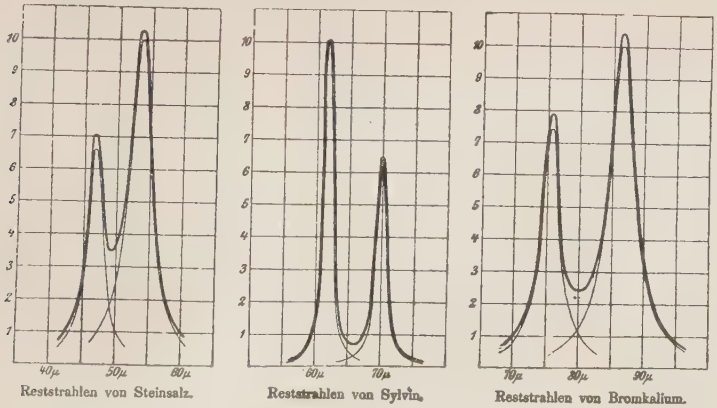


FIG. 279.

for these rays is of especial interest, since the interferometer plates are made of this material, and any selective absorption by them

will exercise a marked influence upon the wave-length determination. The results are recorded in the following table:

Thickness of Plate	Percentage Absorption of Quartz for "Rest-Strahlen" from			
	Rock-salt	Sylvite	KBr	KI
mm.				
.60	35.8	14.8	12.2	—
2.00	67.7	46.	35.5	20
3.03	76.9	54.9	46.8	—
4.03	82.6	69.4	57.6	32

In calculating the percentage of absorption, the loss by reflection at the two surfaces of the plate (26%) has been taken into account. It is clear from the table that the absorption of quartz decreases gradually with increasing wave-length. Strong selective absorption only occurs for the rays of rock-salt. The reflecting power of a number of substances for the rays from bromide of potassium are given in the following table:

Substances	Reflection %
KBr	82.6
KI	29.6
Sylvite	36
Rock-salt	25.8
Fluorite	19.7
Glass	14
Water	9.6

As is to be expected, the reflection of bromide of potassium is very high for its own residual rays. The most important points established by this investigation, apart from the discovery of these very long waves, which has extended the known spectrum over a half octave, are that the mean wave-length of the reflected light increases with the molecular weight, more slowly, however, than the molecular weight, but more rapidly than its square root, and that the refractive index of water for heat-waves 82.6 μ in length, is of the same order of magnitude as in the visible spectrum. The known range of the spectrum now embraces ten full octaves, two of which are in the ultra-violet, one in the visible, and seven in the infra-red.

The Longest Heat-Waves : Focal Isolation. — The longest heat-waves of all have very recently been found by H. Rubens and R. W. Wood, by means of the very simple device shown in Fig. 279 *a*. It depends upon the transparency of quartz for these very long waves combined with its very high refractive index. The radia-

tions from a Welsbach light A (without chimney) pass through a hole 1 cm. in diameter in a screen B made of two sheets of tin, and is received by the quartz lens C. For the longest heat-waves

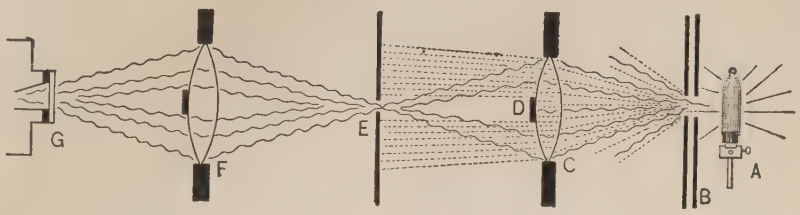


Fig. 279 *a*.

previously measured, Rubens found that the refractive index of quartz was about 2.2, and the screen E, also perforated with a 1 cm. hole, was accordingly mounted at such a distance from the lens that the two screens were at conjugate foci, for radiations for which the refractive index of the lens had this value.

Owing to the much smaller value of the refractive index for the shorter heat and light waves, these actually diverge after leaving the quartz lens. The paths of the two types of rays are indicated in the figure, the long heat rays being represented by wavy lines, the short-wave rays by dotted lines. The aperture E is screened from the central portion of the short-wave ray-bundle by a small coin D fastened to the surface of the lens with wax. A single lens arranged in this way enables us to obtain at once the longest heat-waves of all, but to make matters sure, the second lens F, arranged in the same way, was used to focus the radiation upon the thermo-element of the radio-micrometer G. This still further purified the radiation, though its use was not imperative. To test whether the radiation is pure, *i.e.* whether we have completely eliminated the short waves, we have only to introduce a plate of rock-salt in front of the screen E, which should completely absorb everything.

By introducing the interferometer in the place of the screen E, it was found possible to measure the wave-length of the radiation isolated by the quartz lenses. The maxima and minima obtained as the distance between the quartz plates was increased were very pronounced, but only three could be recorded, owing to the wide spectral range of the transmitted radiation. The curve obtained is shown in Fig. 279 *b*, and its analogy to the intensity curve obtained in all interference experiments with white light is at once apparent. The mean wave-length of the radiation can be determined from the distance between the maxima, and was found to be 107 μ or *more than one-tenth of a millimeter*. There are still longer waves present, as we know from the damping of the intensity curve; in other words 107 μ is merely the centre of a rather broad spectral region isolated by the quartz lenses. We thus have experimental evidence of waves probably as long as 130 μ or possibly 150 μ in the radiation from the Welsbach lamp.

The shortest electro-magnetic waves have been recently obtained by v. Baeyer with a new type of electric oscillator; their wave-length was found to be in the neighborhood of 2 mms., so that the gap remaining between the electric and the optical spectrum is no longer very large. These very long heat-waves have some very curious properties. They pass through thin black paper almost without absorption. It was in fact found possible to isolate them by means of a concave mirror combined with two or three sheets of black paper. The spectrum range obtained in this way was, however, very much wider than the range obtained with quartz lenses. Thick films of smoke absolutely opaque to light were also transparent to them.

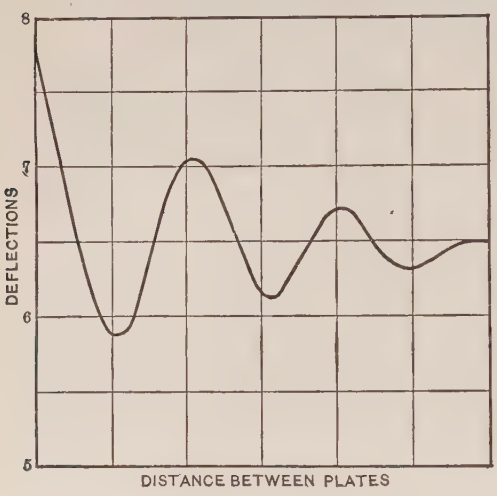


Fig. 279 b.

The absorbing powers of quartz and water for these radiations are shown in the following table, q being the absorption coefficient in the formula $\frac{i}{J} = e^{-qd}$.

Quartz			Water		
Thickness of Plate	Absorption %	q	Thickness of Film	Absorption %	q
.6 mm.	5.4	.105	.0147 mm.	58	.059
2.00 "	18.7	.103	.0294 "	77.5	.051
3.03 "	27.	.104	.0442 "	88.2	.048
4.03 "	33.6	.102			
7.26 "	51.7	.100			
12. "	67.6	.094	Extinction coef. $\kappa = \frac{q\lambda}{4\pi}$		.420
14.66 "	69.4	.081			

The absorption of water was determined by separating the interferometer plates a known distance, measured by sodium fringes, and then allowing a drop of water to creep in between them, watching the fringes to make sure that no change in the distance resulted from capillary attraction. The extinction coefficient of water for the rays from KBr ($\lambda = 87\ \mu$) is $\kappa = .66$, while for the new rays ($\lambda = 107\ \mu$) it is only .42.

For his shortest electro-magnetic waves, v. Baeyer found that even the thinnest water films were absolutely opaque, so that the absorption, which decreases rapidly in passing from 87 μ to 107 μ must begin to increase again somewhere between $\lambda = .1$ mm. and $\lambda = 2$ mm. The investigations with these very long waves is in progress at the time of writing. They can be obtained with an intensity about five times as great as the intensity of the 97 μ waves reflected from the plates of potassium iodide, which makes their study comparatively easy.

Selective Reflection in the Ultra-Violet. — Very little work has been done in this direction. In general the phenomenon of metallic reflection does not occur at the ultra-violet absorption bands. Some investigations, made by Martens, of the ultra-violet dispersion of certain organic compounds, indicate that traces of selective reflection exist in certain cases. Cassia oil, for example, which in the thinnest possible films, pressed between fluorite plates, is opaque to everything below wave-length 3360, exhibits the phenomenon. Light from the cadmium spark, reflected a number of times from its surface, is shown by the spectroscope to consist chiefly of the 2748 line, no trace of the equally strong lines 2573 and 2321 appearing on the photograph. Fourfold reflection from a surface of mono-bromnaphthalin showed the lines 2321 and 2288 only. Bisulphide of carbon, investigated by Flatow, showed a reflection maximum at 2310. This is a subject which requires further investigation, and the work already done should be repeated. In working with ultra-violet rays, we are completely in the dark, and the high dispersion of all transparent substances in this region means wide angular separation of the rays at any refracting surfaces used. Great care must be taken to make sure that the rays of all wave-lengths have a chance to reach the spectroscope. Quantitative measurements of the reflecting power for different wave-lengths should be made at the same time, and the transmission of the thinnest possible films examined. If the substance is in contact with glass, quartz, or some other transparent substance, we may have a selective reflection resulting from the difference between the values of the refractive index. This will modify the color of the reflected light in any case. In many cases the color of the light reflected is totally different from that most strongly absorbed. For example, the surface color of cyanine is purple, and the absorption band is in the yellow. Its purple color is due to its failure to reflect a region of the spectrum bordering the absorption band on the green side, resulting from the very low value of the refractive index. In contact with glass, viewed from the glass side, it reflects yellowish green light. Nitroso-dimethyl aniline, which has an absorption band in the violet, exhibits a brilliant blue reflection when in contact with glass (from the glass side), but practically no trace of color from the surface in contact with air. Selective reflection of this nature is chiefly the result of the high value of the refractive index, and cannot be used as a method of locating the position of the absorption band. Such experiments show us the care we should use in interpreting

results found in the ultra-violet, where effects due to refractive index may be even greater.

Selective Dispersion of an Absorbing Gas. — We will begin by considering the remarkably interesting case presented by the vapor of metallic sodium. This vapor has two very strong absorption bands (the D lines) which profoundly affect the velocity with which light-waves traverse the vapor. The selective dispersion of the vapor was first observed by Kundt, who noticed, when projecting a continuous spectrum upon a screen, a sodium flame having been placed in front of the lens to exhibit the reversal of the lines, that the edges of the spectrum immediately adjacent to the dark absorption lines were slightly curved in opposite directions, indicating abnormally high and low refracting power of the prismatic absorbing flame. The phenomenon has been subsequently studied by Becquerel, Julius, Ebert, and Wood.

An application of the Sellmeier formula, which is identical with the electro-magnetic formula which we have just developed, was made by Lord Kelvin (*Phil. Mag.*, 47, 1899), which will be given presently.

If we write the formula in the form $n^2 = 1 + \frac{m\lambda^2}{\lambda^2 - \lambda^2_m}$, in which $\lambda_m = 5893$, the mean wave-length of the D lines, it will represent fairly well the dispersion of the vapor, the two absorption lines being so close together that they can be considered as one, at least when the vapor is very dense. If we give to λ values slightly larger and slightly smaller than λ_m, we shall find that the denominator grows less as we approach the absorption band from the red side, resulting in a rapid increase of n. When $\lambda = \lambda_m$, the denominator becomes 0 and $n = \infty$. If, however, we approach the absorption band from the side of the shorter wave-lengths, the denominator of the fraction becomes negative, which gives us values of n which are less than unity. The value of the fraction will obviously become greater than one eventually, which will give us for n the square root of a negative quantity. Such a value of the refractive index has no meaning, however, though we may interpret it, as Lord Kelvin does in his paper, by saying that it indicates that no light of such wave-lengths as give an imaginary value to the refractive index can enter the medium: they are selectively reflected. It is perhaps questionable whether this interpretation is allowable, since the formula, as we are using it, has been simplified on the assumption that it is only to be applied to regions of the spectrum for which the substance is transparent, a region which obviously does not contain the wave-lengths in question, owing to the finite width of the absorption bands.

The conception is useful, however, in that it shows us that the failure of light to penetrate a medium may result from other causes than the stirring up of vibrations within the medium which are accompanied by friction. In the latter case the energy is used up in heating the medium; in the former it is thrown back or reflected. It is clear that as the density of the sodium vapor is increased the numerator of the fraction increases. For a very dense vapor the

range of wave-lengths which give to n an imaginary value stretch farther away from the center of the absorption line towards the more refrangible end of the spectrum. We should therefore expect that the dark band seen in the spectrum of the transmitted light would widen out on this side as the density of the vapor increased. As a matter of fact it widens out on both sides. Moreover, it widens in a similar way if we increase the length of the absorbing column, holding the density constant, which makes it seem probable that the broad band is caused by true absorption, and not by selective reflection.

A study of the dispersion of the vapor of metallic sodium has been made by the author, with a view of testing the simplest form of the dispersion formula which we have, namely the one given above.[1]

As the phenomena exhibited by this vapor are among the most beautiful in physical optics, it may be well to consider them in some detail.

The apparatus for showing the selective dispersion of the vapor is very easily prepared. Though glass tubes have sometimes been used in these experiments, their use is discouraged, as they are very liable to crack.

Procure a piece of thin steel tube 25 mms. in diameter and 30 cms. long, drill a small hole near one end and braze in a short piece of brass tubing. This can be done at any bicycle repair shop. The wall of the steel tube should be as thin as possible, to prevent heat conduction, a millimetre or less if possible.

Close one end of the tube with a piece of thin plate glass cemented with sealing wax. To make an air-tight joint, the steel tube should be heated until the wax will melt when brought in contact with it. Spread a layer of the wax around the end of the tube, warm the glass plate, and press it firmly against the wax ring, first softening the latter with the Bunsen flame. Now go over the joint with a minute pointed gas flame burning at the tip of a glass tube drawn down to a point, until the wax runs into close contact with the glass. A joint properly made in this way will hold a cathode ray vacuum for hours.

Mount the tube in a horizontal position in a clamp stand at the height of the slit of the spectroscope, which is to be used for viewing the phenomenon, and introduce eight or ten small pieces of clean sodium, pushing them down the tube one by one, so that they may lie side by side along the middle portion of the tube. Now close the other end with a piece of plate glass, in the same way. The tube should next be placed in communication with an air pump and exhausted. If the pump leaks, it is a good plan to introduce a stop-cock, which can be closed after the exhaustion. A pressure of from one to three cms. of mercury can be used, which can be obtained with a water pump. Strips of wet cotton should be wrapped around the tube near the ends, to prevent the sealing wax from softening, and water should be poured on them from time to time.

[1] Wood, "A Quantitative Determination of the Anomalous Dispersion of Sodium Vapor," *Phil. Mag.*, September 1904.

The under side of the tube is heated by a row of small flames, furnished by a burner, as shown in Fig. 280. The ends are kept cool by wet cotton. The burner is made by drilling a dozen or more minute holes in a piece of brass tubing, closing the ends, and mounting it on a Bunsen burner. On heating the tube the sodium vaporizes, and diffuses gradually to the cooler parts of the tube. The metal usually contains a large amount of hydrogen, and if this is given off in too great quantities it should be removed by the

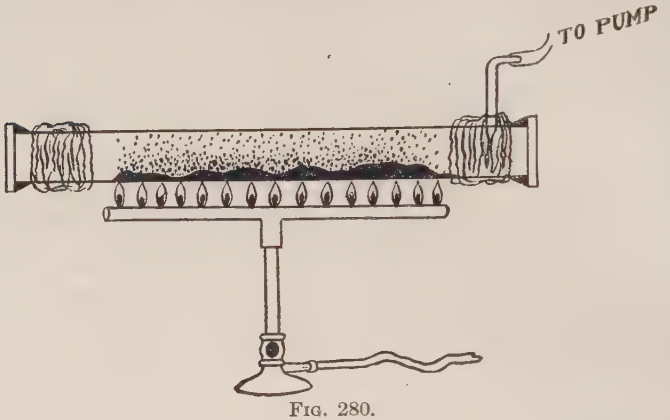


Fig. 280.

pump. A high vacuum should not be used, for it is the presence of hydrogen or nitrogen at low pressure that makes the sodium vapor distribute itself in a non-homogeneous manner. The vapor is given off along the heated floor of the tube, but its diffusion towards the cooler top is interfered with by the residual gas in the tube. The vapor will be found to be very dense along the floor and highly attenuated along the top. This condition can only obtain when some other gas is present to keep the pressure balanced. It is easy to see how this condition comes about. Suppose our residual gas is at a pressure of 1 cm. and we heat the floor of the tube to a temperature such that the vapor pressure of sodium is also 1 cm. If the vapor is given off from the molten metal more rapidly than it

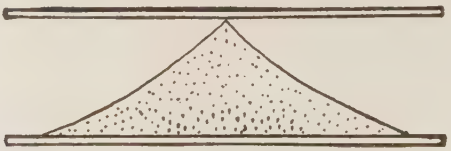


Fig. 281.

can diffuse to the cooler parts of the tube, it will accumulate over the surface of the metal, and may even be thought of as displacing the residual gas entirely in this region. Along the cooler top, where condensation is taking place, the density of the vapor can only be as great as that determined by the temperature here. Here we may have a pressure of 9 mms. of residual gas and 1 mm. of sodium

vapor. Halfway between the top and bottom we may have a mixture of equal parts of gas and vapor, each at 5 mms. pressure.

The sodium vapor is therefore the optical equivalent of a prism, for the amount of it traversed by the light increases as we pass from the top to the bottom of the tube: the sodium tube is thus

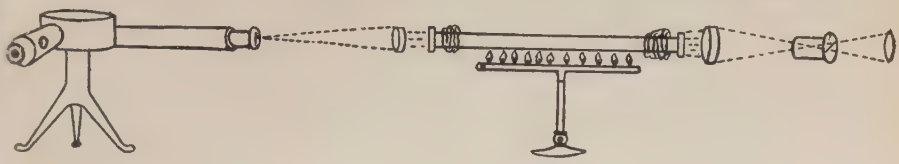


FIG. 282.

the equivalent of a prism with its refracting edge horizontal, resting upon its base. The deviation of the rays will be gradual, however, as in the non-homogeneous media described in the Chapter on Refraction. Observations on the deviation produced by the non-homogeneous cylinder show that the equivalent prism has a form

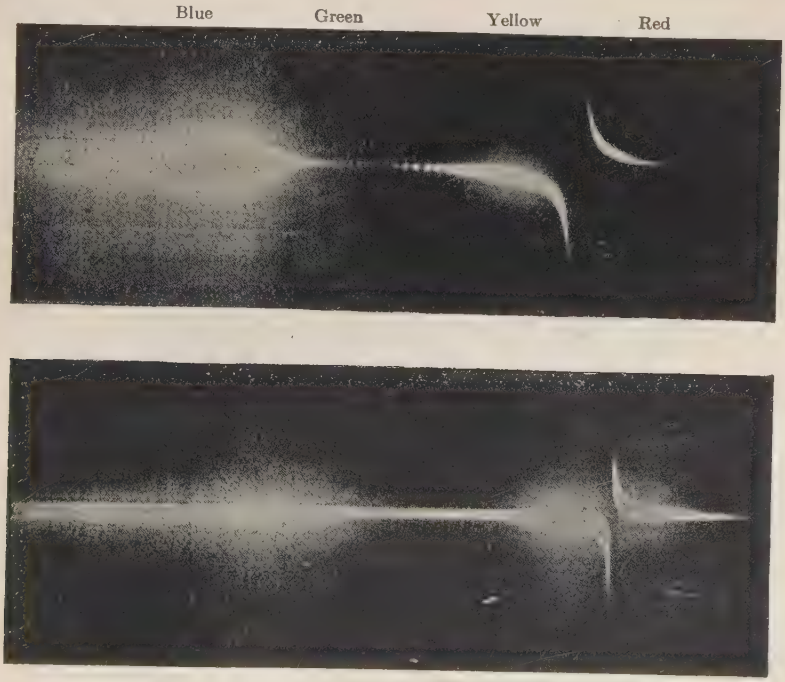


FIG. 283.

similar to that shown in Fig. 281, the density gradient being steeper near the bottom of the tube. To secure good definition it is therefore necessary to place in front of the tube an opaque screen perforated with a wide horizontal slit.

The arrangement of the apparatus is shown in Fig. 282. A horizontal slit is illuminated by focussing the sun or arc upon it, and the emergent rays, rendered parallel by a lens, passed down the sodium tube. A second lens forms an image of the horizontal slit across the vertical slit of the spectroscope. This image should be carefully focussed, so that the spectrum appears as a brilliant narrow band with sharp edges.

On heating the tube, the sodium prism deviates the rays of different wave-length up or down by different amounts, curving the spectrum into two oppositely directed branches. The spectrum on the green side of the *D* lines will be found to bend down in the spectroscope, which means that the rays are deviated upwards in passing through the sodium tube, since the spectroscope inverts the image of its slit. This means that these rays travel faster in the sodium vapor than in vacuo, or the prism acts for these rays like an air prism immersed in water. The red and orange region is deviated in the opposite direction; these rays are therefore retarded by the vapor.

Photographs of the anomalously dispersed spectrum are shown in Fig. 283, the upper for dense, the lower for rare sodium vapor. Its appearance is also shown on the colored Frontispiece, Figs. 2*a* and 2*b*.

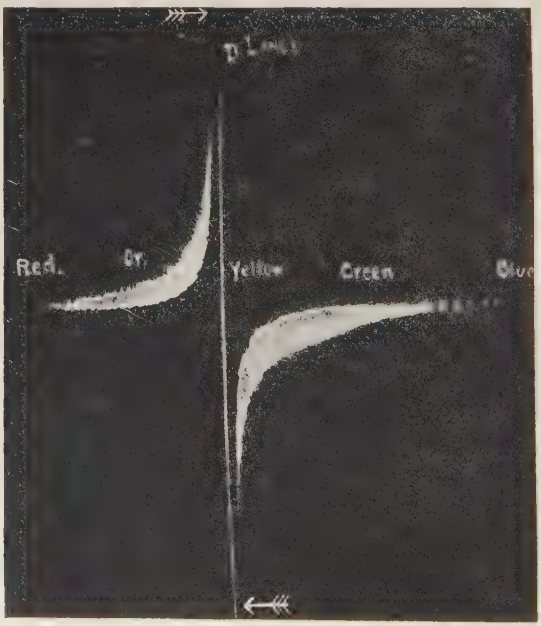


Fig. 284.

It may be found that the performance of the tube is improved by laying a long pad of wet cotton along the top. This makes the temperature gradient steeper and facilitates the formation of the non-homogeneous vapor cylinder. A tube of the dimensions

given should show the phenomenon on as great a scale as the photograph reproduced in Fig. 284. In this case the position of the *D* lines was recorded on the plate by holding a sodium flame in front of the slit, after the exposure was over.

If we remove the spectroscope and place an eye-piece in the position previously occupied by the slit, we shall see the anomalous spectrum produced by the sodium prism, as shown in the colored Frontispiece, Fig. 1.

The development of this spectrum from the horizontal image of the slit, seen when the tube is cold, is most instructive to watch, and shows us at once that the sodium prism deviates some of the rays up and others down.

If the electric arc is employed as the source of light, the extreme violet will be found to occupy the position of the undeviated image of the slit. Then comes the blue, sometimes in contact with the violet and sometimes slightly separated by a fine dark line, owing to the fact that the violet light comes from the fluted carbon band of the arc, which is separated from the blue by a comparatively dark region. Then comes a wide gap corresponding to light absorbed by the sodium vapor in the blue-green region (the channelled spectrum), and above this a beautiful flare of color ranging from blue-green through grass-green to yellow. The red and orange portion of the spectrum is on the other side of or below the undeviated image, forming another brilliant flare of color. It is separated from the violet by a wide dark band, due to the absorption in the vicinity of the *D* lines. If the density of the vapor is increased by heating the tube to a higher temperature, the red flare extends lower down, grows fainter, and finally fades away, owing to the presence of the fluted absorption bands in red. The green and blue persist, however, becoming more widely separated, but finally the green disappears almost entirely. It is best to arrange the gas cock so that the height of the flames can be controlled without leaving the eye-piece, for it is surprising how slight a change is necessary to completely alter the general appearance of the spectrum.

To obtain this spectrum in its greatest splendor, it is best to use a tube of hard glass about 50 cms. long heated in the same way, containing fifteen or twenty small pieces of sodium. The temperature gradient is much steeper in a glass tube and the dispersion much more powerful. The flames may be turned up and down during the experiment, but they should never be turned quite out, for the tube will almost invariably crack on reheating after it has once cooled down.

By employing a quartz spectrograph photographs have been made of the anomalous dispersion at the ultra-violet lines 3303 and 2852, as shown in Fig. 285.

It is much less strong at 3303 and barely noticeable at 2852. These absorption lines all belong to the principal series of sodium, and they are spaced along the spectrum according to a definite law.

Though we can measure the relative indices of the vapor by this method, we have no means of determining the absolute values, f

we have no means of knowing the angle of the prism of vapor which is the equivalent of the non-homogeneous cylinder.

Fig. 285.

Absolute values have, however, been obtained by means of the interferometer, by comparing and measuring the shifts of the fringes obtained by introducing a given amount of sodium vapor into the path of one of the interfering beams. A full description of the methods and apparatus employed will be found in the paper referred to. In brief, it consisted in placing a sodium tube, electrically heated, between two of the mirrors of a Michelson interferometer. Uniform heating was necessary in this case, as no prismatic action was desired. The instrument was illuminated with two sources of light, one a helium spectrum tube which gives us a bright yellow light, D_3, very near the sodium absorption band, the other a spectroscope arranged to furnish a beam of approxi-

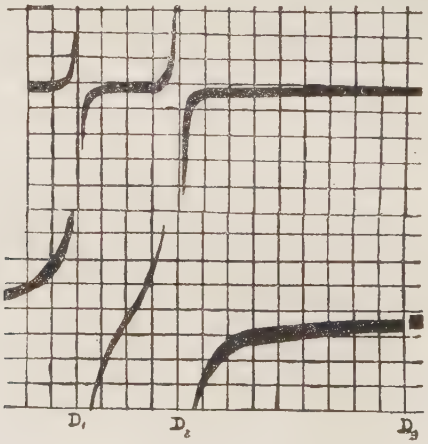


Fig. 286.

mately monochromatic light in any desired part of the spectrum. Two sets of fringes were thus formed, and the drifts of the systems were recorded by two observers as sodium vapor was formed in the tube. To obtain the dispersion very near the absorption band, the helium tube was placed in a powerful magnetic field, which causes the line to become double (with suitable arrangement of the apparatus). The two components were of very nearly the same wave-length, the distance between them being about $\frac{1}{20}$ of the dis-

tance between the *D* lines, yet the dispersion of the vapor was so powerful that the two sets of fringes were displaced at rates so different that the fringes disappeared entirely at regular intervals, owing to their "out-of-step" superposition.

By employing the method of "crossed prisms," relative determinations were made still closer to the *D* lines than the helium line. To see the effect close to and between the *D* lines, the tube should only be slightly heated, and a grating spectroscope employed. As the vapor prism forms, we see presently the portions of the spectrum adjacent to the absorption lines curve away in opposite directions, as shown in Fig. 286. As the vapor becomes denser, the light disappears between the *D* lines, and we have the stage previously described.

Absolute values of the refractive index were obtained by heating the tube to a known temperature, measuring the length of the vapor column, and counting the fringe shift produced when monochromatic light of known wave-length was used to illuminate the instrument.

The Refraction and Dispersion of Sodium Vapor of Great Density. — A knowledge of the absolute value of the refractive index of the vapor, and its dispersion, enables us to compile a table of the refractive indices for all wave-lengths, for vapors of varying density. This has been done for the very dense vapor obtained by heating a vacuum tube containing the metal to the temperature of 644° C. A column of the vapor at this temperature 8 cms. in length examined by transmitted light has a distinct blue color, as a result of the channelled absorption spectrum. The values are given in the following table, and will be spoken of in future as "observed values," to distinguish them from values calculated from the dispersion formula. It must be remembered that sodium vapor as dense as that with which we are dealing in the present case has an absorption band at the *D* lines broad enough to completely cut out everything down to and even below the helium line, at least for all thicknesses with which it is possible to work. On this account we are obliged to calculate the refractive indices within this region from observations made with a less dense vapor, a method which in the present case is probably allowable within certain limits. A thin enough sheet of the vapor would probably transmit light within this region with a velocity indicated by the calculated indices. The question of selective reflection at the surface and refusal to transmit the radiation will be considered presently.

In the following table the wave-lengths are given in the first column, the fringe displacements in comparison with helium light in the second column. These values, with the exception of those in the extreme red, blue-violet, and ultra-violet, were obtained with the interferometer.

In the third column are given the actual fringe shifts which would be found for a layer of vapor 8 cms. thick (effective thickness 16 cms.), and in the fourth column the refractive indices. The indices calculated from the dispersion formula are given in the fifth column, for the sake of comparison with the observed values. More will be said of them in the next section.

For $\lambda = \infty$ the value of n is calculated to be 1.000075, while for $\lambda = 4167$ it is 0.999975, the deviations being the same in both cases, but in opposite directions.

For wave-lengths immediately adjoining the D lines the refractive index has been found to have a value as high as 1.38, as great as that of some liquids, while the dispersion is so great, even at the position of the D_3 line of helium, that, could we form a prism of the vapor giving the same deviation as a 60° glass prism, we could by its aid separate a double line in the spectrum, with components twenty times as close as the sodium lines by an amount as great as the distance between the red and blue of the spectrum formed by the glass prism.

The vapor is remarkable in that all waves on the blue side of the D lines travel in it with a higher velocity than in a vacuum. In other words, they are accelerated. All light of wave-length greater than the D lines is retarded as in ordinary media. This is seen at once from our first experiment, the sodium prism deviating one-half of the spectrum in one direction, the other in the opposite.

λ	Relative Shift	Total Shift	Ref. Index Obs.	Ref. Index Calcul.
7500	2	25	1.000117	
6310	4	50	1.000197	
6200	6	75	1.000291	1.000285
6137	7	88	1.000335	
6055	11	138	1.000523	1.00052
6013	14	175	1.000658	
5977	20	250	1.000934	
5960	25	313	1.001164	
5942	33	413	1.001532	
5916	60	750	1.002972	
5875	100	1250	0.995410	0.9958
5867	67	834	0.996929	0.99692
5858	50	625	0.997711	
5850	40	500	0.998172	0.99815
5843	33	413	0.998492	
5827	25	313	0.998862	
5807	20	250	0.999093	
5750	11	138	0.999505	
5700	9	113	0.999599	
5650	7.4	92	0.999650	
5460	4	50	0.9998294	0.999829
5400	3.6	45	0.9998481	
5300	2.9	36	0.9998807	0.999885
4500	1.4	17.5	0.9999508	0.999965
3610	1.09	13.6	0.9999698	
3270	0.9	11.4	0.9999768	0.999987
2260	0.7	8.7	0.9999877	0.999995

REFRACTIVE INDICES IN THE VICINITY OF THE *D* LINES

λ	*n* Cal.	*n* Obs.
5875	0.9958	0.9954
5882	0.9890	0.9908
5885	0.9830	0.9870
58866	0.9750	0.9740
58884	0.9450	0.9443
58896	0.697	0.614
5991	1.0046	
5904	1.0092	
5901	1.0138	
58994	1.0184	
58976	1.0557	
5897	1.094	
58964	1.386	

Application of the Results to the Dispersion Formula. — The simplest form of the dispersion formula, developed from electromagnetic considerations for a medium with a single absorption band, is

$$n^2 = 1 + \frac{m\lambda^2}{\lambda^2 - \lambda_m{}^2},$$

in which m is a constant, λ the wave-length of the light employed, and λ_m the wave-length at the centre of the absorption band.

The vapor of sodium has, of course, a pair of very close absorption bands (the *D* lines), which are chiefly effective in modifying the refractivity of the medium. The ultra-violet bands affect the dispersion in their immediate vicinity, but their effect can be neglected in comparison with the stronger band, except for wave-lengths very close to them.

The first question to consider is whether we are justified in considering the *D* lines as a single absorption band in the case of the very dense vapor. If we consider the medium as having a single band, and assign to λ_m the value 5893 (a point midway between the *D* lines), we shall find that the observed and calculated values of the refractive index agree very closely, up to within a distance of the band about equal to the distance of the D_3 line of helium. If we attempt to go closer than this, we immediately find discrepancies, which become larger as we approach the *D* lines.

The constant m in the above formula can be determined from a single observation of the refractivity. It was determined from two values, namely, the refractivity for the green line of mercury ($\lambda = 5460$) and that for $\lambda = 5850$, the latter wave-length being quite close to the *D* lines. Since the refractivity has widely different values for these two wave-lengths, we should expect the difference between the two calculated values to be a maximum in this case, in the event of the dispersion being incorrectly represented by the formula. The values for m found in the two cases were 0.000056 and 0.000054, a surprisingly close agreement. The mean value $m = 0.00055$

was taken, and the indices for a number of wave-lengths calculated. Some of these values are given in the table of refractive indices, and they will be found to agree very closely with the observed values. The values calculated between the helium line and the D lines are given in the second part of the table.

An inspection of the formula shows us that, according as we are on the red or blue side of the absorption band, the refractive index (squared) is given by adding to (or subtracting from) unity the value of the constant $m=0.000055$ multiplied by the fraction $\frac{\lambda^2}{\lambda^2-\lambda_m{}^2}$. In the case of all other substances showing anomalous dispersion, aniline dyes for example, to the dispersion of which a formula has been applied, the value of this fraction never exceeds 3 or 4, owing to the impossibility of applying it to wave-lengths very close to the centre of the band. For example, in the case of the dispersion of nitroso-dimethyl aniline, with its strong absorption band at $\lambda=4300$, we cannot obtain accurate data nearer than $\lambda=5000$. In this case $\frac{\lambda^2}{\lambda^2-\lambda_m{}^2}=3.9$.

In the case of sodium vapor the value of the fraction may be several hundred or even thousand. In the case of $\lambda=5882$ the fraction is 367, and yet the observed and calculated values agree closely. For $\lambda=58884$ the fraction is 1940; and for $\lambda=5889.6$ we have a value as high as 3944. The product of these very large numbers and the small fraction 0.000055 give, however, values of the index which are in close agreement with the observed values.

Discrepancies occur in the immediate vicinity of the D lines which can be explained in the following way: To get values in any way consistent with the observed values it was necessary to assign to λ_m the value of the D_2 line, the mean value 5893 being too far removed from the wave-lengths in question to give the requisite steepness to the curve. The calculated values, therefore, apply to a medium with a single band at D_2 and with a constant $m=0.000055$. This gives us a pretty good approximation to the observed curve, but the latter is due to the combined effects of the bands D_1 and D_2, the presence of the D_1 band tending to make the observed curve flatter than the calculated. A more correct approximation could be obtained by assigning to λ_m a value intermediate between D_2 and 5893. The proper method of procedure would, of course, be to make use of two members in the dispersion formula, one for D_1 and the other for D_2, thus:

$$n^2=1+\frac{m\lambda^2}{\lambda^2-\lambda_m{}^2}+\frac{m'\lambda^2}{\lambda^2-\lambda_{m'}{}^2}.$$

If m and m' were each assigned the same value obtained by dividing our original value by 2, in all probability a very close approximation would be obtained in the region in question. This has not been done for two reasons. In the first place it does not appear as if much would be learned by the procedure, and in the second place m and m' are not equal, as is shown by the stronger dispersion near D_2, and until the relative values have been determined we are not

in a position to write the two-member formula accurately. It is doubtful whether anything new would come out of such a determination, and it was on that account not attempted.

Another matter of considerable interest is the question of the indices represented by the square root of a negative quantity in the immediate vicinity of the absorption band on the blue side. Lord Kelvin interprets this as indicating that no light of such wave-lengths enters the medium; in other words, it is metallically reflected.

It is in this way that he has explained the apparent greater broadening of the D lines on the more refrangible side in some of Becquerel's photographs. In the case with which we are dealing the second term of our original formula does not become less than unity until we reach wave-length 58898, which we get by equating $\frac{m\lambda^2}{\lambda^2-\lambda_m^2}$ to unity and solving for λ.

This shows us that, even with a vapor so dense that both D lines run together and broaden out into a wide band, we do not get values of the index which are imaginary until we are within .2 of an Ångström unit of the D line, or, in other words, until we are within a distance of D equal to $\frac{1}{30}$ of the distance between D_1 and D_2.

In the case of the comparatively rare vapor employed by Becquerel we should have to approach much closer than this to get the imaginary values. This makes it appear certain that the greater broadening on the more refrangible side, if it exists, must be assigned to some other cause than imaginary values of the refractive index.

The medium is exceptionally interesting in that its dispersion can be represented throughout the entire range of wave-lengths without taking the extinction coefficient into account, as is always necessary in the case of solids and liquids when in the vicinity of the absorption band.

Selective Reflection by an Absorbing Gas. — Planck's theory of absorption is based upon the supposition that the energy, taken from the oncoming waves, is re-emitted laterally by the resonators. Though this re-emission only occurs in exceptional cases, we do find it in some instances. As will be shown farther on, sodium vapor, when illuminated with sodium light, emits the absorbed wave-lengths without change. The emission is diffuse, however, that is, it is scattered in all directions. Radiation of this nature I have called resonance radiation, to distinguish it from fluorescence, in which case there is a change of wave-length.

It appeared highly probable that if the molecular resonators were packed closely enough together, the secondary wavelets which they emit, having a definite phase relation, would unite into a single wave, and the scattered light would disappear, regular reflection of the light taking its place. Repeated efforts to discover the phenomenon with sodium vapor yielded negative results, since it was impossible to obtain the vapor at great density with a sharply defined surface, on account of its corrosive action upon the transparent walls of the containing vessel.

The phenomenon was finally discovered by heating mercury to

a high temperature in a bulb of fused quartz. The vapor has a very strong absorption line at $\lambda = 2536$ in the ultra-violet.

To separate the images formed by reflection from the inner and outer surfaces of the bulb, the necks were made of very thick-walled tubing, so that the walls of the bulb were prismatic as

Fig. 287.

shown in Fig. 287. These bulbs were made especially for the work by Heraeus of Hanau, and were found most satisfactory.

A good-sized globule of mercury was placed in a bulb which was then highly exhausted and sealed.

It seemed best to begin by using light of exactly the right frequency, that is, of a wave-length identical with that of the absorption line. The light from the mercury arc in a quartz tube shows a strong line of exactly the right frequency. The bulb was mounted

Fig. 288.

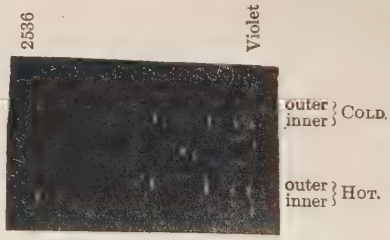


Fig. 289.

in a small tube of thin steel provided with an oval aperture in the side: the ends were closed with disks of asbestos board, one of which supported the neck of the bulb, as shown in Fig. 287. The steel tube was heated by two Bunsen burners, usually to a full red heat, and the mercury arc placed as close as possible to the aperture and a little to one side, so that its image appeared reflected in the tapering neck of the bulb, as shown in the figure. The flame of the burner must play over the aperture to prevent condensation of mercury drops at the point where reflection occurs.

By properly choosing the direction in which the bulb was viewed, the reflection from the outer surface disappeared, and the slit of a small quartz spectrograph was directed towards the bright image of the arc reflected from the inner surface of the wall.

A number of photographs of the spectrum of the reflected light were taken, the first with the bulb cold, the succeeding ones at gradually increasing temperatures. It was found that the relative intensity of the 2536.7 line in the spectrum of the reflected light increased rapidly as the temperature of the bulb increased. A series of these photographs is reproduced in Fig. 288. The time of

exposure was less in the case of the red-hot bulb, in spite of which the line 2536 is very strong while the other lines do not appear at all.

Previous work having shown, however, that mercury vapor emits scattered resonance radiation, when stimulated by the light of the 2536 line, it was next necessary to prove that it was not a diffuse radiation which had caused the increased intensity of the line. A very nice method was found of proving this. If our eyes were sensitive to the ultra-violet light, we should see, in the case of the diffuse emission, the entire surface of the bulb glowing with light, while if the radiation was regularly (*i.e.* specularly) reflected by the vapor, we should see merely the small image of the arc increase in brilliancy.

The steel tube was mounted vertically in such a position that the two images due to reflection from the inner and outer surfaces of the front wall of the bulb appeared one above the other. Inasmuch as the mercury arc consisted of a narrow vertical column of light, these images appeared as narrow vertical lines of light, and could be used in place of the slit of the spectrograph. The arc in this case was placed at a distance of about a metre from the bulb.

The slit tube of the spectrograph was removed, and the instrument placed in such a position that the two reflected images occupied the position previously occupied by the slit. On exposing a plate we obtain two spectra one above the other, the one that of the light reflected from the outer surface, the other that of the light reflected from the inner surface.

Two photographs were taken, one with the bulb cold, the other with the bulb red-hot. The latter showed that the image of the arc as seen in the 2536 light had been increased tremendously in brilliancy by the presence of the mercury vapor. The two photographs are reproduced in Fig. 289, the spectrum image formed by the light of wave-length 2536 being indicated. The light reflected from the inner surface is not as intense as that reflected from the outer, consequently one of the spectra is weaker than the other in each case. The 2536 line is, however, many times brighter in the spectrum of the image formed by the inner surface of the bulb when hot.

This experiment shows that the mercury vapor reflects light of this particular wave-length in much the same way as would a coating of silver on the inside of the bulb.

Experiments were next undertaken to ascertain how nearly the frequency of the light must agree with that of the absorption band in order that metallic reflection should take place.

It was found that the spectrum of the iron arc showed a group of closely packed lines exactly in the region required, and it was accordingly put in place of the mercury arc. The slit was replaced on the quartz spectrograph, and photographs of the reflected image of the arc were taken with the bulb cold and heated to different temperatures. A very remarkable discovery was at once made, for it turned out that the iron line which was metallically reflected (2535.67) was about one Ångström unit on the short wave-length side of the absorption line. As the temperature and vapor density

increased, a second iron line was strongly reflected, this one coinciding almost exactly with the absorption line. It is in reality a double line, with wave-lengths 2536.90 and 2537.21.

To make absolutely sure that no error had occurred, I photographed the spectrum of the iron arc, passing the light through mercury vapor at different densities.

The iron line which first disappeared was the double one, which was not reflected until the mercury vapor was at its greatest density. The line which was metallically reflected by the vapor at a lesser density was not absorbed by the vapor, even when its density was so great that four or five lines on the long-wave-length side of the line first absorbed were completely blotted out. This can be better understood by reference to Fig. 290. In the upper line we have the group of iron lines in question. The next line shows the absorption by Hg vapor when its density is small, the fourth iron line (double, mean λ 2537) having disappeared. Just below is the absorption by dense vapor, the absorption having extended towards the visible region of the spectrum (much farther than can be indicated in the figure), the third iron line being still transmitted, however. Below this is the reflection from Hg vapor at about ten

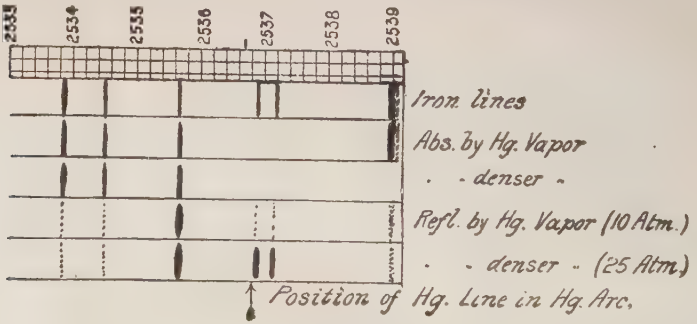


Fig. 290.

atmospheres, the third iron line being strongly reflected. In the lower line we have the reflection by vapor at 25 atmospheres, the third and fourth (double) iron lines appearing strongly reflected.

The explanation of the more powerful reflection of the shorter iron line is probably as follows :

The 2536.7 line shows powerful anomalous dispersion. The refractive index, in its immediate vicinity, on the short-wave-length side is much below unity, probably as low as .5 or even less close to the line. In the case of light going from a rare to a dense medium, a high value of the index for the latter is accompanied by strong reflection. When, however, the ray goes from dense to rare (quartz to mercury vapor), as in the present case, it is a low value of the index for the latter which is accompanied by strong reflection. Now the index for mercury vapor is not far from unity for the entire spectrum with the exception of very narrow regions bordering the 2536 line. On the long-wave-length side the index may be about

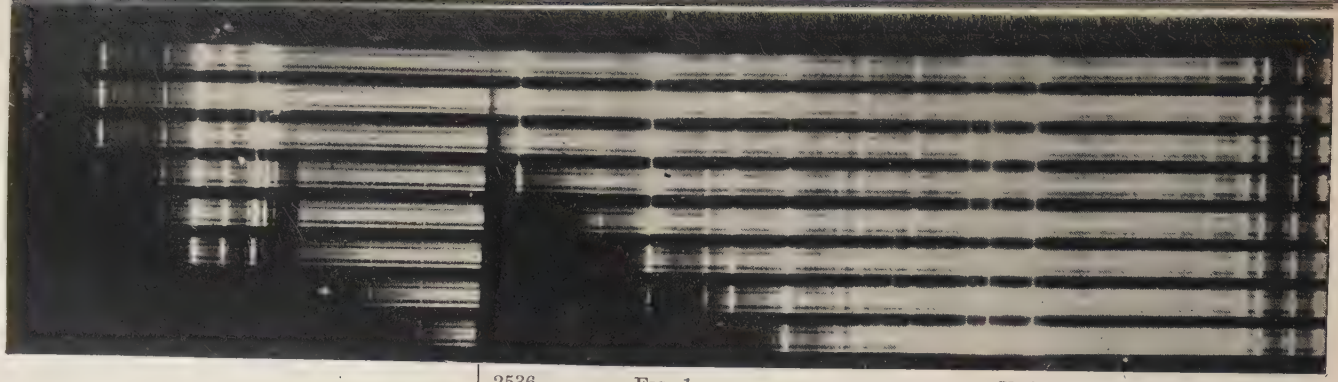


FIG. 1.

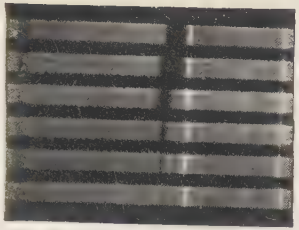


FIG. 2.

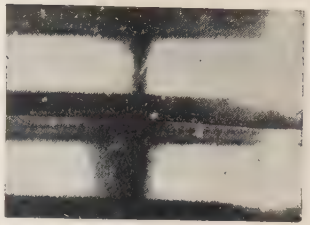


FIG. 3.

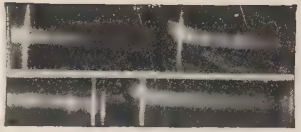


FIG. 4.

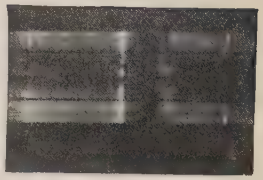


FIG. 5.

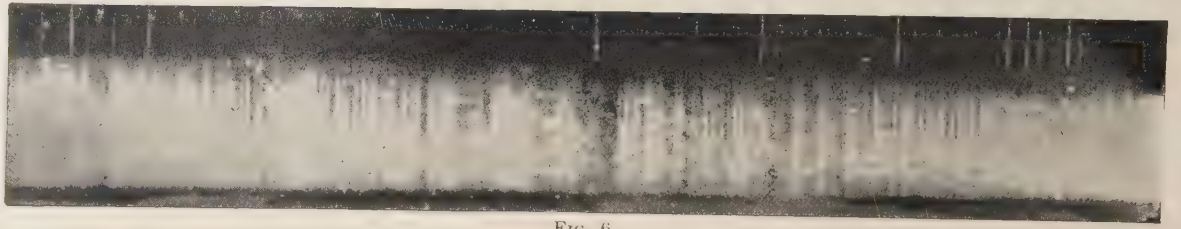


FIG. 6.

that of quartz, consequently there is no reflection at all here, at least none depending upon the relative values of n. On the other side, where the index is considerably below unity, the reflection will be much more powerful than when the bulb is filled with air. This means that the band of metallic reflection will be shifted slightly towards the region of *shorter* wave-length with respect to the true position of the absorption line, as was found to be the case.

An attempt will be made to get rid of this effect by using polarized light, reflected from the quartz-mercury vapor surface at the proper angle. This should give us rest-strahlen of the same wave-length as that of the absorption line.

One very important point in connection with the specular reflection from an absorbing vapor is the very great density necessary before the phenomenon is exhibited. In a gas at atmospheric pressure the molecules are in such close proximity that there are about 80 to the wave-length, *i.e.* 6400 in a square the sides of which are equal to a light-wave in length. This would appear to be more than sufficient for the application of Huygens's principle: experiment shows, however, that the reflection does not occur until we increase this density tenfold. It is probably a question of the suddenness with which the wave is stopped, rather than of packing the resonators close enough together to make the application of Huygens's principle possible. It appears probable that if the wave-train can penetrate to a depth of several wave-lengths into the medium, there will be no regular reflection, regardless of the proximity of the resonators. An analogous case is that of two media of very different refractive indices, between which the transition is gradual instead of abrupt. It has been held by some authorities that reflection will occur in this case since we can divide the transition layer into a large number of planes, each one of which will reflect a small amount. Even if this were the case, we should have complete destructive interference, for the wave-trains reflected from the hypothetical planes would be gradually and progressively displaced with reference to each other and give us zero for a resultant. The same thing may be considered as taking place in the case of the resonators. A rigorous theoretical treatment of the resultant effect of the radiations from a system of closely packed resonators, excited by plane-waves, considering several different depths of penetration, is much to be desired.

Absorption and Selective Dispersion of Hydrogen. — Hydrogen gas, under ordinary conditions, can be regarded as the most transparent substance known. Even the very short waves discovered by Schumann, which are powerfully absorbed by other gases, are freely transmitted by pure hydrogen. Hydrogen gas in the sun and stars shows strong absorption lines, coincident with the emission lines seen when the gas is excited by electrical discharges in vacuum tubes. Many attempts have been made to determine the conditions necessary for the exhibition of absorption in the laboratory.

Various observers have recorded seeing the red hydrogen line reversed in the spectrum of hydrogen tubes, but it is only within the last year or two that the exact conditions necessary for the

exhibition of the phenomenon have been determined and a quantitative investigation made. The absorption only takes place while the gas is in a condition of luminescence, *i.e.* while it is excited by the discharge. Pflüger (*Ann. der Phys.*, 24, 515, 1907) used as a source of light a capillary tube filled with hydrogen at low pressure, excited by powerful discharges from an induction coil with Leyden jars in the secondary circuit. The light from this tube, which showed the hydrogen lines much widened but not reversed, was passed through a wider tube, also containing hydrogen at low pressure and placed in the same electrical circuit. By this arrangement the emission of light by the source was confined to the moments during which the gas in the wider tube was in a condition to absorb. The red line was seen distinctly reversed. Soon after, Ladenburg and Loria (*Verh. der deutschen phys. Ges.*, 10, 858, 1908), using a similar arrangement, reversed both the red and green lines, and

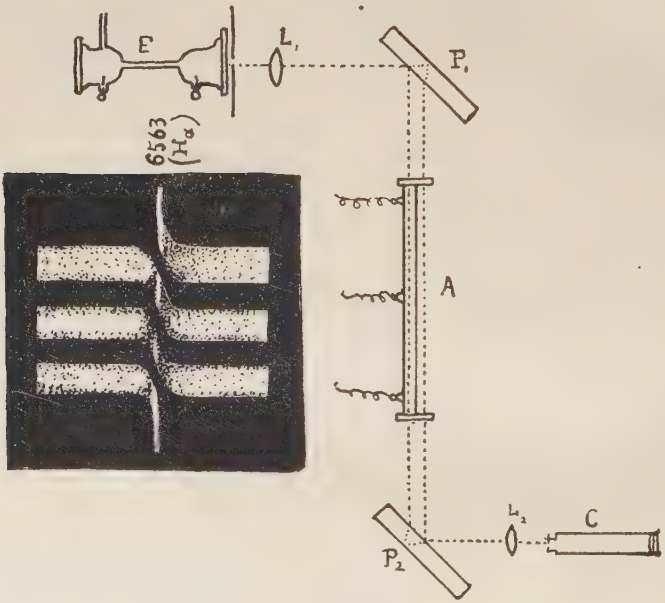


Fig. 291.

obtained photographs showing the selective dispersion and magnetic rotation in the vicinity of the red line.

The apparatus, as arranged for showing the dispersion of the gas, is shown in Fig. 291. Light from a capillary hydrogen tube E, excited by the discharge of a coil with Leyden jars in circuit, is rendered parallel by a lens L, and divided into two beams by the mirror of a Jamin (or other) interferometer P_1. One beam passes down the second (absorption) hydrogen tube A, the other along its side, traversing, however, the projecting parts of the glass plates used for closing the tube. The beams are united by the second Jamin plate P^2 and the horizontal interference fringes focussed

upon the slit of a spectroscope C by the lens L_2. On account of its small bore, the tube E emits, in addition to the hydrogen lines, a good deal of continuous spectrum; consequently we see in the spectroscope a spectrum traversed by horizontal interference bands. If now the absorption tube A, which has a wide bore, and contains hydrogen at 3 or 4 mms. pressure, is excited by being included in the same electrical circuit with E, the interference bands are bent away in opposite directions, to the right and left of the red hydrogen line *Ha*, as shown in the figure. A bright line runs down the centre, since only one of the interfering beams passes down the absorption tube. This experiment shows us that the passage of an electrical discharge through hydrogen gives rise to the formation of dispersion electrons, which are not present in the gas normally.

Hydrogen, then, when ionized, or brought into a state of luminescence by the electrical discharge, has quite different optical properties from ordinary hydrogen. Sodium vapor possesses these properties normally, in the absence of any *electrical* stimulus, and *probably* in the absence of any excitation coming from without, though it is possible that an ionization necessary for selective absorption and dispersion results from the passage through the vapor of the light necessary for observing the phenomena. The dispersion does not, however, depend upon the intensity of the transmitted light, which shows that the effect is very small, if it exists at all.

From the magnitude of the effect in hydrogen, Ladenburg and Loria calculated that the number of dispersion electrons per cubic centimetre was roughly $4 \cdot 10^{12}$, while the number of molecules was $2 \cdot 10^{17}$; in other words only one dispersion electron was formed for every 50,000 molecules. In the case of sodium vapor, Hallo and Geiger found a ratio of 200: 1. It would be extremely interesting to ascertain how long the dispersion electrons persist after the discharge ceases. This could probably be accomplished by arranging matters so that a very small time interval could be introduced between the excitations of the two tubes. If the duration of the discharges could be made brief enough, *i.e.* of the order of magnitude of the duration of a single oscillation of a condenser of small capacity, the necessary time interval might be introduced by putting the source E at a distance, and rendering its light parallel by a lens. If the discharges occured simultaneously, the light from E would reach A a very small fraction of a second later, owing to the finite time required by the light in travelling from E to A. An experiment analogous to this was made by Abraham and Lemoine, and will be described in the Chapter on Electro-Optics.

CHAPTER XV

ABSORPTION OF LIGHT

The transmission of light through a material medium is always accompanied by a certain amount of absorption, regardless of the color or wave-length of the light. Media which we commonly speak of as transparent, if not employed in too great thickness, transmit without appreciable absorption the range of wave-lengths comprised within the region of the visible spectrum.

In general, however, they exercise powerful absorption in the infra-red and ultra-violet regions, and if a sufficiently great thickness is employed, absorption will be found present even in the range of visible radiations. Pure water, which is one of the most transparent substances which we have, in long columns appears distinctly blue, showing that it absorbs more or less completely the red end of the spectrum. The same is true of most varieties of glass. The definition " transparent " is thus seen to be purely arbitrary, there being no such thing in nature as a perfectly transparent substance.

The character of the absorption exerted by any substance can be best observed by receiving the transmitted light on the slit of a spectroscope, when dark regions will be observed in the spectrum, corresponding in position to the wave-lengths absorbed. If the absorbing medium is moulded into the form of a wedge, which is placed in contact with the slit of the instrument, we can observe at once the effect of increased thickness, the form of the absorption curve being pictured in the spectrum. In general, it will be found that as the thickness increases, the absorption band widens out. One edge of the spectrum shows us the absorption of a thin layer, the other edge that of a thick layer, intervening portions corresponding to intermediate thicknesses. The resultant curve is sometimes symmetrical, but more often not so, and we shall see, when we come to consider the theory of absorption, that the form of this curve depends upon a number of different factors. The absorption spectra of about 150 aniline dyes have been photographed and published in the form of an atlas by Uhler and Wood. A wedge-shaped layer of the liquid was used, contained in a quartz cell which was placed in contact with the slit of a large grating spectroscope. Photographs were in this way obtained showing the position and forms of all absorption bands, both in the visible and ultra-violet regions. These photographs are extremely useful in the preparation of screens for absorbing particular regions of the spectrum. Three of these pictures are reproduced in Fig. 292, and show the absorption of nitroso-dimethyl-aniline, auramine, and potassium permanganate. We shall first examine the phenomenon of absorption in a general way, and then in its relation to other closely re-

lated phenomena, such as dispersion, emission, and the transformation of the absorbed radiations into other types of energy.

At the beginning of the subject we shall find it convenient to distinguish between two types of absorption: general, in which the absorbing power is very nearly the same for all wave-lengths, at least over a fairly wide range; and selective, when the absorbed region is more or less limited in extent. The absorption of metal films and lamp-black represents the first type fairly well. The

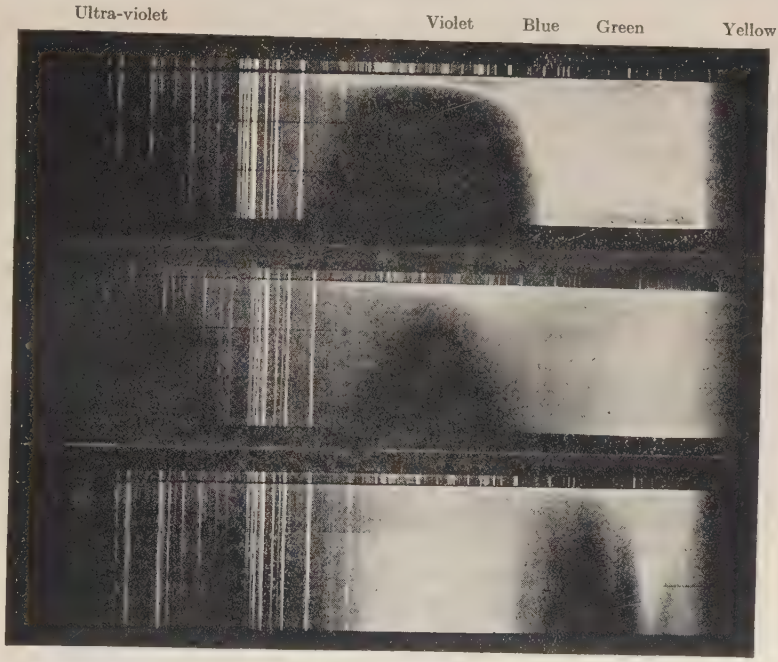


FIG. 292.

light transmitted through thin layers differs but slightly in its composition from the original light, and exhibits therefore but little color. Of course there are exceptions, for, as is well known, thin films of gold transmit an excess of green light, while silver is fairly transparent to the ultra-violet. Aniline dyes, and, in fact, all colored media, represent the second type, certain colors being freely transmitted, while others are strongly absorbed. When we come to consider the theory of the phenomena, we shall see that the causes of the absorption are radically different in the two cases, though in many cases both conditions may occur simultaneously in one and the same medium.

Laws of Absorption: Lambert's and Beer's. — Lambert's law states that each layer of equal thickness absorbs an equal fraction of the light which traverses it. If we consider layers of the thickness of a single molecule, we can say that each molecule absorbs an equal fraction of the light which passes by it.

In a solution, then, the absorption depends upon the concentration and thickness of the layer traversed. Unit layer and unit concentration absorb in the same degree as a layer of thickness 2 with half the concentration.

Calling the absorption coefficient of unit concentration a, the thickness d, and the concentration c, we have, if J_0 is the original intensity,

$$J = J_0 a^{dc} \text{ (Beer's law).}$$

No exceptions have ever been found to Lambert's law which could not be attributed to experimental error. Beer's law holds, however, only when the absorbing power of a molecule is not influenced by the proximity of its neighbors, which is not always the case, especially for gases, as we shall see presently.

Coefficient of Transmission: Dichromatism. — If the absorbing medium is homogeneous, the quantity of light of a given wave-length which is absorbed will be proportional to the thickness of the medium traversed. If we represent the intensity of the light that enters the front surface of the medium by I, the intensity after transmission through unit thickness can be represented by Ia, in which a is a fraction depending on the nature of the medium and the wave-length of the light. If the same fraction is absorbed by each successive layer, it is clear that the intensity, after traversing a thickness ϵ of the medium, will be Ia^{ϵ}, the quantity a being called the coefficient of transmission.

The coefficient of transmission varies with the color, and the emergent light is therefore colored. In the case of most absorbing media the color of the transmitted light does not depend to any great degree on the thickness, the depth or saturation merely increasing. In some cases, however, the color depends on the thickness, thin layers, for example, appearing green, and thick layers red. Such substances are said to exhibit dichromatism. Some of the aniline dyes, or mixtures of them, show the phenomenon. Thin layers of a solution of cyanine appear blue, thick layers red. The addition of a little nitroso-dimethyl aniline to the solution gives us a green-red dichromatic liquid, as has been shown by Pflüger. The explanation of the change of tint is very simple. Suppose we have a substance which absorbs the yellow and blue. The transmitted light then consists of a mixture of red and green. Let us assume, as is usually the case, that the intensity of the green is greater than that of the red. Writing for these intensities $I_g > I_r$, and assuming that the coefficient of transmission of the green is less than that of the red, $a_g < a_r$, it follows that for small thicknesses $I_g a_g^{\epsilon}$ will be greater than $I_r a_r^{\epsilon}$, while the reverse will be true for thick layers. This is at once apparent if we call the original intensities of the green and red 100 and 50, and the coefficients of transmission .5 and .8, and calculate the intensities of the transmitted colors for several different thicknesses. They will be equal for a thickness ϵ given by the equation

$$I_g a_g^{\epsilon} = I_r a_r^{\epsilon},$$

or taking the logarithms of both sides,

$$\epsilon = \frac{\log I_g - \log I_r}{\log a_r - \log a_g}.$$

For this thickness the intensities of the red and green will be equal, and the color of the transmitted light will appear to be yellow, for a mixture of red and green light produces the sensation of yellow when mixed in proper proportions. An excellent mixture for illustrating this can be formed by dissolving "brilliant green" and "naphthaline yellow" in hot Canada balsam and pressing the mixture between two glass plates in the form of an acute prism. The balsam should be previously boiled down until a drop solidifies on cooling, and the dyes should not be added until the fluid has cooled somewhat, otherwise they are apt to decompose. The thin edge of the wedge will appear green, the thick edge red, and the intermediate portions, where we have equality of transmission, yellow.

If some of the same mixture is moulded into a prism of 20 or 30 degrees angle, the mechanism of dichromatism can be beautifully shown by observing a lamp flame through it. The prism will show the red image well separated from the green, and the latter will be found to be extinguished more rapidly than the former as the prism is moved laterally before the eye.

Our equations for color show us as well that the color of the transmitted light, for a given thickness, will vary with the composition of the original light. If the plate of stained balsam is examined by gas-light and then by daylight, it will be found that parts of it will appear red in the former and green in the latter case. A solution of cyanine and nitroso-dimethyl aniline in alcohol appears red by lamplight and bottle-green by daylight. The same phenomenon is exhibited by the gem Alexandrite, found in the Urals.

Body Color and Surface Color. — The colors of most natural objects result from absorption. The light penetrates their surfaces and then suffers internal reflections or refractions and emerges robbed of the rays which are most strongly absorbed. If this is to happen, it is clear that the substance must not be homogeneous, otherwise the reflections and refractions, which return the unabsorbed light, will not occur. It is thus incorrect to say that colored pigments reflect certain colors more strongly than others. If the pigment particles formed a continuous and homogeneous medium, no color whatever would appear in the reflected light, which would be white. If any color appeared, as it might if the pigment were a very powerful absorbent, it would be the tint complementary to the one exhibited by the powder. Cases of this nature we shall consider presently.

Since pigments produce color by absorption, it is at once apparent why a mixture of two pigments does not exhibit the color which we should obtain if we actually mixed the colored lights which they appear to reflect. The light reflected from the mixture is the residual color which remains after the dual absorption has taken place. For example, if we mix yellow light and blue light, we get

white, while a mixture of a blue and yellow pigment appears green. The reason of this is, that the yellow pigment absorbs the blue and violet, the blue pigment the red and yellow, the mixture absorbing everything except the green.

The nature of pigments can be well studied by preparing a number of beads of fused borax, colored with varying amounts of cobalt. If we powder a bead which appeared bright blue by transmitted light, we shall find that the powder is white, the reason being that the light in this case does not penetrate a sufficient thickness of the absorbing medium. A bead colored so dense as to appear black will, however, furnish us with a blue pigment when it is reduced to powder. Pigments, then, are very powerful absorbing media, and, if they could be obtained in homogeneous masses, would be intensely opaque, even in fairly thin sheets.

If we go on increasing the absorbing power, we shall finally observe a phenomenon of a different nature. The color, instead of being absorbed, is selectively reflected. Substances which possess this property are said to exhibit surface color. The aniline dyes are excellent examples. A dye which in solution absorbs green light, appearing purple by transmitted light, in the solid state reflects green light selectively. Absorption is, however, not the only factor which determines this selective reflection, and we often find misleading statements in text-books on optics, it being frequently stated that the wave-lengths most copiously reflected are the ones most strongly absorbed. This is by no means the case. Cyanine, for example, has a strong absorption band in the yellow, while the color of the selectively reflected light is purple, not so very different in hue from that of the transmitted light. If we examine the spectrum of the reflected light, we find a very dark band in the green, the centre being not far from wave-length .0005. The distribution of intensity in the rest of the spectrum is not very different from what it would be in the case of reflection from glass, which shows that the peculiar color of the dye is not so much due to a very powerful reflection of certain waves as it is to its almost complete refusal to reflect a certain region of the spectrum.

In the last chapter we have seen that, in the case of absorbing media, the reflecting power depends both upon the refractive index and the coefficient of absorption. Now, absorbing media have a high refractive index on the red side of the absorption band and a low index on the blue side, consequently the spectrum of the reflected light will be brightest on the red side of the absorption band, since for these wave-lengths we have a large coefficient of absorption and a high refractive index. On the blue side, however, the low value of the index diminishes the reflecting power more than the augmentation due to the powerful absorption. The hue of the surface color thus depends on the refractive index of the medium in which the substance is immersed, for it is the relative and not the absolute refractive index with which we are concerned. Cyanine in contact with glass exhibits a yellowish green surface color, much more nearly resembling the hue of the absorbed light. If the dye could be brought in contact with a transparent substance having

the same dispersion, the wave-lengths selectively reflected would be identical with those absorbed, since in this case the relative refractive index would be unity for all wave-lengths.

An excellent way of showing the variable reflecting power of a film of cyanine is to compare it with glass in different parts of the spectrum. A little of the melted dye is pressed between two plates of hot glass, which are separated when cold. A spot is selected where the film has a good optical surface, and this spot is left on the glass, the rest being cleaned off. By holding the plate in the spectrum formed by a prism or grating, the reflecting power of the two surfaces can be studied. In some parts of the spectrum the cyanine reflects more strongly than glass; in other regions the reverse is true, while at wave-length .0005 the cyanine refuses to reflect to such a degree that the film appears as a black spot on the blue field reflected by the glass.

It is interesting to note the difference in the surface color of the dye when the reflection takes place at the surface in contact with the glass. A very convenient way of showing the yellowish green color in this case is to press out a film of the molten dye on one surface of a prism of 8 or 10 degrees angle.[1] In this way the light reflected from the dye can be obtained uncontaminated with the light reflected from the first glass surface. The method is analogous to that employed by Lippmann in mounting his color photographs. The calculation of the curve of reflected intensities under these conditions makes a good exercise for the student.

Influence of Temperature on Absorption Bands. — As early as 1831 Brewster found that the colors of absorbing media were often changed by an elevation of temperature. The observations of Schönbein in 1852 that many substances such as mennige manganese oxide and mercury oxide became more strongly colored at high temperatures and less so at low temperatures.

He found that sulphur loses its color at $-50°$ and bromine at $-70°$.

Still more recently (1903) Dewar observed that solid fluorine, which is at first yellow, becomes white at a temperature of $-253°$. Chlorine, bromine, and iodine behave in the same way.

A study was made by Pulfrich [2] in 1892 of the effects of increased temperature on the optical properties of various kinds of glass as influenced by temperature changes. The expansion of the glass resulting from increased temperature was the only factor recognized in changing its refractive index, previous to this work.

Pulfrich found, however, that the migration of the ultra-violet absorption band towards the visible spectrum, with increasing temperature, exerted a great influence, increasing the refractive index and dispersion. The two factors oppose each other, and the one or the other can preponderate.

[1] A suitable prism can be made in half an hour by grinding down a piece of thick plate window glass. A strip of thick glass cemented along one edge will be all that is necessary to make the glass take the required form. Grind on a piece of glass with very coarse emery at first, then use finer grades, polishing with rouge at the end. Small scratches do no harm, and a high polish is not necessary.

[2] *Wied. Ann.*, 45, p. 609.

It would seem as if this circumstance might furnish us a means of making a glass free from temperature effects, which would be most useful for the construction of prisms.

The usual effect of an increase of temperature is a shift of the absorption band towards the red. A very good way of showing this is to place a small piece of metallic thallium in a glass tube and heat it strongly in a blast lamp. The metal oxidizes and unites with the glass, forming a glass which is blood-red while hot, and pale lemon-yellow when cold. Oxide of cerium is deep yellow when hot, and white when cold. This can be shown with a Welsbach mantle, illuminated with sun or arclight, and heated by brushing it lightly with a Bunsen flame. Some very interesting results have recently been obtained by Jean Becquerel, who examined the effects of very low temperatures on the narrow absorption bands of the minerals tysonite and xenotime, which contain didymium and erbium.

Some of the absorption bands, which at ordinary temperatures are forty Ångström units in width, at the temperature of liquid air break up into separate bands, which are sometimes not over two or three units wide, *i.e.* half the distance between the *D* lines of sodium. There was usually a slight shift towards the region of shorter wave-lengths, though in some instances the shift was in the opposite direction.

Influence of Solvent on Position of Absorption Bands. — Kundt (*Pogg. Ann.*, Jubelband, page 615, 1874) made an extensive study of the influence of the nature of the solvent upon the position of the absorption bands of the dissolved substance, and established the following law, which has been known as Kundt's law.

If one transparent solvent has a higher refractive and greater dispersive index than another, the absorption bands of a dissolved substance lie nearer the red end of the spectrum when it is dissolved in the former than when it is dissolved in the latter.

Kundt endeavored to determine whether the shift of the absorption band towards the red was due to the increased refractive index of the solvent or its greater dispersive power, but as dispersion and refractivity go hand-in-hand, so to speak, he was unable to arrive at any definite conclusion.

There appears to be evidence, also, that the position of an absorption band depends upon the concentration of the solution. The salts of didymium exhibit very sharp and intense bands, and Becquerel (*Compt. Rend.*, 102, page 106, 1886) found that one of the bands occurred at $\lambda = 5790$ when the refractive index of the solution was 1.439 and at 5745 when the index was 1.345.

A similar result was found by Stöckl in the case of potassium permanganate, except that the bands were shifted towards the blue in the dilute solution.

The very marked effect of the solvent upon the position of the absorption bands is well brought out in Fig. 7, page 15, in the case of the salts of cobalt. Very extensive observations have been published by H. C. Jones in the Carnegie Institute bulletins, one of especial interest being the remarkable advance up the spectrum

of the ultra-violet band of a solution of copper chloride caused by the addition of calcium chloride. The edge of the band is very sharp and it can be moved up gradually from $\lambda = 35$ to $\lambda = 52$ (see Fig. 7).

Very little is known about the subject from a theoretical point of view. An excellent résumé of the whole subject will be found in the 3d volume of Kayser's *Spectroscopy*, and the work just referred to has yielded a great deal of new material.

Absorption of Light by Gases and Vapors. — The absorption bands of solids and liquids, with the one or two exceptions mentioned, are broad and more or less ill defined. Gases and vapors, on the contrary, usually exhibit absorption lines of extremely small width, the spectrum of the transmitted light being crossed with fine black lines. This type of absorption was first observed by Brewster (*Pogg. Ann.*, xxviii.) in the case of nitric oxide (NO_2), in the absorption spectrum of which he found over 2000 dark lines, resembling the Fraunhofer lines in the solar spectrum. Similar lines are shown by many other vapors, bromine and iodine, for example. The vapor of sodium shows a pair of lines in the yellow, corresponding in position to the *D* lines of the solar spectrum: if the vapor is denser, as when evolved by heating the metal in an iron or glass tube, a host of other lines appear in the red and green portions of the spectrum, while at a full red heat practically all of the red, yellow, and green is absorbed, the color of the transmitted light being deep violet. Vapors also exhibit broad bands resembling those shown by liquids and solids. The yellow vapor of nitroso-dimethyl aniline has a broad absorption band in the violet, and shows no trace of any fine lines. Other gases show both types of bands simultaneously, chlorine, for example, which has a broad band in the violet and a large number of fine lines in the blue, green, and yellow regions.

Many vapors, which under ordinary circumstances show no trace of absorption, and appear colorless, exhibit the lines when great thicknesses are used. Jannsen observed them in the spectrum of light transmitted through a tube 37 metres long filled with dry steam.

Absorption by Sodium Vapor. — A very complete study of the absorption of this substance has been made by the author. The absorption is of two types. A Balmer series, which is characterized by a continuously decreasing distance between the lines, the spacing being represented by an empirical formula due to Balmer. The *D* lines form the first member of this series, the others being found in the ultra-violet. But seven members were known previously, these having been observed in the emission spectrum of the sodium arc, for in many cases substances which have absorption lines have corresponding emission lines. By studying the absorption of the vapor of the metal in a long steel tube filled with hydrogen, and heated to a high temperature, the number of lines in the series was raised to 48, the largest number ever found for any substance exceeding even that shown by hydrogen in the sun and certain stars, which shows about 30 lines.

The sodium series is shown in the Chapter on Magneto-Optics, sec-

tion on "Zeeman Effect and Spectral Series," Fig. 1, an enlargement, and Fig. 2, a contact print. Only the lower end of the series is shown in these two figures. The complete absorption spectrum of the vapor at different densities is shown in Fig. 3, the *D* lines falling at the right-hand edge. The series terminates in a *head* in which the lines are no longer resolvable. The last 22 lines, or nearly one-half of the entire series, fall within a region narrower than the distance betwee the *D* lines. These absorption lines affect the optical properties of the vapor, and selective dispersion probably occurs at all of them, though it has been observed only at the first three or four.

In addition to the Balmer series of lines there are a number of very complicated channelled absorption bands, which appear to be without influence upon the dispersion. One is located in the red and the other in the blue-green region, though with dense vapor they practically meet in the yellow. There are, in round numbers, about 6000 absorption lines in these bands in the visible spectrum. In the ultra-violet, each Balmer line, at least the first five or six, are accompanied by similar channelled absorption bands, the one surrounding the first ultra-violet line being shown in Plate VI., Fig. 6. In the visible spectrum the regions transmitted between the absorption lines are as narrow apparently as the emission lines of the iron arc which was photographed on the same plate.

This shows us that an absorbing mechanism may sift out from white light radiations, nearly, if not quite, as homogeneous as those emitted by glowing vapors in the arc or in vacuum tubes.

When we come to the subject of fluorescence we shall discover that it is possible to analyze this complicated spectrum by exciting certain groups of lines at a time, and we shall see further that there is some mechanical or electrical connection between the mechanism producing the Balmer lines, and the one which gives rise to the channelled spectra.

Effect of Density on Absorption Bands. — It has usually been assumed that the absorption produced by a given quantity of gas is independent of its density — in other words, that the absorption is a function only of the number of molecules lying in the path of the light. The important discovery was made by Ångström that the compression of a gas increased the intensity of its absorption, the mass of the gas traversed remaining the same. The experiment was made with carbonic acid, which has two strong absorption bands in the infra-red at wave-lengths 2.8 μ and 4.3 μ. The apparatus consisted of a long glass tube divided by a rock-salt plate into two compartments, 3 and 30 cms. in length respectively. The two compartments could be placed in communication by a glass tube furnished with a stopcock. The investigation was carried on in the following way:

(*a*) The two compartments were exhausted and the spectrum of the transmitted light investigated with a thermo-element.

(*b*) The small compartment was filled with carbonic acid at a pressure p_1 and the absorption a_1 determined, the other compartment remaining vacuous.

(*c*) The stopcock between the two compartments was now opened, and the gas was allowed to fill the entire tube. The pressure is now $p_2 = p_1 \frac{3}{3}$. The absorption a_2 was now determined and found to be less than a_1.

(*d*) A non-absorbing gas was now introduced into the tube until the total pressure of the gas mixture had the same value p_1, as that of the carbonic acid in experiment *a*. The absorption in this case a_3 was found to be equal to a_1, from which the following law was deduced: As the pressure decreases, the product of pressure and

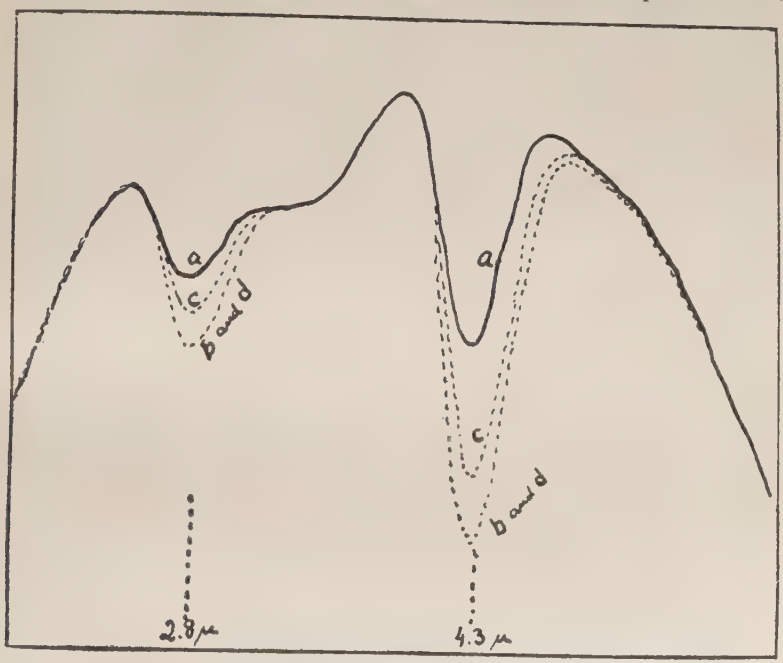


FIG. 293.

length of column remaining constant, the absorption decreases, returning, however, to its original value when an amount of a non-absorbing gas is added sufficient to give to the mixture a pressure equal to the original pressure. The absorption curves obtained in this way are shown in Fig. 293, in which the letters *a*, *b*, *c*, and *d* refer to the cases mentioned. Similar results were obtained with carbon monoxide. The presence of the CO_2 bands in the curve *a* are due to the gas normally present in the air of the room. Some experiments made by the author have shown that similar effects can be observed in the visible and ultra-violet regions with metallic vapors. In the experiments upon sodium vapor which were commenced in 1900, it was frequently observed that when the vapor was formed in a high vacuum, the admission of hydrogen gas caused a very marked widening of the *D* lines. No very definite conclusions could be drawn from this, however, since under the

conditions of the experiment it seemed probable that the presence of the hydrogen increased the actual amount of sodium vapor present in the tube by preventing its distillation to the cooler parts of the tube. Shortly after the publication of Ångström's paper, results of which there could be no doubt were obtained with mercury vapor. A small amount of mercury was introduced into a steel tube 3 metres long, the ends of which were closed with quartz plates. The tube was highly exhausted, and the light from a cadmium spark, rendered parallel by a quartz lens, was passed through the tube and focussed on the slit of a small quartz spectrograph. Even at room temperature the absorption line at wavelength 2536 could be photographed without difficulty. On admitting air to the tube, the absorption band became much heavier and wider. This experiment is not open to the objections which can be raised in the case of the experiment with sodium vapor, where only the central portion of the tube was heated, and there appears to be no question but what the same amount of mercury vapor is present in each case, the widening of the absorption band resulting from the increased pressure due to the admission of air. Ångström's work on carbonic acid, which has been verified by the author's work on sodium and mercury vapor, enables us to lay down the following facts regarding gas absorption:

1. Beer's law does not hold in general for gases.
2. If to a gas at a definite volume we add a foreign gas which does not act upon it chemically, its absorption is increased.
3. The absorption of a gas mixture is greater than the sum of the absorptions of its separate constituents, each one taken under its partial pressure.
4. The absorption of a gas mixture is, on the contrary, equal to the sum of the absorptions of its constituent parts, if each part is taken at a pressure equal to the total pressure of the mixture.

The absorption spectra investigated by Ångström were all band spectra, and he left the question open as to whether similar laws held for line spectra. The results obtained with sodium and mercury vapor appear to indicate that this is the case.

Absorption by Hydrogen Gas. — Reference has already been made in a preceding chapter to the work of Pflüger and of Ladenburg and Loria. An observation recorded by the author several years ago enables us to show the absorbing power of the gas for light of a wave-length corresponding to that of the red line without the aid of any special apparatus. A vacuum tube is arranged for end-on observation having a bore of about 3 mms. The pressure of the hydrogen in the tube should be 6 or 7 mms., and it should be excited by a large induction coil with a spark gap in the circuit. It will be noticed that the color of the discharge, seen through the side of the tube, is rose-red, but is bluish white when viewed end-on. This is due to the fact that the long column of glowing gas exercises a powerful absorption for the light of the red line, and little or no absorption for the other rays. That the dependence of color upon the direction in which the observation is made is not a result of the difference of total intensity can be

shown by viewing the end-on discharge through a pair of Nicol prisms, by which the intensity can be reduced until it is equal to that of the discharge, seen through the side of the tube. The

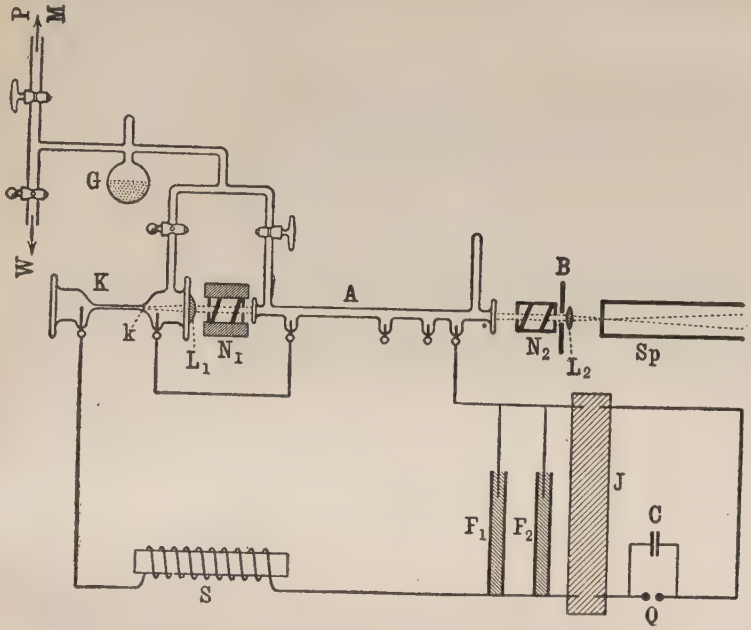


Fig. 294.

apparatus used by Ladenburg and Loria is represented in Fig. 294 which will be readily understood from what has been said in the previous chapter. The intensity of the light from the source K can be regulated at will by turning the Nicol prism N_1. Starting with the two Nicols crossed, by which the light from the source is cut out, one sees in the spectroscope only the bright emission lines from the discharge in the absorption tube A, which contains hydrogen at one mm. pressure. On turning the Nicol, one observed the spectrum of the source K, which is nearly continuous on account of the high pressure of the gas and the small diameter of the tube. Two dark lines appear at the edges of the bright lines H_a

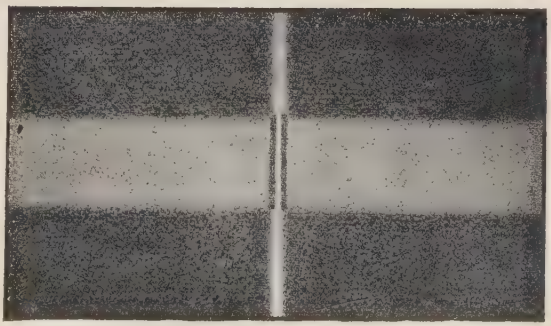


Fig. 295.

and H_β where they cut across the continuous spectrum, as shown in Fig. 295. By further turning of the Nicol, these lines widen and finally obliterate the bright line lying between them. In the second case, with the absorbing layer at a pressure of 25 mms. and 55 mms. in length, the dark line appeared in the middle of the bright line and gradually widened until the bright line was completely extinguished (Fig. 296).

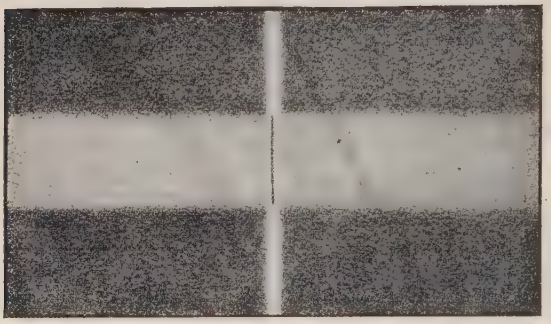


Fig. 296.

Form of Absorption Bands: Influence of Other Media. — Absorption bands are often very unsymmetrical in form, one of the most remarkable examples being found in the case of mercury vapor, which has been studied by the author (*Phil. Mag.*, August 1909).

The absorption spectrum of the vapor is shown on Plate VI., Fig. 1. A small globule of mercury was placed in a quartz bulb 3 cms. in diameter, which was thoroughly exhausted and sealed. The spectrograms were taken in succession, the temperature of the bulb being gradually raised. There are three distinct absorption bands, all in the ultra-violet region. The one at wave-length 2536 appears first as a pair of fine lines, not unlike the *D* lines in appearance, one of them ($\lambda = 2539.4$) relatively much weaker than the other ($\lambda = 2536.7$). As the density of the vapor increases, they fuse together, forming a single band which then widens in a remarkable manner towards the region of longer wave-length, its boundary on the other side remaining almost fixed in position.

If the bulb contains air or any other chemically inert gas at atmospheric pressure, the band (2536) widens symmetrically at first, attaining a width of about 8 Ångström units. Beyond this point a further increase in the density of the mercury vapor causes a widening in one direction only, as is the case when the vapor is *in vacuo*.

If, instead of sealing the mercury up in a bulb filled with air, we place it in a quartz flask provided with a long neck and gradually raise the flame below the flask, we get a remarkable series of spectrograms shown in Fig. 2. The lower spectrum was taken first, the upper last. The band widens and then appears to drift towards the longer wave-lengths, without further increase in width. This apparent drift is due to the expulsion of the air by the boiling mercury, the band contracting on the short wave-length side, as it widens on the other.

This action of the air in modifying the appearance, and in some cases the apparent position of the absorption band, cannot be

attributed to chemical action, for the same effect was found with hydrogen, nitrogen, and helium. As we shall see presently, the vapor is deprived of its power of fluorescing when it is mixed with another gas.

It was found, by making a more careful study, that the effect of air upon the band at 2536 was a little more complicated than was at first supposed. *In vacuo* the broadening is almost entirely in the direction of longer wave-lengths. If air is present, a hazy band appears on the short wave-length side of the line, and if the time of exposure and vapor density are just right, the band and line are separated by a narrow strip slightly lighter than the band. Photographs of the band obtained with the mercury vapor *in vacuo* and in air are shown in Fig. 3, Plate VI. The selective dispersion of the vapor at the 2536 line is shown in Fig. 4, and the magnetic rotation in Fig. 5. The reader is referred to the original paper for a fuller discussion.

Absorption by Porous Surfaces. — The absorption of light at surfaces formed of lampblack or finely divided metals such as platinum black is accompanied by very little reflection. The question naturally arises as to why a metal with a high reflecting power can, under certain conditions, appear nearly dead black. The roughness of the surface will not account for the fact, for matt surfaces of electrolytically deposited silver appear as white as plaster. Chemically precipitated silver, on the other hand, appears black.

The phenomenon is to be referred to the condition of the surface. Consider a bunch of polished steel needles, turned with their points towards the light. Rays falling upon the surface formed by the points will be reflected down into the interstices between the needles, and practically none of it will escape or be reflected back. A portion is absorbed at each reflection, and the large number of reflections reduce the intensity rapidly to zero. Surfaces of lampblack (soot) and platinum black can be considered as porous, the pores acting as light "traps." The energy penetrates into the spongy mass by multiple reflection, and is speedily transformed into heat by absorption. If the pores are closed up by compressing the mass, its reflecting power is increased or wholly restored. If, too, the angle of incidence is too large to admit of downward reflection into the mass, the light is more or less completely reflected. A surface of smoked glass reflects very perfectly at large angles of incidence, and at the same time yields a sharply defined image of the source of light, as we have seen in the Chapter on Huygens's Principle.

Absorption by Metals. — While metals possess in general a high reflecting power, a considerable portion of the incident energy penetrates the surface and is absorbed. In the case of steel, for example, nearly one-half of the light is lost by absorption, while even silver absorbs 5%. In the majority of cases the absorption is general, that is, it is not confined to a narrow range of wave-lengths, as is the case with the substances which we have examined thus far. It is to a certain extent selective, however, as is best illustrated by

gold-leaf, which appears green by transmitted light, and thin films of chemically deposited silver, which appear blue. These silver films are fairly transparent to a limited range of ultra-violet radiations, which lies just beyond the limit of the visible spectrum. It is probable that the absorption of metals is due to the presence of two types of electrons — conducting ones which are free to move indefinitely under the influence of a steady electric force, and non-conducting ones, which are similar to those which we have considered in the Chapter on Dispersion. We shall examine the effects of these two types more in detail in the Chapter on the Optical Properties of Metals.

Absorption Spectra of the Rare Earths. — While the absorption spectra of solids and liquids show in general only broad diffuse bands, some marked exceptions occur in the case of salts of the rare earths, erbium, praeseodymium and neodymium, europium, holmium, etc. These show both in solution and in the solid state very narrow bands or even sharp lines. Most remarkable is the absorption spectrum of a solution of europium, which is shown in Fig. 297. The chloride of this element has finally been obtained in the pure state by Urbain, after 6000 fractional crystallizations, a labor of 10 years. Two of the absorption lines in the green are as sharp and narrow as the Fraunhofer lines in the solar spectrum. Neodymium has an equally fine line in the ultra-violet. The bands of erbium are broader. They appear not only in the absorption spectra of solutions, but even in the spectrum of the oxide when illuminated by sunlight. Dip a platinum wire into a concentrated solution of ebrium chloride, and heat it in a Bunsen burner. Hold the oxide bead in the sunlight and examine it with a spectroscope. If the bead is heated white-hot in the flame, *bright* bands appear in place of the dark ones. These substances will be more fully discussed in the Chapter on Radiation.

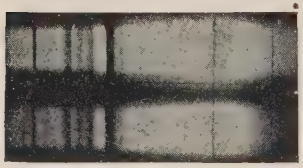


FIG. 297.

Theories of Absorption. — In the Chapter on Dispersion we have seen that the presence of electrons of definite periods of vibration, gives to a medium the property of absorbing radiations, the frequencies of which agree with those of the electrons.

The introduction of a term which expressed the vibration of the electron as accompanied by friction was sufficient to explain the absorption of energy. Such a treatment is rather loose, however. If the energy is transformed into heat by this assumed " friction," we must explain how the average molecular velocity (which is our definition of temperature) is raised. Clearly, friction, as we ordinarily understand it, occurring within the molecule cannot affect the velocity of the molecule. Moreover, a vibration of the electron excited by the light-waves can be considered as affecting the molecular velocity only, in virtue of some action occurring at the moment when two molecules are in collision. It is conceivable that, at the moment of impact, the energy stored in the molecule in the form of vibrational

energy of the electron may be consumed in increasing the velocity of rebound of the molecules. Practically nothing is really known about the exact nature of the transformation of absorbed radiations. It can be shown, however, that a damping of the vibration of the electron may result from molecule impacts, the result being similar to that which would follow if its vibrations were accompanied by friction. The introduction of the friction term into the equations is thus not wholly unwarranted. The absorbed energy may be spent in effecting chemical changes within the substance, as we shall see in the Chapter on the Transformation of Absorbed Radiations. There is, however, another action which may well be expected to happen, namely, a remission of energy by the vibrating electron in the form of ether waves of the same period as those absorbed.

Unfortunately, experimental confirmation of such re-emission is very meagre. Fluorescence is a totally different phenomenon, for in this case the lengths of the emitted waves are different from those of the exciting ones. The vapor of sodium, however, appears to exhibit the phenomena, for when illuminated by a powerful beam of sodium light, it scatters in all directions a feeble light of apparently the same wave-length. Mercury vapor behaves in the same way when illuminated with the light of the 2536 line. These cases will be further considered under "Fluorescence."

If the electrons become centres of radiation, giving back their energy to the ether, it would appear at first sight as if no absorption would result, for by Huygens's principle the secondary waves originating from their vibration would reconstitute a wave of a type similar to the exciting wave. We are obliged to assume, however, that the electrons would send out energy in *all* directions, however; consequently this case would differ from that in which we determined the resultant of the secondary disturbances on a wave-front, in that we should have a back-wave travelling in the reversed direction, as well as a forward wave. A theoretical treatment of absorption and dispersion has been given by Planck,[1] based solely on this assumed radiation of the electrons. The reduction in the intensity of the advancing wave can be explained perfectly by it, but there is no true absorption as ordinarily understood, the energy being sent back in the opposite direction. This lateral radiation is assumed by Planck to be the only cause of the damping of the vibration of the electrons, an assumption which is hardly justifiable if we require a complete explanation of all of the phenomena of absorption, but which is perfectly allowable if we wish merely to find out how far such a radiation can account for the observed effects.

One great advantage of this conception is, that it neither involves the introduction of any new constant into the equations, nor ascribes the damping to some action of which the physical significance is obscure.

There are, however, strong objections which can be brought up against the theory. In the first place, as has already been said, this lateral emission is only found in one or two rare instances. If

[1] *Berlin Acad.*, Berlin, 1903–04.

the electrons lie close enough together, we might explain this by applying the principle of Huygens to their radiations, a lateral emission failing to take place for the same reason that a beam of light radiates no lateral disturbance. In this case there must be a return of a wave of the same type as the exciting wave, *i.e.* selective reflection. This occurs, as we have seen, in the case of solid films of strongly absorbing media, giving rise to surface color. If the molecules are too far apart for the application of the above principle, then the light should be scattered in all directions. Yet solutions of strongly absorbing media show absolutely no trace of such a lateral emission.

Planck's treatment is instructive, however, in that it gives us a clear idea of the effect of such secondary radiations arising from the electrons, upon the propagation of the exciting wave.

It might at first sight appear as if such a treatment would lead us to the conclusion that the resultant of all the wavelets coming from the electrons would be identical with the original wave, which is the same thing as saying that there is no absorption. Planck has, however, taken into account the fact that the phase of a resonator lags a quarter of a period behind that of the exciting waves, and that there is in addition a quarter-period difference between the phase of the resonator and that of the wave which it emits. The resultant wave emitted by the collection of resonators will thus be half a wavelength behind the exciting wave, which will gradually be reduced in intensity by interference with the wave originating in the resonator system. The resonators, however, emit spherical disturbances; consequently there will be an envelope propagated in the backward direction, and since there is no primary wave travelling in this direction there will be nothing to interfere with its propagation. The medium thus sends back towards the source of light a frequency corresponding to the frequency of the vibrating electrons. This is nothing more than selective reflection.

In the opinion of the author this holds, however, only for a single layer of resonators. When we have a large number uniformly distributed in space, the propagation of a wave in the backward direction, originating in the mass taken as a whole, appears to be out of the question, the case being analogous to the absence of reflection at a boundary when the transition in refractive index is gradual. (See selective reflection by a gas, in the Chapter on Dispersion Theory.)

While Planck's treatment of the subject cannot very well be given in full, we can study to advantage some of the results to which it leads. His final equations, expressing n the refractive index, and κ the extinction coefficient, are

$$n^2 = \frac{\sqrt{(\alpha^2+\beta^2-\alpha)^2+\beta^2}+(\alpha^2+\beta^2-\alpha)}{2\,(\alpha^2+\beta^2)},$$

$$\kappa^2 = \frac{\sqrt{(\alpha^2+\beta^2-\alpha)^2+\beta^2}-(\alpha^2+\beta^2-\alpha)}{2\,(\alpha^2+\beta^2)},$$

in which $\alpha = \dfrac{\lambda_0{}^2-(1-g)\lambda^2}{3\,g\lambda^2}$ and $\beta = \dfrac{\sigma\lambda_0}{3\,\pi g\lambda}$.

λ is the wave-length of the incident light, λ_0 the wave-length in vacuum which the resonators would emit if thrown into vibration, σ the logarithmic decrement of the resonators, and $g = \frac{\sigma N \lambda_0^3}{4\pi^3}$, N being the number of resonators in unit volume.

Since the last quantity depends on the distribution of the resonators in space, we are able to trace the relation existing between the refraction and absorption and the density of the medium. Planck's equation has shown that the form of the extinction curve depends on N, *i.e.* on the proximity of the resonators, and that its maximum lies on the longer wave-length side of λ_0^2. The equation calls for three different types of curves, according to the values assigned to $\frac{g}{\sigma}$. Consider first the case where $\frac{g}{\sigma}$ has a large value, which will occur when N is large. The curve found in this case shows that the extinction coefficient rises gradually with increasing λ, attains the value 1 for $\lambda^2 = \frac{\lambda_0^2}{1+\frac{g}{2}}$, the value $\sqrt{2}$ for $\lambda^2 = \lambda_0^2$, reaching its maximum just before $\lambda^2 = \frac{\lambda_0^2}{1-g}$; beyond this point it descends much more rapidly than it rose, the curve being unsymmetrical (Fig. 298).

Planck defines the region of metallic absorption as the region within which $\kappa > 1$. It extends from $\lambda^2 = \frac{\lambda_0^2}{1+\frac{g}{2}}$ to $\lambda^2 = \frac{\lambda_0^2}{1-g}$. Its width is seen to depend only upon g, while the value of the maximum absorption depends upon σ as well. Considering λ_0 and σ as constant, and increasing g by bringing the resonators closer together, we obviously increase the width of the band of metallic absorption, the band widening unsymmetrically, however. It spreads towards the region of shorter wave-lengths, but cannot pass the point determined by $\lambda^2 = \frac{2}{2}\lambda_0^2$, while in the other direction there is no limit. At the same time the point of maximum extinction is shifted towards the region of longer wave-lengths, and the maximum value of κ is increased.

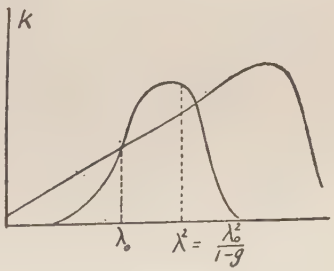


FIG. 298.

The interesting question now arises as to whether the shift of the maximum point may bring it about that the curve for a large value of N, instead of completely enclosing the curve for a small value of N, may cut the latter at two points, as shown in Fig. 298. If this were the case, we should have the value of the extinction coefficient decreased for a certain value of λ to the left of λ_0, by increasing the density of the medium, a circumstance which would be in viola-

tion of Beer's law of the proportionality between extinction and density. Planck finds this to be the case, for $\kappa = \sqrt{2}$ for $\lambda = \lambda_0$ regardless of the value of g. The family of curves obtained by assigning to N different values thus pass through a common point situated at λ_0. For any two curves there is a second point of intersection, and between these two points an increase of density is accompanied by a decrease in the value of the extinction coefficient.

If $\frac{g}{\sigma}$ has a small value, as will be the case when N is small, the curve is found to be symmetrical, with its maximum at λ_0. Metallic reflection does not occur, κ being less than unity for all values of λ, and if g is gradually increased, κ increases proportionally for all wave-lengths, and Beer's law of absorption is followed. The absorption band is narrow, and no shift is produced by increasing the density. A third type of curve, intermediate between the other two, is found for intermediate values of $\frac{g}{\sigma}$.

Planck's treatment differs from that of Drude and Lorentz in that the damping is referred solely to radiation. Drude's formula calls for a maximum value of κ for wave-length λ_0, and an increase in the width of the band of metallic reflection towards the region of shorter wave-lengths only.

Sufficient experimental observations are not yet availalbe to make a choice between the different treatments possible. An experimental proof of Planck's theory would be difficult, since he assumes at the outset that the resonators are at rest and separated by distances which are large in comparison to their dimensions.

A mathematical discussion of absorption by Lamb (*Camb. Phil. Soc. Trans.*, vol. xviii., Stokes Commemoration, 1900) is extremely interesting as showing the enormous checking power which a single resonator (gas molecule) can exert upon advancing radiation of a frequency very nearly, but not quite, that of its own free period. Lamb draws attention also to the fact that it has not been possible to represent the dissipation of radiant energy by an absorbing medium except vaguely by means of a frictional coefficient. He considers, as does Planck, that the energy is scattered, *i.e.* remitted by the resonator, which he assumes to be a spherical molecule of enormous specific inductive capacity, with one or more free periods of vibration.

The main result of his investigation is stated as follows: "For every free period of vibration (with a wave-length sufficiently large in comparison with the diameter of a molecule) there is a corresponding period (almost exactly but not quite coincident with it) of maximum dissipation for incident waves. When the incident waves have precisely this latter period, the rate at which energy is carried outwards by the scattered waves is, in terms of the energy-flux in primary waves,

$$\frac{2n+1}{2\pi}\lambda^2, \quad \ldots\ldots\ldots \quad (1)$$

where λ is the wave-length and n is the order of the spherical har-

monic component of the incident waves which is effective. In the particular case of $n=1$, this is equal to $.477\,\lambda^2$. Hence in the case of exact synchronism, each molecule of a gas would, if it acted independently, divert per unit time nearly half as much energy as in the primary waves crosses a square whose side is equal to the wave-length. Since under ordinary atmospheric conditions a cube whose side is equal to the wave-length of sodium light would contain something like 5×10^6 molecules, it is evident that a gaseous medium of the constitution here postulated would be practically impenetrable to radiations of the particular wave-length."

"It is found, moreover, on examination that the region of abnormal absorption in the spectrum is very narrowly defined, and that an exceedingly minute change in the wave-length enormously reduces the scattering."

It may be remarked that the law expressed by the formula (1) is of very general character, and is independent of the special nature of the conditions to be satisfied at the surface of the sphere. It presents itself in the elastic-solid theory; and again in the much simpler acoustical problem, where there is synchronism between plane-waves of sound and a vibrating sphere on which they impinge.

CHAPTER XVI

OPTICAL PROPERTIES OF METALS

The laws which govern the optical behavior of metals are very different from those which hold in the case of transparent substances and substances showing strong selective absorption. Metals exhibit in general a very high reflecting power, and in some cases, also, a powerful absorbing action. We must distinguish carefully between absorption and reflection. If we examine a thin film of silver deposited on glass, we find that it transmits little or no light, and we might therefore come to the conclusion that the metal absorbs strongly. A little further investigation will show, however, that over 90% of the light has been reflected, the remaining 10% having been absorbed. Platinum, however, has a much lower reflecting power, combined with equally great opacity; consequently we may regard platinum as possessing a stronger absorbing power than silver. Gold is, however, as compared to silver and platinum, extremely transparent, ordinary gold-leaf transmitting no inconsiderable amount of green light.

While metallic absorption is not in general characterized by such marked selective action as is the case with the colored media which we have studied, the phenomenon of selective absorption is by no means absent, as the strong coloration of the light transmitted by gold-leaf proves: silver, too, while it appears to favor equally all wave-lengths in the visible spectrum, is fairly transparent to ultra-violet radiation comprised within the range .305–.320: its reflecting power is correspondingly low for these same wave-lengths.

The colors which metals exhibit are due to a selective reflecting power, which is especially marked in the case of gold and copper. If two gilded glass plates are mounted parallel at a distance of a centimetre or so, with the reflecting surfaces opposed and a beam of light caused to travel back and forth between them, suffering a number of reflections, the surface color is still more marked, the filament of an incandescent appearing as red as if seen through ruby glass. This method of bringing out the surface color is analogous to the method of Rubens and Nichols for isolating long heat-waves by multiple reflections from surfaces of quartz and rock-salt. The same phenomenon can be seen in a less marked degree by looking into the interior of a gilded goblet, the bottom of which appears of a fairly deep-red color.

Glass plates can be easily gilded by exposing them to the discharge from a gold cathode in a vacuum tube; plates gilded by the application of gold-leaf, in the manner employed for lettering on windows, would doubtless answer as well, and can be readily obtained from a sign-painter.

The phenomenon of elliptical polarization is exhibited in a high degree when light polarized in an azimuth of 45° is reflected obliquely from a metal surface.

In the case of transparent substances, we have seen that this result can occur if the surface is contaminated with a film, but in the case of metallic reflection the presence of a surface film is not necessary. The first attempt to establish equations for the reflection of light by metallic surfaces was made by MacCullagh, who assumed that the reflection was of a nature similar to that of total reflection in the case of transparent substances, and assigned therefore to the metals an imaginary refractive index. Cauchy also developed an equation practically identical with MacCullagh's, and more elaborate treatments were subsequently given by Beer, Eisenlohr, and Lundquist (*Pogg. Ann.*, xcii., page 402; civ., page 368; clii., page 398).

Against the methods employed by these investigators, Ketteler raised the objection that the development of the equation for the reflected wave necessitated the existence of a longitudinal disturbance in the ether, against the existence of which there exists the strongest experimental evidence. Ketteler (*Wied. Ann.*, B. i. and iii.) developed an equation along different lines, and by adopting different boundary conditions avoided the necessity of a longitudinal wave. These earlier treatments, based upon the elastic-solid theory, have been completely supplanted by the methods of the electro-magnetic theory, which are much more intelligible for the reason that the physical actions which are going on are definitely specified. According to our present views, we regard metals as substances in which electrons exist, which are capable of continuous movement under the action of a steady electro-motive force. Heretofore we have regarded the electrons as bound to positions of equilibrium by forces of restitution, experiencing only a slight change of position under the action of a steady electric force. Upon the removal of the force, the electron returns to its original position.

We will now investigate the behavior of free electrons under the action of the rapidly alternating electrical forces of light waves.

Electro-Magnetic Theory of Absorption. — We have already discussed the propagation of waves in a medium which is a perfect insulator, in which the current is proportional to $\frac{\partial X}{\partial t}$. In such a medium the current may consist of two parts, a displacement current in the ether represented by $\frac{1}{4\pi}\frac{\partial X}{\partial t}$, and a convection current due to the motions of the electrons inside the atoms. It is clear that the current will cease as soon as the electric force ceases to vary; for example, if the electric force rises from 0 to X, the electron will be displaced a certain amount, the motion constituting a convection current, but if the force then remain steady, there will be no further motion of the electron, and the current will cease.

We will now investigate the propagation of waves in a medium

which is not a perfect insulator. In such a medium a current will be set up under the influence of a steady electric force, which will be proportional to the force X instead of to the rate of change of X. We may think of this current as due to the motion of free electrons, which will drift along under the influence of the force, giving rise to a conduction current represented by σX, in which σ represents the absolute conductivity measured electro-statically. The current in an imperfect insulator will then be made up of two parts, one proportional to $\frac{\partial X}{\partial t}$ and the other proportional to X, the former vanishing under the influence of a steady field.

If we have a periodic electric force, as in light-waves, both currents will be present, and we may have absorption or a transformation of energy from two distinct causes. If the vibration of the electrons which are not free is accompanied with something akin to friction, there will be a heating similar to the heating of the dielectric of a condenser when it is rapidly charged and discharged. This type of absorption has been discussed in the Chapter on Dispersion. There may in addition be an ohmic heating, similar to the heating of wires by steady currents. This we may think of as due perhaps to the impacts of the free electrons or changes in the potential energy, as electrons are torn-off atoms under the influence of the electric force.

If we limit ourselves to plane-polarized plane-waves, we may write for the current parallel to the x axis

$$j_x = \frac{\epsilon}{4\pi}\frac{\partial X}{\partial t} + \sigma X. \tag{1}$$

The current is thus seen to be made up of two parts: a displacement current in the ether, resulting only from an electrical force which changes with the time, and a conduction current proportional to X which does not depend for its existence upon fluctuations of X. It will be remembered that the modern theory of metallic conduction regards the electrical current as a streaming motion of negatively charged electrons, which are free to move in the metallic conductor under the action of a steady electro-motive force.

No further modifications than the one introduced into equation (1) are needed, and the fundamental Maxwell equations $\frac{4\pi j_x}{c} = \frac{\partial \gamma}{\partial z} - \frac{\partial \beta}{\partial y}$, etc., and $\frac{1}{c}\frac{\partial \alpha}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$, etc., still hold, if we write the permeability $\mu = 1$.

This we are justified in doing, even in the case of the strongly magnetic metals such as iron, nickel, and cobalt, for experiments indicate that the magnetic molecules are unable to follow the very rapid changes involved in the case of light-waves.

The boundary conditions may be written as before, $X_1 = X_2$, $Y_1 = Y_2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$.

For the present we shall concern ourselves only with the absorp-

tion due to ohmic heating, *i.e.* resulting from the term σX, which represents the conduction current.

If $j_x = \frac{\epsilon}{4\pi}\frac{\partial X}{\partial t}$, we have $\frac{\partial^2 X}{\partial t^2} = \frac{c^2}{\epsilon}\frac{\partial^2 X}{\partial z^2}$ (eq. (12)).

If $j_x = \sigma X$, $\frac{\partial X}{\partial t} = \frac{1}{\sigma}\frac{\partial j_x}{\partial t}$, and since $\frac{\partial j_x}{\partial t} = \frac{c^2}{4\pi}\frac{\partial^2 X}{\partial z^2}$ Chapter on Reflection Theory

and $\left(\frac{4\pi j_x}{c} = \frac{\epsilon}{c}\frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}\right)$, (eq. 7, 8 and 12)

$$\frac{\partial X}{\partial t} = \frac{c^2}{4\pi\sigma}\frac{\partial^2 X}{\partial z^2}.$$

From (1) we then have

$$\frac{\epsilon}{c^2}\frac{\partial^2 X}{\partial t^2} + \frac{4\pi\sigma}{c^2}\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial z^2}.$$

When applied to harmonic motion, this equation has for its solution

$$X = Ae^{i\frac{2\pi}{T}(t - mz)}, \text{ in which } m \text{ is complex.} \qquad (2)$$

Differentiating (2), $\frac{\partial X}{\partial t} = i\frac{2\pi}{T}X$, and substituting in (1),

$$j_x = \frac{\epsilon}{4\pi}\frac{\partial X}{\partial t} + \sigma\frac{\partial X}{\partial t}\frac{T}{i\,2\pi} = \left(\frac{\epsilon}{4\pi} + \frac{\sigma T}{i\,2\pi}\right)\frac{\partial X}{\partial t}.$$

Multiplying the numerator and denominator of the second term in the parenthesis by $2i$ gives us $j_x = \left(\frac{\epsilon}{4\pi} - \frac{2i\sigma T}{4\pi}\right)\frac{\partial X}{\partial t}$,

$$j_x = \frac{\epsilon - 2i\sigma T}{4\pi}\frac{\partial X}{\partial t}.$$

For perfect insulators $j_x = \frac{\epsilon}{4\pi}\frac{\partial X}{\partial t}$, the only difference being that in the case of absorbing media the real constant ϵ of the equation for insulators is replaced by the complex constant $\epsilon' = \epsilon - 2i\sigma T$.

Substituting this complex dielectric constant in equation (12), page 359, gives us

$$\frac{\epsilon'}{c^2}\frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial z^2}. \qquad (3)$$

Differentiating (2), $\frac{\partial^2 X}{\partial t^2} = -\frac{4\pi^2}{T^2}X$, $\frac{\partial^2 X}{\partial z^2} = -\frac{4\pi^2 m^2}{T^2}X$,

and substituting in (3), $\frac{\epsilon'}{c^2}\frac{4\pi^2}{T^2}X = \frac{4\pi^2 m^2}{T^2}X$,

$$\frac{\epsilon'}{c^2} = m^2, \qquad (4)$$

and since ϵ' is complex, m is also complex.

Now m has the dimension of a reciprocal velocity, and we may write $m = \frac{1-i\kappa}{V}$, in which V is the velocity of propagation of the wave in the absorbing medium.

Substituting this value in (2),

$$X = Ae^{\frac{2\pi i}{T}\left(t-\frac{(1-i\kappa)}{V}z\right)} = Ae^{\frac{2\pi i}{T}\left(t-\frac{z-i\kappa z}{V}\right)} = Ae^{\frac{2\pi it}{T}-\frac{2\pi iz}{TV}-\frac{2\pi\kappa z}{TV}},$$

in which $TV = \lambda$.

$$X = Ae^{-2\pi\kappa\frac{z}{\lambda}}e^{2\pi i\left(\frac{t}{T}-\frac{z}{\lambda}\right)}. \qquad (5)$$

In this expression $Ae^{-2\pi\kappa\frac{z}{\lambda}}$ represents the amplitude, which clearly decreases as z increases.

After traversing a thickness equal to the wave-length λ, the amplitude has decreased by the amount $e^{-2\pi\kappa}$. The constant κ is the measure of the absorption, and is called the absorption index.

If we call $\frac{c}{V} = n$ the refractive index of the medium, we have from equation (4)

$$\frac{\epsilon'}{c^2} = \left(\frac{1-i\kappa}{V}\right)^2,$$

$$\epsilon' = \frac{c^2}{V^2}(1-\kappa^2-2i\kappa);$$

$$\therefore \epsilon' = n^2(1-\kappa^2-2i\kappa); \qquad (5)$$

and since $$\epsilon' = \epsilon - i2\sigma T = n^2 - n^2\kappa^2 - 2n^2i\kappa,$$

we get by equating the real and imaginary parts,

$$(5a) \qquad n^2(1-\kappa^2) = \epsilon, \quad n^2\kappa = \sigma T.$$

This last relation is not in agreement with facts, however, as we shall see presently. The reason of this is that, in the present treatment, we have not taken into account the influence which the vibrating electrons have upon the propagation of the disturbance. We have shown that in the case of imperfect insulators we have a complex dielectric constant, due to the conduction term σX. In the treatment of dispersion we have seen that a complex dielectric constant results from the presence of vibrating electrons, even when the term σX is absent, *i.e.* when no conducting electrons are present, and absorption will occur if the vibration of the electron is accompanied with friction. There are thus two distinct types of absorption — one caused by conducting electrons, the other by vibrating electrons, the motion of which is accompanied by friction.

Metallic Reflection. — Consider now the case in which plane-polarized light is incident on a polished metal surface, the plane of polarization making an angle of 45° with the plane of incidence. Referring to the treatment of reflection given previously, and making use of the same symbols, we have in the present case

$E_p = E_s$, and

$$\sin \chi = \frac{\sin \Phi}{\sqrt{\epsilon'}}. \tag{6}$$

This is simply the previous equation with the complex dielectric constant ϵ' substituted for ϵ.

By equation (24), Chap. on Refl. $\frac{R_p}{R_s} = -\frac{E_p \cos(\Phi+\chi)}{E_s \cos(\Phi-\chi)}$,

we have, since $E_p = E_s$, $\frac{R_p}{R_s} = -\frac{\cos(\Phi+\chi)}{\cos(\Phi-\chi)}$,

in which R_p and R_s are complex quantities.

Let $R_p = R_p e^{i\delta_p}$ and $R_s = R_s e^{i\delta_s}$, then

$$\frac{R_p}{R_s} = \rho e^{i\Delta}, \text{ in which } \delta_p - \delta_s = \Delta \text{ and } \rho = \frac{R_p}{R_s};$$

$$\therefore \frac{R_p}{R_s} = \rho e^{i\Delta} = -\frac{\cos(\Phi+\chi)}{\cos(\Phi-\chi)}. \tag{7}$$

Since the right-hand member of the above equation is complex, Δ must differ from zero, and there is a phase-difference between the two components of the reflected light, which produces elliptical polarization.

We will now determine how this phase-difference and the accompanying elliptical polarization vary with the angle of incidence.

$$\rho e^{i\Delta} = -\frac{\cos\Phi\cos\chi - \sin\Phi\sin\chi}{\cos\Phi\cos\chi + \sin\Phi\sin\chi},$$

multiplying this equation by the denominator and transposing the terms gives us

$$\frac{1+\rho e^{i\Delta}}{1-\rho e^{i\Delta}} = \frac{\sin\Phi\sin\chi}{\cos\Phi\cos\chi} = \tan\Phi\frac{\sin\chi}{\cos\chi}$$

$$= \tan\Phi\frac{\sin\Phi}{\sqrt{\epsilon'}\cos\chi} = \frac{\tan\Phi\sin\Phi}{\sqrt{\epsilon'}\cos^2\chi} = \frac{\tan\Phi\sin\Phi}{\sqrt{\epsilon'\left(1-\frac{\sin^2\Phi}{\epsilon'}\right)}};$$

$$\therefore \frac{1+\rho e^{i\Delta}}{1-\rho e^{i\Delta}} = \frac{\sin\Phi\tan\Phi}{\sqrt{\epsilon'-\sin^2\Phi}}. \tag{8}$$

At normal incidence $\Phi = 0$, $\rho e^{i\Delta} = -1$, *i.e.* $\Delta = 0$ and $\rho = -1$, or the wave is reflected with a change of sign, but with no phase-difference between the components; the light therefore remains plane-polarized. The reflected waves form, by interference with the incident waves, a system of stationary waves, and since the reflection is accompanied by a change of sign, we shall have a node at the reflecting surface.

At grazing incidence $\Phi = 90$, $\rho e^{i\Delta} = 1$, *i.e.* $\Delta = 0$ and $\rho = 1$, or reflection occurs without change of sign and without elliptical polarization.

The ellipticity will be greatest for the angle of incidence for which $\Delta = \frac{\pi}{2}$. At this angle we have $e^{i\Delta} = i$, since $e^{i\Delta} = \cos\Delta + i\sin\Delta = 0 + i$.

This angle is termed the angle of principal incidence, and we will designate it by $\bar{\Phi}$.

$$\frac{1+i\bar{\rho}}{1-i\bar{\rho}} = \frac{\sin\bar{\Phi}\tan\bar{\Phi}}{\sqrt{\epsilon'-\sin^2\Phi}}. \tag{9}$$

Multiplying this equation by its complex conjugate

$$\frac{1-i\bar{\rho}}{1+i\bar{\rho}} = \frac{\sin\bar{\Phi}\tan\bar{\Phi}}{\sqrt{\epsilon''-\sin^2\Phi}},$$

in which ϵ'' is the complex conjugate of ϵ'.

The left-hand member thus becomes $\frac{1+\rho^2}{1+\rho^2} = 1$, and we have

$$\sin^4\bar{\Phi}\tan^4\bar{\Phi} = (\epsilon'-\sin^2\Phi)(\epsilon''-\sin^2\Phi),$$

or, substituting the value which we have found for ϵ',

$$\sin^4\bar{\Phi}\cdot\tan^4\bar{\Phi} = n^4(1+\kappa^2)^2 - 2n^2(1-\kappa^2)\sin^2\bar{\Phi} + \sin^4\Phi. \tag{10}$$

In the case of metals, $n^2(1+\kappa^2)$ has a value much greater than unity (from 8 to 30), consequently it is sufficient if we take only the first term of the right-hand member;

$$\therefore \sin\bar{\Phi}\tan\bar{\Phi} = n\sqrt{1+\kappa^2}. \tag{11}$$

We can derive this expression from eq. (8) if we disregard $\sin^2\Phi$ in comparison to ϵ,

$$\frac{1+i\rho}{1-i\rho} = \frac{\sin\bar{\Phi}\tan\bar{\Phi}}{\sqrt{\epsilon'}},$$

in which $\sqrt{\epsilon'} = n(1-i\kappa)$, see eq. (5).

Multiplying this by its complex conjugate,

$$1 = \frac{\sin^2\bar{\Phi}\tan^2\bar{\Phi}}{(n-ni\kappa)(n+ni\kappa)} = \frac{\sin^2\bar{\Phi}\tan^2\bar{\Phi}}{n^2+n^2\kappa^2},$$

$$\sin^2\bar{\Phi}\tan^2\bar{\Phi} = n^2+n^2\kappa^2 = n^2(1+\kappa^2);$$

$$\therefore \sin\bar{\Phi}\tan\bar{\Phi} = n\sqrt{1+\kappa^2}.$$

If by means of a Babinet compensator, which annuls the phase-difference introduced by the metallic reflection at any angle of incidence, we convert the elliptical vibration into a plane-polarized one, the plane of polarization will make an angle Ψ with the plane of incidence, and we have the relation $\rho = \tan\Psi$.

If we can establish equations connecting n and κ with Φ, Δ and Ψ, it is clear that we can determine the refractive index n and the extinction coefficient of a metal by observations made on the state of polarization of the reflected light. Such methods are called katop-

tric methods in contrast to the dioptric methods employed in the case of transparent substances. They have been largely used in the determination of the optical properties of intensely opaque matter. They are obviously not as reliable as the dioptric methods, for, as we have seen, the ellipticity of the reflected light is profoundly affected by the presence of surface films, and we can seldom be certain that such films are not present in the case of metallic surfaces.

We will now establish relations between n and κ, and Φ, Δ, and Ψ.

We require first an expression for $\frac{1-\rho e^{i\Delta}}{1+\rho e^{i\Delta}}$, in terms of Ψ and Δ, which expression we shall substitute in eq. (8).

Since $\rho = \tan\Psi = \frac{\sin\Psi}{\cos\Psi}$, we may write the above expression,

$$\frac{1-\tan\Psi(\cos\Delta+i\sin\Delta)}{1+\tan\Psi(\cos\Delta+i\sin\Delta)} = \frac{\cos\Psi-\sin\Psi(\cos\Delta, \text{ etc.})}{\cos\Psi+\sin\Psi(\cos\Delta, \text{ etc.})}.$$

Writing $\cos\Psi = a$, $\cos\Delta\sin\Psi = b$, $\sin\Delta\sin\Psi = c$, the above expression is of the form

$$\frac{a-b-ic}{a+b+ic}.$$

Multiplying the numerator and denominator by $a+b-ic$ gives us

$$\frac{a^2-b^2-c^2-2aic}{a^2+b^2+c^2+2ab},$$

in which we substitute the values for a, b, and c, and find (writing for $\cos^2\Psi - \sin^2\Psi$ its equivalent $\cos 2\Psi$), since $\cos^2\Delta+\sin^2\Delta = 1$ and neglecting $\sin^2\Phi$,

$$\frac{1-\rho e^{i\Delta}}{1+\rho e^{i\Delta}} = \frac{\cos 2\Psi - i\sin 2\Psi\sin\Delta}{+\sin 2\Psi\cos\Delta} = \frac{n(1-i\kappa)}{\sin\Phi\tan\Phi}, \text{ see eq. (5)};$$

$\therefore$ equating the real and imaginary parts,

$$\frac{n\kappa}{\sin\Phi\tan\Phi} = \frac{\sin\Delta\sin 2\Psi}{1+\cos\Delta\sin 2\Psi}, \quad \frac{n}{\sin\Phi\tan\Phi} = \frac{\cos 2\Psi}{1+\cos\Delta\sin 2\Psi},$$

and, dividing the first by the second,

$$\frac{n\kappa}{n} = \frac{\sin\Delta\sin 2\Psi}{\cos 2\Psi},$$

$$\kappa = \sin\Delta\tan 2\Psi, \quad n = \sin\Phi\tan\Phi\frac{\cos 2\Psi}{1+\cos\Delta\sin 2\Psi},$$

$$n^2(1+\kappa^2) = \sin^2\Phi\tan^2\Phi\frac{1-\cos\Delta\sin 2\Psi}{1+\cos\Delta\sin 2\Psi}.$$

If the light is incident at the angle of principal incidence $\bar{\Phi}$, the corresponding angle $\bar{\Psi}$ is called the principal azimuth. In this case

$$\kappa = \tan 2\bar{\Psi} \text{ (since } \sin\Delta = 1).$$

Determination of Principal Incidence and Azimuth. — Since reflection at the angle of principal incidence converts circularly polarized into plane-polarized light, we can easily determine the angle by reflecting a circular vibration, obtained by means of a quarter-wave plate or Fresnel rhomb, from the metallic surface, and determining the angle of incidence at which the reflected light can be quenched by a Nicol prism. The principal azimuth is the angle which the direction of the plane vibration (short diagonal of the Nicol) makes with the plane of incidence. Or we may start with plane-polarized light and employ a Babinet compensator (with its wedges set so as to displace the central fringe through a distance corresponding to a phase-difference of a quarter of a period) to analyze the reflected light, observing the angle of incidence at which the central fringe returns to the central position. The principal azimuth is determined by observing the angle through which the second Nicol has to be rotated to make the central fringe black. Colored light, obtained by passing sunlight through colored glasses or solutions, should be employed, as the optical constants are a function of the wave-length.

Determination of the Change of Phase by Reflection. — The change of phase produced by perpendicular reflection at a silver surface is best determined by means of the Michelson interferometer, one of the back mirrors being coated over half its surface with a film of silver. The shift between the fringes on opposite sides of the dividing line gives us the measure of the phase-difference between disturbances reflected from glass and silver, if we take into account the slight shift due to the shortening of one optical path by the material thickness of the film. The thickness can be determined by silvering the plate in the manner shown in Fig. 299. The upper half a, b is first heavily silvered, the lower portion being covered with a glass plate. The left-hand portion a, c is then treated in a similar manner. The phase-change due to reflection will be the same at the surfaces a and b, the shift of the fringes at the boundary being due to the difference in thickness, which is obviously the thickness of c. The shift at the boundary between c and d is next observed, attention being paid to the direction of shift in each case. From these data it is easy to calculate the phase-change produced by reflection, if we remember that reflection at a glass surface produces a change of half a period. (See Mann's *Manual of Advanced Optics.*)

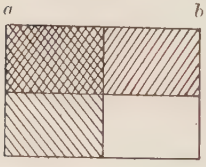


FIG. 299.

Absorption and Reflection by Metals. — The ratio of the reflected to the incident intensity at normal incidence is called the coefficient of reflection. For the ratio of the amplitude we have in the case of transparent media (see page 368),

$$\frac{R_p}{E_p} = \frac{n-1}{n+1}.$$

In the present case we write for n, the square root of the complex dielectric constant ϵ', since $\sqrt{\epsilon'} = n(1-i\kappa)$.

In the present case, then, we have

$$\frac{\mathbf{R}_p}{E_p} = \frac{R_p e^{i\delta_p}}{E_p} = \frac{n(1-i\kappa)-1}{n(1-i\kappa)+1}.$$

Multiplying this equation by its complex conjugate, we obtain for the coefficient

$$R = \frac{n^2(1+\kappa^2)+1-2n}{n^2(1+\kappa^2)+1+2n}.$$

In the case of metals $2n$ is small in comparison to $n^2(1+\kappa^2)$. If we neglect it, we get $R=1$. In the case of silver 95 % of the incident light is reflected, *i.e.* $R=.95$. The reflecting power will be greatest for the wave-lengths for which κ has its greatest value; thus gold reflects red light much more powerfully than green. The best determinations of the reflecting power of different metals are those made by Rubens and Hagen (*Ann. der Physik*, i. 352, 1900; 8, page 1, 1902). The values which they found for a number of the more common metals are given in the table on next page, which will be found useful for reference in all optical work involving the reflection of light from metallic surfaces.

It is apparent from the table that the reflecting power of silver in contact with glass is somewhat less than that of silver in contact with air.

The same is true for mercury.

An easy way of exhibiting the loss of light by reflection from a metal is to half fill a test-tube with mercury and plunge it in a jar of clean water; the light reflected from the metal will appear quite dull in comparison with the light totally reflected at the glass air surface.

In the above table there will be found in the last column the values of R calculated from observations by katoptric methods (yellow light), *i.e.* calculated from Φ, Ψ, and Δ. They will be found in rough agreement with the values observed by Rubens.

In the case of silver the minimum at $\lambda=316$ is very remarkable, the reflecting power of the metal for this wave-length being about that of glass for yellow light.

The highest reflecting power appears to be possessed by metallic sodium, for which $R=99.7$ according to Drude. This value was calculated, however.

Effect of Thickness of Film. — The most recent investigations of the reflecting and absorbing power of metals has been made by Hagen and Rubens.

The films were deposited upon quartz plates, which permitted of investigations in the ultra-violet region, and their thickness determined by two different methods, weighing the film both before and after its conversion into the iodide and the interference method of Wernicke.

They first investigated the dependence of reflecting power upon the thickness.

Reflecting Powers in Per Cents of Various Metals and Alloys (Hagen and Rubens)

For λ =	251	288	305	316	326	338	357	385	420	450	500	550	600	650	700	800	900	1000	1200	1500 μμ	*R* Calculated
Silver	34.1	21.2	9.1	4.2	14.6	55.5	74.5	81.4	86.6	90.5	91.3	92.7	92.6	93.5	94.6	96.3		96.6		98.4	95.3
Platinum	33.8	38.8	39.8	—	41.4	—	43.4	45.4	51.8	54.7	58.4	61.1	64.2	66.3	69	70.3	73.6	75.5	77.7	79	70.1
Nickel	38	43	44	—	45	46	49	49	56	59	55	63	65	66	69	70	72	73	76	81	
Steel	33	35	37	—	40	—	45	48	52	54	55	55	55	56	58	58	—	62	68	72	58.5
Gold	39	34	32	—	28	—	28	27	29	33	47	74	84	89	92	95	96	97	97	97	85.1
Copper	26	24	25	—	25	—	27	28	33	37	44	48	72	80	83	89	92	93	95	94	73.2
Brashear alloy — 68 Cu, 32 Sn	30	38	42	—	—	—	51	53	56	60	63	64	64	66	67	71	74	74	75	76	
Schröder alloy — 66 Cu, 22 Sn, 12 Zn	40	48	50	—	54	—	57	60	62	63	63	63	64	65	67	71	73	75	76	76	
Magnalium — 69 A 31 Mg	67	70	72	—	75	—	81	84	83	83	83	83	83	82	83	84	—	84	—	83	
41 Cu, 26 Ni, 24 Sn, 8 Fe, 1 Sb	36	37	37	—	39	—	43	44	44	47	49	48	48	50	55	63	—	71	—	81	
Glass backed by silver										79 to 85	81 to 86	82 to 88	82 to 88	83 to 89	84 to 89						
Mercury													75	(Lord Rayleigh)							78.4
Glass backed by mercury										73	71	71	70	71	73						

The results for gold are shown graphically in Fig. 300, the ordinates representing reflecting power in percentages and the abscissæ the thicknesses of the film. We see from the curves that the reflecting power increases with the thickness, reaching its maximum value at about 80 $\mu\mu$ or .00008 mm., which is about one-eighth of the wave-length of red light, after which it remains constant. For red light this maximum value is 90 %; for green light it is less than 50 %. This explains the yellow color of gold and the green color of gold-leaf by transmitted light. In the ultra-violet it is necessary to use somewhat thicker films to get the full value of the reflecting power.

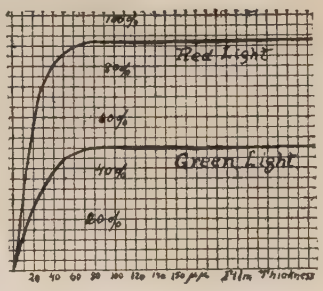


Fig. 300.

The absorption constant a we may define as the reciprocal of the thickness (measured in μ) which will reduce the intensity of the entering radiation to $\frac{1}{10}$ of its original value. If d is the thickness of a film, J the intensity of the entering radiation, and i the intensity of that which emerges, we have

$$\frac{i}{J}=10^{-ad} \text{ and } a=\frac{1}{d}\log\frac{J}{i}.$$

The extinction coefficient κ we define as follows:

$$\kappa=\frac{a\lambda}{4\pi M},$$

in which M is the modulus of the natural system of logarithms. Wernicke measured a and κ by using two films of different thickness d_1 and d_2 deposited on the same plate. If the loss by reflection is the same for the two films, and the ratio of the two transmitted intensities is b, we have

$$a=\frac{1}{d_1-d_2}\log b.$$

The thickness of the films was determined by transforming them into the transparent iodide of silver, by exposure to fumes of iodine, and measuring the thickness by an interference method.

To determine the light actually absorbed, we must measure the intensity of the transmitted light and compare it, not with the intensity of the incident light, but with that of the incident minus the reflected light.

Hagen and Rubens measured their intensities absolutely with the thermo-element and galvanometer. If p is the ratio of the intensities of the transmitted and incident light, $\log p$ must be a linear function of the thickness, providing that it is in every case capable of giving the maximum reflection.

Let J_0 = intensity of incident light,
J = intensity of light which enters film,
i = intensity of transmitted light,
R = reflecting power,

then $J_0(1-R) = J$,

and since $p = \frac{i}{J_0}$ and $a = \frac{1}{d}\log\frac{J}{i}$,

it follows that $p = \frac{i}{J}(1-R)$ and $\log p = \log\frac{i}{J} + \log(1-R)$

$$\log p = -ad + \log(1-R).$$

If we determine p for various thicknesses, and plot the values of $\log p$ as ordinates with the values of d as abscissæ, the points will lie in a straight line. Moreover, the tangent of the angle which this line makes with the axis along which we measure d, will be the value of the absorption constant a.

This was done for a number of metals, and the linear relation found in each case, which proves that the law of absorption holds for metals. The values of a and κ for the three metals silver, gold, and platinum for different wave-lengths are given in the following table:

λ	Silver		Gold		Platinum	
	a	κ	a	κ	a	κ
2.5 μ			37	17	28	13
2			42	15	30	11
1.5	45	12	41	11	32	8.9
1.2	47	10	40	8.8	33	7.3
1.0	44	8	37	6.9	35	6.5
.8	42	6.2	35	5.2	36	5.3
.7	43	5.5	32	4.1	37	4.8
.65	40	4.8	30	3.6	38	4.5
.60	38	4.2	26	2.9	38	4.1
.55	37	3.8	23	2.3	38	3.8
.50	35	3.2	22	2.	38	3.5
.45	31	2.6	21	1.7	37	3
.42	30	2.3	22	1.7	39	3
.385	25	1.8	25.8	1.8	39	2.7
.357	19	1.3	26.4	1.7	39	2.5
.338	13	.86				
.332	9	.55				
.326	7.5	.45	25	1.5	39	2.3
.321	7.2	.42				
.316	7.8	.45				
.310	11	.62				
.305	14	.79				
.288	19	1.00				
.251	22	1.00				
.221	16	.68				

Optical Constants of the Metals. — These may be determined by various methods. n and κ may be determined directly by dioptric methods, accurate measurements involving great difficulties, however, or they may be calculated from determinations of Φ and Ψ, the angles of principal incidence and azimuth.

Kundt was the first to determine directly the refractive indices of the metals. He employed exceedingly acute prisms deposited on glass by means of the cathode discharge or chemical means, and actually measured the deviation of light produced by them. In some cases n was found to be less than unity; in other words, the light was propagated in the metal at a higher velocity than *in vacuo*.

Drude gives the following table for the constants of a number of the more common metals:

Metal	nk	n	$\bar{\Phi}$	$\bar{\Psi}$	R
Silver	3.67	.18	75°	43°	95 %
Gold	2.82	.37	72°	41°	85
Platinum	4.26	2.06	78°	32°	70
Copper	2.62	.64	71°	39°	73
Steel	3.40	2.41	77°	28°	58
Sodium	2.61	.005	71°	45°	99.7
Mercury	4.96	1.73	79°	35°	78.4

If we compare the values given in the above table with the equation $\epsilon = n^2(1-\kappa^2)$, we find that ϵ is negative in every case, since $\kappa = \tan 2\Psi$, and $2\bar{\Psi}$ is greater than 45° in every case, *i.e.* $\kappa > 1$.

A negative dielectric constant has no meaning, however. Neither does the relation $n^2\kappa = \sigma T$ hold; for example, in the case of mercury $\sigma T = 20$ and $n^2\kappa = 8.6$, while in the case of silver σT is larger and $n^2\kappa$ is smaller than in the case of mercury. As we shall see presently, these discrepancies are due to the fact that we have only taken the conducting electrons into account. The trouble is similar to the one which we experienced when we established the relation $n = \sqrt{\epsilon}$ before taking the electrons into account at all.

The Dispersion of Metals. — We will now extend the treatment by considering that there are present in the metal two types of electrons, conducting and non-conducting, the motion of the latter being resisted by forces of restitution, as in the case of transparent substances.

For the conducting electrons we can put the constant θ (eq. (1), Elect. Mag. Dispersion Theory) equal to infinity, since $\frac{1}{\theta}$ is proportional to the force of restitution.

The equation of motion of these electrons can be written

$$m\frac{\partial^2 \xi}{\partial t^2} = eX - re^2\frac{\partial \xi}{\partial t}, \quad \ldots \ldots \quad (1)$$

or if we call $j_x = eN\frac{\partial \xi}{\partial t}$, the current due to them,

$$\frac{m}{e^2N}\frac{\partial j_x}{\partial t} + \frac{r}{N}j_x = X, \quad \ldots\ldots\ldots \quad (2)$$

in which m is the mass of the electron, e the charge, and N the number in unit volume.

By eq. (2) (this chapter), $\frac{\partial X}{\partial t} = \frac{i}{\tau}X$, in which $\tau = \frac{2\pi}{T}$;

$$\therefore X = -i\tau\frac{\partial X}{\partial t}, \text{ since } \frac{1}{i} = -i.$$

For periodic changes we can write $j_x = -\tau i\frac{\partial j_x}{\partial t}$, and if we substitute these values in eq. (2), we obtain

$$j_x\left\{\frac{i}{\tau}\frac{m}{e^2N} + \frac{r}{N}\right\} = -i\tau\frac{\partial X}{\partial t},$$

which equation can be easily brought into the form of eq. (4), (Elect. Mag. Disp. Theory), by transposing the terms

$$j_x = \frac{1}{4\pi}\frac{\partial X}{\partial t}\left\{-\frac{\tau e^2 N i\tau 4\pi}{im + e^2 r\tau}\right\},$$

$$j_x = \frac{1}{4\pi}\frac{\partial X}{\partial t}\left\{\frac{4\pi\tau N}{ir - \frac{m}{\tau e^2}}\right\},$$

the complex quantity in the brackets corresponding to the dielectric constant ϵ of the earlier formula.

As we have seen in the Chapter on Dispersion, the dielectric constant of a medium containing non-conducting electrons is represented by

$$\epsilon' = 1 + \sum_h \frac{\theta_h'}{1 + i\frac{a_h}{\tau} - \frac{b_h}{\tau^2}}.$$

The dielectric constant ϵ'' resulting from the presence of both types of electrons is therefore

$$\epsilon'' = 1 + \sum_h \frac{\theta_h'}{1 + i\frac{a_h}{\tau} - \frac{b_h}{\tau^2}} + 4\pi\tau\sum_h \frac{N}{ir} + \frac{m'}{\tau},$$

in which

$$m' = -\frac{m}{e^2}.$$

If we are dealing with vibrations, the periods of which are far removed from the free periods of the non-conducting electrons, so that $a_h = 0$, and if we write $\epsilon'' = n^2(1 - \kappa^2 - 2i\kappa)$, we obtain by equat-

ing the real and imaginary parts of the above equation,

$$n^2(1-\kappa^2)=1+\sum_h \frac{\theta_h'}{1-\left(\frac{\tau_h}{\tau}\right)^2}-4\pi\sum\frac{m'N}{r^2+\left(\frac{m'}{\tau}\right)^2}$$

and

$$n^2\kappa=2\pi\tau\sum\frac{rN}{r^2+\left(\frac{m'}{\tau}\right)^2}.$$

Consider now the first of these two equations. In the case of transparent substances, where we neglect a_h, $\kappa=0$, and we have

$$n^2=1+\sum\frac{\theta_h'}{1-\left(\frac{\tau_h}{\tau}\right)^2}$$

for wave-lengths far removed from the natural period of the electron.

In the present case we have a third term to take into account, so that even in the case of these wave-lengths we may have $\kappa>1$ (which means very heavy absorption), since the right-hand member can become negative as a result of the third term. This term will have large negative values for small values of r. Now r represents something which opposes the motion of the conducting electrons, which, in the Chapter on Dispersion, we called for convenience "friction." Small values of r consequently represent high conductivity, and the most opaque metals will be those of the highest electrical conductivity. This point will be more clearly brought out in the section following.

The discrepancy between theory and experiment as represented by the equation deducted at the beginning of the chapter, $n^2\kappa=\sigma T$, is cleared up b the second of the two equations, if we write it in the form which i will take when we have infinitely long waves ($\tau=\infty$). We must do this if we are to fit our optical equations to ordinary electrical measurements, made with very slow periods, or without any periodicity whatever.

For the conductivity we can write $\sigma=\frac{N}{r_1}+\frac{N^2}{r_2}$, since it is proportional to the number of conducting electrons and inversely as the frictional force which opposes their motion. Substituting these values, we get at once

$$n^2\kappa=2\pi\tau\sum\frac{N}{r}=\sigma T.$$

As has already been pointed out, $n^2\kappa=8.6$ in the case of mercury, while $\sigma T=20$.

If r is small, as in this case, and if we are dealing with small values of τ, as when measuring $n^2\kappa$ by optical methods, we cannot neglect $\frac{m'}{\tau}$ in comparison to r: our second member will then be smaller than in the previous case, since the denominator is larger, and we shall have $n^2\kappa<\sigma T$. It is worthy of remark that the conductivity

of electrolytes is not sufficient to cause appreciable absorption of light, the conductivity of the best electrolytes being only about $\frac{7}{10000000}$ that of mercury. For these σ will be of the order of magnitude 7×10^{11}, while for light-waves $T=2\times10^{-15}$, therefore $\sigma T=14\times10^{-4}=.0014$. By our formula $n^2\kappa$ is never greater than σT, generally much smaller, which shows us that κ must be very small in the present case.

Relations between Optical and Electrical Properties of Metals. — Very intimate relations have been established between the optical and electrical properties of metals by the recent work of Rubens and Hagen (*Ann. der Physik*, 11, 873, 1903). Previous to this important investigation many discrepancies existed between theory and experiments. The optical properties of metals could not be represented by Maxwell's theory in its original form. One of the most difficult things to account for was the comparatively great transparency of some metals for light-waves. Moreover, the reflecting power appeared to stand in no definite relationship with the electrical conductivity. Kundt, moreover, found that the refractive indices of his metal prisms for red light arranged themselves in the order of the specific resistances of the metals, whereas on Maxwell's theory the reverse should hold true.

The work of Rubens and Hagen has shown conclusively that these discrepancies have resulted from the employment of too short waves. As soon as the optical work was carried on in the remote infra-red region of the spectrum between wave-lengths 4 μ and 12 μ, most perfect agreement with the theory was found.

They found, for example, that platinum and bismuth, which have a low conductivity, are much more opaque in the visible spectrum than gold and silver, which are much better conductors. This is contrary to what would be expected on Maxwell's theory. In the infra-red, however, it was found that they were much more transparent. This effect is not difficult to show. A thin quartz plate is lightly silvered chemically, the action being stopped when the film appears blue by transmitted light. On a similar plate a perfectly opaque film of bismuth is deposited by the cathode discharge. If the rays from a Welsbach light, without its chimney, are allowed to fall upon a sensitive thermopile, it will be found that the silver plate practically cuts off all of the long heat-waves, while the bismuth is fairly transparent for them.

Maxwell's theory gives for the reflecting power of a metal of conductivity σ, for electro-magnetic waves of period T, the formula $R=100-\frac{200}{\sqrt{\sigma T}}$ (see Drude's *Physik des Aethers*, p. 574). Introducing in place of the conductivity σ (measured in electrostatic units) the conductivity x, *i.e.* the reciprocal of the resistance, measured in ohms, which a conductor of 1 mm. cross section and 1 metre in length would have, and in place of T the wave-length λ, the formula takes the form

$$R=100-\frac{36.5}{\sqrt{x\lambda}}.$$

The intensity of the radiation which penetrates the surface is given by

$$100-R=\frac{36.5}{\sqrt{x\lambda}},$$

$$\text{or } (100-R)\sqrt{x}=\frac{36.5}{\sqrt{\lambda}}=C\lambda.$$

This equation shows that the product of $(100-R)\sqrt{x}$ is a constant for a given wave-length and is independent of the nature of the metal.

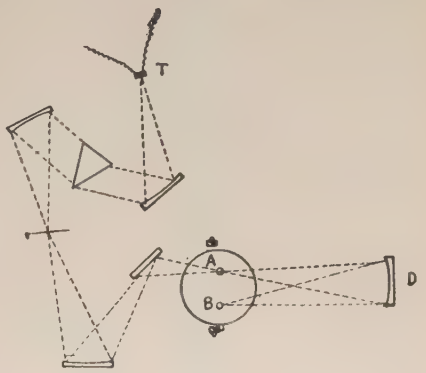


Fig. 301.

The reflecting powers were measured by means of the apparatus shown in Fig. 301. A Nernst lamp B was fixed on a turn-table in such a position that its image A, formed by the concave mirror D, was symmetrically located with respect to the centre of the turn-table. By turning the table the filament of the lamp could be brought into the position previously occupied by its image. The surface of the mirror D was composed

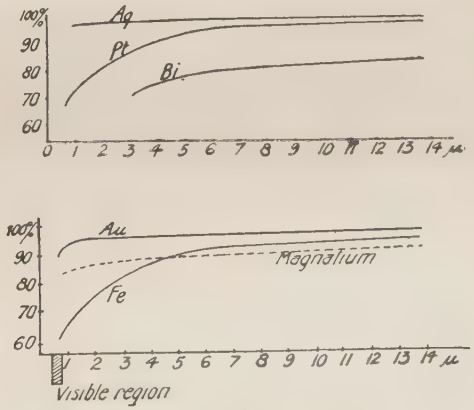


Fig. 302.

of the metal under investigation. The rays from the filament or its image were focussed upon the slit of a reflecting spectrometer, furnished with a fluorite prism, and the spectrum thrown upon the thermopile at T. It is clear that by this arrangement we have a means of comparing the incident with the reflected energy, since the loss by reflections from the surfaces of the optical parts of the instrument is the same in each case. They investigated a large number of metals and alloys of known conductivity, obtaining curves of the type shown in Fig. 302.

It is worthy of note that magnalium, the most brilliantly reflecting

alloy which we have, is surpassed by iron, in reflecting power, for wave-lengths greater than 4 μ. These curves indicate that for infinitely long waves the metals would reflect 100 % of the incident intensity, as is the case with electro-magnetic waves of slow period.

According to theory the product $(100-R)\sqrt{x}$ should be a constant for a given wave-length.

This relation was verified in a remarkable manner as is shown by the following table, which is for $\lambda = 12\,\mu$:

	$\sqrt{x}$	$100-R$	$(100-R\sqrt{x})$
Silver	7.85	1.15	9.0
Copper	7.56	1.6	12.1
Gold	6.43	2.15	13.8
Platinum	3.04	3.5	10.6
Nickel	2.92	4.1	12.0
Steel	2.24	4.9	11.0
Bismuth	.916	17.8	16.3

The mean value of the product $(100-R)\sqrt{x}$ was found to be 19.4 for $\lambda = 4\,\mu$, 13 for $\lambda = 8\,\mu$ and 11 for $\lambda = 12\,\mu$. The theoretical values given by $c = \frac{36.5}{\sqrt{\lambda}}$ for these same wave-lengths are 18.25, 12.9 and 10.54 respectively.

These results show that, in the case of metals, the resonance of electrons (which did not enter into Maxwell's theory) plays no part in the case of waves of length greater than 4 μ, and that the discrepancies which occur when visible light is employed are in all probability due to the fact that, for these higher frequencies, the vibrations of the electrons are excited.

In the hope of finding still better agreement Rubens and Hagen then employed the " Rest-strahlen " from fluorite, $\lambda = 25.5\,\mu$, but in this case the reflecting power was so nearly 100 % for all metals, that it was not expedient to determine the coefficient of penetration $(100-R)$ from observations of the reflecting power.

On account of the relation between emission and absorption, expressed by Kirchoff's law, we can determine $(100-R)$ by comparing the emission from the substance of waves of given length, with the corresponding emission of a perfect black body. Twenty different metals were investigated by this method. The radiations from the black body (see Chapter on Laws of Radiation) and from the metal, both heated to the same temperature, were allowed to fall alternately upon a thermopile, after reflection from four fluorite surfaces, which eliminated all wave-lengths except 25.5 μ. The temperature employed was 170°, maintained by an electrically heated bath of aniline, and the conductivity of the metal used in the calculations was of course the conductivity for this temperature. The results appear in the following table:

TEMPERATURE 170°

	$\sqrt{x}$	EMISSION $J=100-R$	$(100-R)\sqrt{x}$
Silver	6.26	1.13	7.07
Copper	5.70	1.17	6.67
Gold	5.21	1.56	8.10
Aluminium	4.52	1.97	8.91
Zinc	3.19	2.27	7.24
Cadmium	2.86	2.55	7.29
Platinum	2.44	2.82	6.88
Nickel	2.29	3.20	7.33
Tin	2.24	3.27	7.32
Palladium	2.11	3.58	7.53
Steel	1.81	3.66	6.62
Mercury	.957	7.66	7.33
Bismuth	.716	25.6	18.3

It is at once clear that a further increase in the length of the wave has improved matters, as is shown by the more nearly constant value of the products in the last column. The mean value is 7.34, while that calculated is $c_{25\mu} = \dfrac{36.5}{\sqrt{25.5}} = 7.23$.

These results show that it is possible to determine the specific electrical conductivity of the metals by purely optical means, if we limit our observations to waves of sufficient length.

Relation between the Emission Coefficients of Metals and the Temperature. — Inasmuch as the optical properties of metals, at least for long waves, depend upon their electrical conductivity, we should expect their emission and reflection coefficients to vary with the temperature, except perhaps in the case of the alloy "constantan," the electrical resistance of which is nearly independent of temperature. Hagen and Rubens[1] have made a very exhaustive investigation of the subject and have found a most perfect agreement with theory. Up to 5 μ there was very little change in the optical properties of metals, resulting from an increase of temperature. Beyond this point, however, *i.e.* with longer waves, the emission coefficient was found to increase with the temperature, the effect being most pronounced in the case of metals having a large temperature coefficient of electrical conductivity. The metals were heated to known temperatures, and their emission compared with that of a black body at the same temperature. Constantan showed very little change with the temperature, as was to be expected.

In studying the corresponding changes in the reflecting power, two metal plates were mounted side by side in an electrically heated oven, and the light reflected to and fro from one to the other. In this way the very slight change in the reflecting power with the temperature was magnified. The reader should consult the original papers, as many extremely interesting facts were ascertained.

[1] *Sitz. Ber. der Preus. Akad.*, XVI, 1909; XXIII, 1910.

CHAPTER XVII

ROTARY POLARIZATION

We have seen that, in general, when a ray of plane-polarized light is passed through a crystal in the direction of its optic axis, there is no double refraction, and the light emerges with its plane of polarization unchanged. The discovery was made by Arago in 1811 that a rotation of the plane occurs when light is transmitted through quartz in a direction parallel to the optic axis. If two Nicol prisms are placed in front of a sodium flame and so oriented as to completely extinguish the light, the introduction between the prisms of a quartz plate, cut perpendicular to the axis, causes the field to become bright again. On turning the analyzing Nicol through a certain angle the light can be completely extinguished, showing that it is still plane-polarized, but that the plane of polarization has been rotated through an angle, which is measured by the angle through which the Nicol has been turned. If white light is used the field appears colored, the colors changing as the analyzing Nicol is rotated, the light never disappearing entirely, as was the case with the monochromatic sodium flame. This is due to the fact that the different colors are rotated by different amounts, the phenomenon being termed rotatory dispersion.

Fig. 303.

The amount of rotation was found to be proportional to the thickness of the crystal section, which shows that the action occurs within the medium and not at the surfaces. Moreover, some crystals were found to rotate the plane to the right, while others turned it to the left. The former are termed right-handed or dextrogyrate, the latter left-handed or laevogyrate. Simple inspection of the crystals is sufficient to determine their character. One can usually find small hemihedral planes which cut off the obtuse angles at the base of the hexagonal pyramid. The character of the crystal is determined by the relative positions of the surfaces s and x. If s and x lie to the right of R, the surface of the hexagonal pyramid which both touch, the crystal is dextrogyrate, if to the left, laevogyrate, as in the upper figure. The surface x is often absent, in which case we can determine the nature of the crystal by the striæ on the surface of s, which always run towards the position which x would occupy if present, as shown in the figures (303).

The direction of rotation of the plane is not affected by turning the plate around, consequently if the ray is reflected back through the

crystal, the plane of polarization is turned back into its original position. This point is very important in connection with the magnetic rotation, which we shall discuss in a subsequent article, for in the case of substances which acquire rotatory power by being placed in a strong magnetic field, the direction of rotation depends on the direction of the magnetic field, and reflection back through the plate doubles the rotation.

It was subsequently found that many liquids and solutions possessed like quartz the power of rotating the plane of polarization, though in a much less degree. The rotation of the red rays by a quartz plate 1 mm. thick amounts to about 18°, while that due to a layer of turpentine of equal thickness is about one quarter of a degree.

Rotatory Dispersion. — The phenomenon of rotatory dispersion was investigated by Biot, who found that the rotation was nearly, though not exactly, proportional to the inverse square of the wave-length. Still more accurate measurements were made by Brock, who obtained the following values for a plate 1 mm. thick, for wave-lengths corresponding to those of the principal Fraunhofer lines:

	B	*C*	*D*	*E*	*F*	*G*
ρ,	17° 30′	17° 24′	21° 67′	27° 46′	32° 50′	42° 20′
$\rho\lambda^2$,	7238	7249	7511	7596	7622	7841

The values of $\rho\lambda^2$ are seen to increase with decreasing wave-length, which shows that Biot's law is only approximately followed. Brock's experiment, which is well worth repeating, consisted in passing sunlight through two Nicols, between which a quartz plate was mounted, and then analyzing the light with a spectroscope. On turning the analyzing Nicol, a dark band entered the spectrum from the red end, and passed slowly down towards the violet as the Nicol was rotated. By setting the centre of the band on a Fraunhofer line and reading the position of the Nicol, the rotation for the wave-length of the line was determined.

Stefan calculated from Brock's measurements the following empirical formula $\phi = -1.581 + \frac{8.0403}{\lambda^2 10}$, of which the first member represents the departure from Biot's law. If the law held rigorously it would be equal to zero. Cauchy's formula for ordinary dispersion is

$$n = A + \frac{B}{\lambda^2},$$

in which, however, A is always positive.

The rotatory power of quartz must be kept in mind in all experiments with polarized ultra-violet light. If quartz plates or lenses are used, the planes of polarization of the different lines in the spectrum will be in every conceivable direction owing to the enormous rotatory dispersion. The difficulty can be overcome by employing equal optical paths of right and left-handed quartz. Lenses can be made which do not rotate the plane, by combining two plano-convex lenses, one of right, the other of left-handed

quartz. Optical apparatus is frequently designed without paying due reference to this phenomenon. The following table will be found extremely useful in all polarization work involving the use of quartz plates or lenses:

ROTATION OF THE PLANE OF POLARIZATION BY A QUARTZ PLATE 1 MM. THICK. FOR FRAUNHOFER AND CADMIUM LINES.

λ	δ	λ	δ	λ	δ
A 7600	12°.668	*h*	47°.481	*P*	74°.571
a	14 .304	*H* 3968	51 .193	*Q*	78 .579
B 6870	15 .746	*K* 3934	52 .155	Cd 12. 3252	80 .459
C 6563	17 .318	*L*	55 .625	*R*	84 .972
D_1 5890	21 .684	*M*	58 .894	Cd 17. 2748	121 .052
D_2 5896	21 .727	Cd 9. 3612	63 .628	Cd 18. 2572	143 .266
E 5270	27 .543	*N*	64 .459	Cd 23. 2314	190 .426
F 4861.5	32 .773	Cd 10. 3466	69 .454	Cd 24. 2266	201 .824
G 4340.6	42 .604	*O*	70 .587	Cd 25. 2195	220 .731
		Cd 11. 3403	72 .448	Cd 26. 2147	235 .972

If we employ a plate a centimeter or more in thickness the spectrum will be found to be crossed with a number of dark bands which are very nearly equidistant, and which move down the spectrum as the Nicol is turned. The angular rotation for wave-lengths corresponding to two adjoining bands differs of course by 180°.

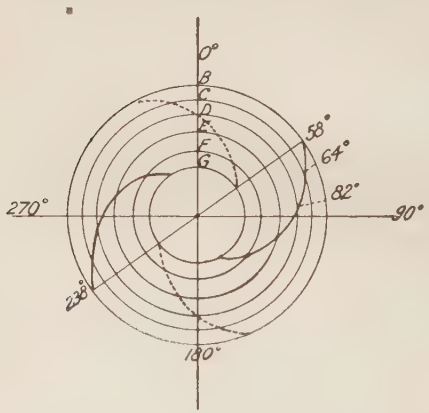


FIG. 304.

We can represent by a geometrical construction the position in the spectrum of the maxima and minima for any position of the analyzing Nicol. Describe around a point (Fig. 304), which we may consider as the source of light seen through the Nicols, a series of concentric circles with radii proportional to the wave-lengths of the principal Fraunhofer lines, the system representing a circular spectrum, such as would be seen by viewing the source of light through a diffraction grating revolving rapidly in its own plane. We will now represent the case of a quartz plate 3.75 mm. thick, for which the rotations for the specified wave-lengths are as follows, if Φ is the rotation for a 1 mm. plate:

	B	*C*	*D*	*E*	*F*	*G*	*H*
3.75Φ,	58° 1′	64° 5′	81° 7′	104° 1′	123° 9′	159° 6′	191° 8′

Let the diameter 0°–180° represent the direction of the vibration of the light after it has passed the first Nicol. The quartz plate will rotate the red (B) through an angle of 58° 1′; consequently we mark two points on the B circle at 58° and 238°, this diameter representing the position of the vibration plane of the analyzing Nicol when the red is most copiously transmitted. Doing the same for the other colors and connecting each set of points by a line, we obtain two spirals, which represent graphically the rotations of the different colors. The loci of planes perpendicular to these planes are given by joining points 90° away from the first point. We obtain in this way two other spirals (dotted in diagram) rotated through an angle of 90° with respect to the first. For thinner plates the spirals will have less curvature and be shorter, while for very thick plates they will have one or more complete convolutions.

These maximum and minimum spirals can be shown experimentally by means of a very ingenious device due to Mach. A brilliant point source of light, a Nicol and the quartz plate, are set up in line, followed by a Nicol, and a nearly direct vision prism or transmission grating mounted in a tube which can be set in rapid rotation by means of a pulley (Fig. 305). The revolving prism spreads out the source of light into the circular spectrum, which we represented in our geometrical construction, from which the colors disappear in different regions as the Nicol rotates with the prism. The circular spectrum is seen traversed by a pair of intensely black spirals, the direction of rotation of which depends on whether we use a plate of dextro or laevorotatory quartz. This extremely beautiful experiment can be projected, though the colors are naturally much more vivid when viewed subjectively.

Fig. 305.

Fresnel's Explanation of the Rotation. — A theory was formulated by Fresnel to account for rotatory polarization, which was based upon the fact that a ray of circularly polarized light is propagated without change through a quartz crystal in a direction parallel to the optic axis. A linear vibration can be regarded as the component of two oppositely polarized circular vibrations, and Fresnel made the assumption that the plane-polarized light, upon entering the crystal, was decomposed into two oppositely polarized circular vibrations, which were propagated with unequal velocities. This inequality in the velocity of propagation will produce a rotation of the resultant, which will amount to 90° after a thickness has been traversed such that one circular disturbance is half a wave-length ahe d of the other. Let two points moving in opposite directions around a circle represent the two circular vibrations. The resultant linear vibration will be represented by the line joining the

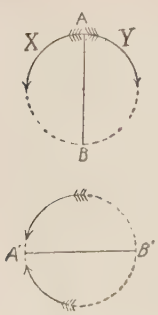


Fig. 306.

two points at which the moving points pass each other. Let x and y be the disturbances which in the upper diagram have the resultant AB (Fig. 306). In the lower diagram the disturbance y is half a period ahead of x, and the resultant is $A'B'$.

The existence of these two circular components was shown experimentally by Fresnel, who reasoned that if there existed in reality two circularly polarized disturbances which travelled with different velocities, they should be refracted by different amounts on emerging into the air through an oblique surface; in other words, quartz ought to show feeble double refraction in a direction parallel to the optic axis, and the two images produced thereby should be circularly polarized. Fresnel first tried a single surface, but the effect was too slight to be noticed. By the ingenious device of building up a compound prism composed of alternate prisms of right and left-handed quartz the effect was found (Fig. 307). This method permitted the use of very oblique refracting surfaces, since only the small differences in the velocities came into play, while in the case of a single prism we have the refractive index with respect to air to consider, and if the prism be too large, light cannot be made to traverse it, owing to total reflection. By Fresnel's device the separation is increased at each surface, as will be seen from the following consideration. The difference between right and left-handed quartz lies in the fact that the right-handed circular component is the faster in the former, the slower in the latter, that is, the R prisms in Fig. 307 act as the rarer, the L prisms as the denser media with respect to this component. The reverse holds true for the left-handed component, the R prisms being the denser in this case. The former component is therefore bent down and the latter up, the angular separation increasing at each surface. If the two images of the source seen through the compound prism are examined with a Nicol, they are seen to remain unaltered when the Nicol is rotated. This means that the compound crystal prism has either depolarized the light completely or transformed it into circularly polarized light.

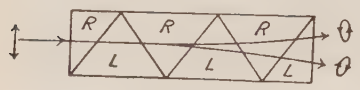


Fig. 307.

The introduction of a quarter-wave plate of mica causes the images to disappear in succession as the Nicol is rotated, which proves the two images to be circularly polarized, and in opposite directions, the mica plate transforming them into plane-polarized images, the planes of polarization being at right angles.

Cornu has recently shown that with a single 60° prism of quartz the separation of the rays can be shown, the angle amounting to 27″ for sodium light. This means that even a quartz prism so cut that the rays travel along the optic axis yields double images. To remedy this defect Cornu devised a prism consisting of two right-angled prisms, the one of right, the other of left-handed quartz (Fig. 308). It is apparent that the optical path parallel to the base of the prism is the same for the two circular components; conse-

Fig. 308.

quently the double refraction, which would otherwise cause a doubling of the spectrum lines, is eliminated.

Prisms of this type are used in practically all of the quartz spectrographs constructed at the present time.

Babinet's Experiment. — The difference between the velocities of right and left-handed circularly polarized rays in quartz was shown in a different way by Babinet, who made use of the fringe system produced by a pair of Fresnel mirrors. The incident light, which was monochromatic and plane-polarized, passed, before falling on the mirror, through a quartz plate which was covered with two $\frac{\lambda}{4}$ plates of mica, one of which produced right, the other left-handed circular polarization. The two oppositely polarized circular disturbances passed through the same quartz plate and fell, the one upon the mirror, the other upon its neighbor. Since, however, a path-difference between two interfering rays which are circularly polarized in opposite directions does not affect the intensity of the illumination, but only the direction of the resultant plane-polarized vibration into which they unite, no maxima and minima fringes are produced; there exists, however, a fringe system differentiated not by intensity but by the position of the plane of polarization. If the path-difference is zero the plane coincides with that of the light originally; if there is a path-difference of $\frac{\lambda}{2}$ the plane is rotated through 90°. The system when viewed through a Nicol prism becomes visible, owing to the extinguishing of the light in those regions where the plane of vibration is crossed with the plane of vibration of the analyzer. If, now, the beams of right-handed and left-handed light traverse the crystal with equal velocities, the fringes should occupy the same position as in the original Fresnel experiment. If, however, one travels faster than the other there will be a path-difference, and a corresponding shift of the fringes. This was found to be the case, for when the plane of the incident light was rotated through 90°, the two quarter-wave plates exchanged properties, and the fringes shifted in position. In this experiment we have the equivalent of two circularly vibrating sources, of the same period, but with opposite directions of rotation. Changing the plane of the incident light by 90° is equivalent to reversing the revolutions of the sources.

Unequal Absorption of the Circular Components. — The existence of the two circular components was shown in another way by Dove,[1] who found that in colored crystals of quartz (amethyst) the two disturbances were absorbed in different amounts. A similar phenomenon has been more recently observed by Cotton, who found that the strongly colored solutions of certain tartrates absorb right and left-handed circular vibrations in unequal amounts. These cases will be referred to again when we consider the theory of rotatory polarization.

Other Rotatory Crystals. — Descloizeaux found that crystals

[1] *Pogg. Ann.*, cx.

of cinnabar have a rotatory power similar to quartz, but fifteen times stronger, some crystals being right-handed, others left. Sulphate of strychnia and sulphate of aethylendiamin have the same property.

The curious discovery was made by Marbach that crystals of sodium chlorate show the phenomenon of rotatory polarization in all directions, sections 1 mm. thick turning the plane (yellow light) 3.7 degrees, no matter how they are cut from the crystal.

Sulphate of strychnia has the power of rotating both in the crystalline state and in solution. As we shall see presently, many solutions exhibit the phenomenon of rotatory polarization, but in general the substances crystallize with two optic axes, the double refraction which occurs in all directions masking any rotatory power which may be present. If, however, the substance can be obtained as an amorphous solid the rotatory power is preserved. Quartz in an amorphous condition (fused, for example) does not have the rotatory power, neither have solutions of quartz in potash. This makes it quite certain that in the case of quartz and other active crystals, the optical activity depends on the arrangement of the molecules in the crystal, while in the case of substances which are active in the dissolved state the property depends on the arrangement of the atoms making up the molecule.

Rotatory Polarization in Convergent Light. — The behavior of quartz plates cut perpendicular to the optic axis was investigated both theoretically and experimentally by Airy in 1831.[1] Inasmuch as only circularly polarized rays are propagated without change parallel to the axis, and plane-polarized rays perpendicular to the axis, Airy made the hypothesis that in any other direction the only form of vibration capable of being propagated without change was an elliptical one, assuming that a plane-polarized ray incident in a direction inclined to the axis was decomposed into two oppositely polarized elliptical vibrations which travelled through the crystal with different velocities. Airy further assumed that the ellipses were similar, and that the path-difference between the two elliptically polarized rays was the same as in ordinary non-rotatory uniaxal crystals, *increased* by an amount which was independent of direction, and inversely proportional to the square of the wave-length. This means that in active crystals the two wave-surfaces are completely separated, the very small distance between them and the points where they intersect the optic axis being the increment referred to above. The wave-surfaces are shown in Fig. 309.

Fig. 309.

Rotatory Polarization of Liquids. — The rotation of the plane of polarization by liquids was accidentally discovered by Biot in 1815, while experimenting upon the effect of the surrounding medium upon the colors of thin crystalline plates. The changes which he observed in the colors of plates immersed in oil of tur-

[1] *Camb. Trans.*, iv., part i., pages 79–198.

pentine led him to try the effect of the oil alone on polarized light. He found that the turpentine behaved in a manner similar to a quartz plate cut perpendicular to the axis, turning the polarization plane to the left by an amount proportional to the thickness of the fluid traversed, and approximately inversely proportional to the square of the wave-length.

On examining other substances for this property he found that a large number of fluid organic compounds behaved in a similar way, some turning the plane to the left, others to the right. The rotatory power was, however, very much less than that of quartz, a column of turpentine 100 mm. long turning the plane of sodium light 37°. An equal length of a solution of santonid or parasantonid in a mixture of chloroform and bisulphide of carbon, which is the most active liquid substance known, gives a rotation of 446°, while a quartz plate of equal thickness would turn the plane 2167°, or six complete revolutions.

Biot found further that a mixture of an active substance with an inactive one, chemically inert towards it, had a rotatory power proportional to the amount of active substance present; in other words, the rotation was simply proportional to the number of active molecules in the path of the light, and wholly independent of the proximity of similar or different molecules.

The neutral effect of dilution with an inactive substance made the investigation of solid substances possible by bringing them into solution in some optically inactive liquid.

In this way the number of active organic substances was enormously increased, many sugars, gums, albumens, alkaloids, fruit acids, etc., being added to the list. We shall see presently, however, that in some cases the solvent, even if optically inactive, is not without influence on the rotatory power.

Rotatory Power and Change of State. — Biot found that substances were which active in solution preserved their property in the solid state, provided they were prevented from crystallizing. Sugar and tartaric acid can be obtained as amorphous solids, and are optically active in this state. If the substance crystallizes, the double refraction completely masks the rotatory polarization.

Bringing the substance into the vaporous state does not affect its activity, as Biot ascertained in 1818 by means of a tube 15 meters long filled with the vapor of turpentine. His apparatus took fire before the completion of the experiment and was destroyed, and it remained for Gerney to make careful measurements of the rotatory powers of vapors. He found that the specific activity was the same in the vaporous state as in the liquid, *i.e.* a long column of vapor has the same rotatory power as the short column of liquid into which it condenses (the cross sections being, of course, equal).

Rotatory Dispersion of Liquids. — The rotatory dispersion of liquids was found by Biot to be similar to that of quartz. That the increase in rotation is not strictly proportional to the inverse square of the wave-length he showed by filling a tube with a mixture of dextro-rotatory and laevo-rotatory liquids in such proportion that perfect compensation was secured for a single

color. If the law of the inverse squares of λ was strictly followed, the compensation would be perfect for all wave-lengths, which was found not to be the case, the light appearing colored through the analyzer. The departure from the law is different for different liquids, being very small for the oils and very large in the case of a solution of camphor in alcohol.

Molecular Rotation and the Influence of the Solvent on the Rotatory Power. — The fact that the rotation produced by a given substance appeared to be proportional to the number of molecules in the path of the light, gave rise to the opinion that the rotatory power was inherent in the molecule, and led to the term molecular rotatory power. This we may define as the amount of rotation produced by a column of the solution 1 dm. in length containing 1 gram of the substance per cubic centimeter. If we dissolve p grams of the substance in q grams of the solvent, the density being δ, then $\frac{p}{p+q}\delta$ is the amount of the substance contained in unit volume of the solution, and if we fill with this solution a tube of length l and observe a rotation for some particular wave-length, we have

$$\rho = [\rho]\frac{p}{p+q}l\delta,$$

in which $[p]$ is a constant for the substance, and is defined as the molecular rotatory power. This constant may also be defined as the rotation produced by a thickness of 1 dm. of the pure substance divided by the density of the substance. Biot found that the rotation was not strictly proportional to the amount of dissolved substance, and that it varied, moreover, with the nature of the solvent. If the change in the rotatory power with changing concentration is continuous, a formula may be deduced by which we may determine the molecular rotatory power of the pure substance from observations of solutions.

This matter was very carefully investigated by Landolt, who worked with fluid substances, so that the activity of the pure substance could be directly determined, and then compared these values with the values calculated from observations made with the substance dissolved in various inactive solvents.

He found that the molecular rotatory power could be expressed as a function of the quantity of active substance contained in solution. If q represents the weight of the solvent in 100 parts by weight of the solution, an equation of the following form could be applied:

$$(\rho) = A + Bq + cq^2.$$

The constants could be determined by making observations with solutions of various concentration, the constant A being the molecular rotation of the pure substance ($q=0$).

In this way both the effects of the nature of the solvent and the degree of concentration are eliminated.

For example, the value obtained with sodium light for pure oil of

turpentine was 37°.01, the equations obtained with alcohol, benzol, and acetic acid as solvents being

1. *Alcohol.*

$$[\rho]_0 = 36°.974 + .004816\, q + .000133\, q^2.$$

2. *Benzol.*

$$[\rho]_0 = 36°.970 + .02153\, q + .00006673\, q^2.$$

3. *Acetic Acid.*

$$[\rho]_0 = 36°.894 + .02455\, q + .0001369\, q^2.$$

In some cases concentrated solutions are dextro-rotatory; dilute, laevo-rotatory. Such is the case with malic acid, which is represented by the equation

$$[p]_0 = 5°.891 - .08959\, q,$$

right-handed rotation being regarded as positive.

For $q = \frac{5.891}{.0896} = 65.7$ we have an optically inactive solution.

Anomalous Rotatory Dispersion. — In the case of solutions of tartaric acid in water, the dispersion at first increases with decreasing wave-length, reaches a maximum, and then decreases. Measurements made by Arndtsen [1] for 50 parts of crystallized acid in 50 parts of water gave the following values:

C	*D*	*E*	*b*	*F*	*e*
11°9	13°	14°	13°.7	13°.3	10°.3

The formula for this substance is $[\rho] = A + Be$, in which e is the percentage of water, the value for A and B for the various colors being

	C	*D*	*E*	*b*	*F*	*e*
$A =$	+2°.748	1°.95	°0.153	−.832°	−3°.598	−9°.657
$B =$	+9°.446	13°.03	17°.514	19°.147	23°.977	31°.437

From this it is apparent that the pure substance in an amorphous state is dextro-rotatory for all wave-lengths on the red side of a point in the spectrum a little below the *E* line, and laevo-rotatory for wave-lengths below this point, a circumstance which had already been noticed by Biot in the case of amorphous plates of the acid, made by fusing the crystals and pouring the liquid on a glass plate.

In the case of active substances which show strong selective absorption, we may have true anomalous rotatory dispersion when we cross the absorption band. Such cases are not to be confused with the one just mentioned, which in all probability owes its peculiarity, as we shall see when we take up the theory of the rotation, to the

[1] *Pogg. Ann.*, Bd. cv.

presence of both a dextro- and laevo-rotatory system within the molecule. We have a somewhat remote analogy in the achromatic prism. Considered as a single dispersing system it may be said to show anomalous dispersion, the outstanding colors due to imperfect compensation being arranged in anomalous order, as we have seen.

Double Refraction of Active Liquids. — The division of the plane-polarized ray into two oppositely polarized circular disturbances was shown experimentally by F. v. Fleischl [2] by a method identical with the one employed by Fresnel in the case of quartz. A long narrow trough was divided into 22 prismatic compartments by means of oblique partitions of plane-parallel glass (Fig. 310). These compartments were filled alternately with dextro- and laevo-rotatory liquids of the same refractive index, the best results having been obtained with oil of orange and a mixture of the oils of turpentine and ricinus. A small source of polarized-sodium light appeared doubled when viewed through the compound-fluid prism, and examination with a quarter-wave plate and Nicol prism showed the two images to be circularly polarized in opposite directions.

Fig. 310.

Theory of Rotatory Polarization. — The first attempt to bring the phenomenon of rotatory polarization within the range of mathematical analysis was made by M'Cullagh in 1836. Making no assumptions regarding the ultimate physical structure of media which had the power of rotating the plane of polarization, he investigated the changes which were required in the equations of wave-motion in doubly refracting substances, to make them include the phenomenon of rotation. His treatment would be out of place here, as it is purely mathematical, and is of no assistance in forming an idea of the possible cause of the rotation. He found, in brief, that if the introduction of a third derivative into the equations of wave-motion was made, the equations broke up into expressions representing circular vibrations which were propagated with different velocities.

The equations of wave-motion in doubly refracting media are

$$\frac{d^2\xi}{dt^2} = b^2\frac{d^2\xi}{dz^3}, \quad \frac{d^2\eta}{dt^2} = b^2\frac{d^2\eta}{dz^2}$$

for polarized vibrations propagated along the optic axis, which coincides with the z axis of coördinates.

M'Cullagh modified these equations as follows:

$$\frac{d^2\xi}{dt^2} = b^2\frac{d^2\xi}{dz^2} + c\frac{d^3\eta}{dz^3}, \quad \frac{d^2\eta}{dt^2} = b^2\frac{d^2\eta}{dz^2} - c\frac{d^3\xi}{dz^3},$$

the introduction of the third derivations being purely arbitrary.

Solving these equations, he found that they represented right and left-handed circular vibrations, the former being propagated faster or slower according as the constant c was taken positive or negative.

[2] *Ber. Wien. Akad.*, 1884.

The resultant plane-polarized vibration was rotated by an amount

$$\rho = \frac{2\pi^2 c V^2}{b^4 \lambda^2},$$

in which λ and V represent wave-length and velocity of propagation in the crystal. If we disregard dispersion, *i.e.* consider V independent of λ, the expression shows that the rotation varies as the inverse square of the wave-length, as Biot originally believed it to be. M'Cullagh introduced the same arbitrary modifications into the expressions representing rays propagated in other directions than that of the optic axis, and showed that they represented elliptical vibrations, as had been imagined by Airy in his treatment of the subject. His treatment of the whole subject is given in Verdet's *Optics*, together with a theoretical investigation of the rotatory power of liquids, based upon the somewhat fanciful supposition that they are made up of, or contain molecules, which, separately considered, act like crystal films, breaking a linear vibration up into two elliptical vibrations.

Physical Explanation of the Rotation. — In the case of rotation by crystals we can refer the phenomenon to the crystalline structure. Ewell [1] has shown that twisted gelatine cylinders show rotatory polarization. If we imagine a bar of elliptical cross-section which has been twisted torsionally, we have a rough analogy, which may help us to understand how a spiral arrangement of the axes of maximum and minimum elasticity may account for the rotation of a plane-polarized vibration.

In the case of liquids and solutions, however, we must necessarily refer the rotatory power to the structure of the molecule. All rotatory liquids contain carbon, and their power has been ascribed by Le Bel and Van't Hoff to the quadrivalence of this element.

If the four atoms or radicals, which are in combination with the carbon atom, form the corners of a regular tetrahedron, we can arrange them in two different ways, such that one is the looking-glass image of the other, and yet no amount of turning enables them to be brought into coincidence. One of these we may consider dextro-, the other laevo-rotatory. Right and left-handed spirals have similar geometrical properties.

The earliest attempts to explain rotatory polarization were based upon an experiment made by Reusch, who found that, if thin mica plates were superposed, each plate having its principal section turned through a definite angle either to the right or left, with reference to the principal section of the plate below, the combination imitated the behavior of a quartz plate cut perpendicular to the axis, rotating plane-polarized light to the right or left according as the pile of plates were built up clockwise or counter-clockwise. The thinner the plates and the greater their number, the more nearly the pile imitated a rotatory crystal. It was quite natural, in view of this very suggestive experiment, to ascribe a somewhat similar structure to quartz, but the efforts to explain the rotation of liquids

[1] *American Journal of Science*, 8, 89, 1899.

on the assumption of molecules having a laminated structure was pushing the analogy too far. The modern ionic theory is capable of explaining both the natural rotation of liquids and solids, and the magnetic rotation discovered by Faraday.

We will begin with the case of rotatory liquids, following the admirable treatment given by Drude.

We assume that the electrons, which are set in vibration by the light-waves, are forced to move back and forth over a spiral path instead of along straight lines. The force of restitution is supposed to act along the axis of the spiral, and to vary with the displacement, precisely as we assumed in the case of the electrons which caused dispersion. It is clear that the electron will be urged along its spiral path, the axis of the spiral being parallel to x, not only by the force X, but also by the forces Y and Z. If the spiral is oriented as in Fig. 311, a positive electric force X will displace the electron to the right, regardless of its position. The y component will aid or oppose the x component according as the electron is on the lower or upper side of the spiral. If it is at B, a positive Y will carry it to the right, if at A to the left. If Y has the same value at both points, there will be compensation, but if Y varies with z, there will be an outstanding effect for each revolution which the electron makes, which will be represented by $-\frac{\partial Y}{\partial z}$, if Y increases with z. In the same way the z component, which aids X when the electron is at D and opposes X when it is at C, exerts an outstanding effect represented by $\frac{\partial Z}{\partial y}$. The necessary modification which we must make in our previous conception of the electron's motion is therefore that it moves not only as the result of the force X at the point which it occupies, but also as the result of the values which the y and z components have in its immediate vicinity. With this modification our equation for the motion of the electron becomes

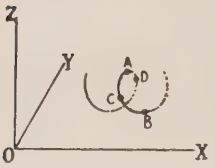


FIG. 311.

$$m\frac{\partial^2\xi}{\partial t^2}=e\left[X+f'\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)\right]-\frac{4\pi e^2}{\theta}\xi-re^2\frac{\partial\xi}{\partial t}.$$

For the condition shown in Fig. 311, f' is of course negative, and its value depends on the diameter of the spiral path and its pitch. If the pitch is small, the electron is obliged to make a larger number of revolutions in travelling a given distance along the x axis, and the resultant effect of the y and z components of the electric force is greater than when the pitch is large. If the pitch is infinite, the spiral degenerates into a straight line and $f'=0$.

The electric convection current due to the motion of the electrons along the x axis we will designate, as before, by

$$(j_x)_1=e_1N\frac{\partial\xi}{\partial t}.$$

For periodic changes we have $\xi = Ae^{i\frac{t}{\tau}}$, from which we get, as before,

$$e_1\xi = \frac{1}{4\pi}X' \frac{\theta_1}{1+\frac{i}{\tau}a_1-\frac{b}{\tau^2}}, \text{ in which } X' = X + f'\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right).$$

This gives us $$(j_x) = \frac{\theta N}{4\pi\left(1+\frac{ia}{\tau}-\frac{b}{\tau^2}\right)}\frac{\partial}{\partial t}\left(X+f'\left[\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right]\right).$$

For periodicities not too near the free period of the electron we neglect the friction term $\frac{a}{\tau}$. Adding to $(j_x)_1$ the quantity $(j_x)_0 = \frac{1}{4\pi}\frac{\partial X}{\partial t}$, which represents the displacement current in the ether, we get for the total current

$$j_x = \frac{1}{4\pi}\frac{\partial X}{\partial t} + \frac{\theta N}{4\pi\left(1-\frac{\tau_h^2}{\tau}\right)}\frac{\partial}{\partial t}\left[X+f'\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)\right], \quad . \quad . \quad (1)$$

which, if we consider that we have electrons of different free periods, and write, for abbreviation,

$$\epsilon = 1+\sum\frac{\theta_h N_h}{1-\left(\frac{\tau_h}{\tau}\right)^2}, \qquad f = \sum\frac{\theta_h f_h' N_h}{1-\left(\frac{\tau_h}{\tau}\right)^2},$$

becomes $$j_x = \frac{1}{4\pi}\frac{\partial}{\partial t}\left\{\epsilon X + f\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)\right\}.$$

The fundamental Maxwell equations now take the form,

$$\left.\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\left(\epsilon X+f\left[\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right]\right) &= \frac{\partial \gamma}{\partial y}-\frac{\partial \beta}{\partial z},\\
\frac{1}{c}\frac{\partial}{\partial t}\left(\epsilon Y+f\left[\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right]\right) &= \frac{\partial \alpha}{\partial z}-\frac{\partial \gamma}{\partial x},\\
\frac{1}{c}\frac{\partial}{\partial t}\left(\epsilon Z+f\left[\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right]\right) &= \frac{\partial \beta}{\partial x}-\frac{\partial \alpha}{\partial y},
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

which, if we differentiate successively with respect to x, y, and z and add, give

$$\frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)=0.$$

The magnetic equations remain unchanged:

$$\frac{1}{c}\frac{\partial \alpha}{\partial t} = \frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}, \quad \frac{1}{c}\frac{\partial \beta}{\partial t} = \frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}, \quad \frac{1}{c}\frac{\partial \gamma}{\partial t} = \frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}. \quad . \quad (3)$$

By elimination of α, β, γ from the fundamental equations (2) and (3) we get

$$(10)\qquad \left.\begin{aligned} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\epsilon X + f\left[\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right]\right)&=\Delta X,\\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\epsilon Y + f\left[\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right]\right)&=\Delta Y,\\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\epsilon Z + f\left[\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right]\right)&=\Delta Z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

If we are dealing with plane-waves propagated along the z axis we can write

$$(11)\qquad X=Me^{\frac{i}{\tau}(t-pz)},\quad Y=Ne^{\frac{i}{\tau}(t-pz)},\quad Z=0,$$

in which p represents the reciprocal of the wave-velocity.

Differentiating and multiplying by ϵ gives us

$$\epsilon\frac{\partial^2 X}{\partial t^2}=-\frac{\epsilon M}{\tau^2}e^{\frac{i}{\tau}(t-pz)},\ \text{a similar expression for}\ \frac{\partial^2 Y}{\partial t^2}\ \text{and}\ \epsilon\frac{\partial^2 Z}{\partial t^2}=0,$$

$$\frac{\partial Y}{\partial z}=-\frac{i}{\tau}Npe^{\frac{i}{\tau}(t-pz)},$$

$$\frac{\partial Z}{\partial y}=0,\quad \frac{\partial^2}{\partial t^2}\left(\frac{\partial Y}{\partial x}\right)=\frac{Npi}{\tau^3}e^{\frac{i}{\tau}(t-pz)}.$$

Substituting in the first of equations (4),

$$\frac{1}{c^2}\left[-\frac{\epsilon M}{\tau^2}e^{\frac{i}{\tau}(t-pz)}+f\frac{iNp}{\tau^3}e^{\frac{i}{\tau}(t-pz)}\right]=\Delta X.$$

Now $\quad \frac{\partial^2 X}{\partial x^2}=0,\quad \frac{\partial^2 X}{\partial y^2}=0,\quad \frac{\partial^2 X}{\partial z^2}=-\frac{M}{\tau^2}p^2e^{\frac{i}{\tau}(t-pz)};$

$$\therefore\ e^{\frac{i}{\tau}(t-pz)}\frac{1}{c^2}\left[-\frac{\epsilon M}{\tau^2}=f\frac{iNp}{\tau^3}\right]=-\frac{M}{\tau^2}p^2e^{\frac{i}{\tau}(t-pz)},$$

$$\frac{1}{c^2}\left[\epsilon M-f\frac{iNp}{\tau}\right]=Mp^2,$$

or

$$\epsilon M-\frac{i}{\tau}fpN=Mp^2c^2,$$

$$\epsilon N+\frac{i}{\tau}fpM=Np^2c^2.$$

Multiplying the first equation by N, and the second by M, and subtracting, gives us

$$-iN^2=iM^2\ \text{or}\ M^2=-N^2\ \text{or}\ M=iN,\ M=-iN.$$

Substituting these values in $\epsilon-p^2c^2=-\frac{i}{\tau}fp\frac{M}{N}$ gives

$$\epsilon-p^2c^2\frac{fx}{\tau},\quad \epsilon-p^2c^2=-\frac{fp}{\tau},$$

equations which show us that we have two different values of p, *i.e.* that we are dealing with two waves of different velocities. To find these velocities, we solve the two quadratic equations by completing the squares, and get at once

$$p' = \frac{1}{V'} = -\frac{f}{2\tau c^2} + \frac{1}{c}\sqrt{\frac{f^2}{4\tau^2 c^2} + \epsilon}, \quad p'' = \frac{1}{V''} = +\frac{f}{2\tau c^2} + \frac{1}{c}\sqrt{\frac{f^2}{4\tau^2 c^2} + \epsilon}.$$

The nature of these waves we find from the equations connecting N and M, the amplitudes. In $M = iN$, if M is real, N must be imaginary, *i.e.* if the wave has a real amplitude along the x axis, it has an imaginary amplitude along y. The physical significance of this can be at once found, if we remember that, as before, only the real part of equation (11) is to be taken.

Writing now equations (11) in the form

$$X = M\cos\frac{1}{\tau}(t - pz) + Mi\sin\frac{1}{\tau}(t - pz),$$

$$Y = N\cos\frac{1}{\tau}(t - pz) + Ni\sin\frac{1}{\tau}(t - pz),$$

and substituting for N its equivalents $\frac{M}{i}$ and $-\frac{M}{i}$, we get, if we confine ourselves to the real parts only,

$$X = M\cos\frac{1}{\tau}(t - pz), \quad Y = M\sin\frac{1}{\tau}(t - pz) \text{ (for } iN = -M),$$

and $$X = M\cos\frac{1}{\tau}(t - pz), \quad Y = -M\sin\frac{1}{\tau}(t - pz) \text{ (for } iN = M),$$

which represent circularly polarized waves, which are propagated with the velocities given by the expressions which we have already deduced for p' and p''.

The amount of the rotation computed from the above equations is

$$\partial = \frac{z}{\tau}\frac{p'' - p'}{2} = \frac{f}{2\tau^2 c^2} z = 2\pi^2 \frac{f}{\lambda^2} z,$$

in which λ is the wave-length of the light (in vacuum).

This expression shows us in the first place that the amount of rotation is proportional to f which we have written for $\sum \frac{\theta_h f_h' N_h}{1 - \left(\frac{\tau_h}{\tau}\right)^2}$.

The sign of f depends on whether τ is larger or smaller than τ_h, consequently we should expect the rotation to have its largest value when τ is very nearly equal to τ_h and to change its sign when we cross the region of the spectrum defined by $\tau_h = \frac{T_h}{2\pi}$, that is the centre of the absorption band caused by the electrons in question.

This is in perfect agreement with the experiments of Cotton, who

found anomalous rotatory dispersion in the case of certain strongly absorbing active substances.

It will be remembered that certain cases of anomalous rotatory dispersion have been cited, which did not appear to be due to the presence of an absorption band.

We are now in a position to explain this curious phenomenon.

Real and Spurious Anomalous Rotatory Dispersion. — The rotation of the plane of polarization as a function of the wave-length of the light is given by the formula

$$\delta = \frac{k}{\lambda^2} \sum \frac{\theta_h f_h' N_h}{1 - \left(\frac{\tau_h}{\tau}\right)^2} = \sum \frac{k_h}{\lambda^2 = \lambda_h^2},$$

the summation being restricted to the electrons which produce optical activity.

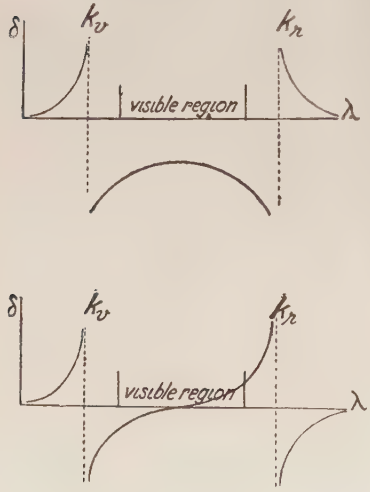


FIG. 312.

Now the sign of k_h depends on whether the electron for which it stands is dextro or laevo-rotatory, consequently if we have an infra-red dextro-rotatory electron and an ultra-violet laevo-rotatory one, the sign of the rotation for all wave-lengths comprised between the two will be the same, since $\lambda^2 - \lambda_h^2$ is negative if $\lambda_h > \lambda$ and positive if $\lambda > \lambda_h$. In this case the rotation will have a minimum value somewhere near the middle of the visible spectrum. If the sign of k_h is the same for both electrons, the rotation will be zero near the middle of the visible spectrum, and will be of opposite sign on either side of this point, increasing as the absorption bands are approached. This is the condition in the case of solid tartaric acid, as we have seen. The rotatory dispersion curves for the two cases just considered are shown in Fig. 312.

Rotatory Dispersion. — The variation of the rotation with the wave-length is given by the formula

$$\delta = \frac{k}{\lambda^2} \sum \frac{\theta_h f_h' N_h}{1 - \left(\frac{\tau_h}{\tau}\right)^2}.$$

If $\left(\frac{\tau_h}{\tau}\right)^2$ is small in comparison to 1, as will be the case when the period of the electrons is small in comparison to that of the waves (ultra-violet electrons), the terms summed will give us a constant, and we can write our formula $\delta = \frac{k'}{\lambda^2}$, which is identical with Biot's empirical formula.

For a limited range of the spectrum this formula often represents the dispersion with a fair degree of accuracy. If, however, we extend our observations over a wide range of wave-lengths, we must use the complete formula.

If we apply the formula

$$\delta = \sum \frac{k_h}{\lambda^2 - \lambda_h^2}$$

to the observations which have been made with quartz, we shall find that much light is thrown upon the nature of the different electrons which give rise to absorption and dispersion. These observations cover a range extending from wave-length 2.14 μ in the infra-red to .219 μ in the ultra-violet. As we have seen in the Chapter on Dispersion, we have three absorption bands which have to be taken into account, two in the infra-red at $\lambda_2 = 8.5\,\mu$ and $\lambda_3 = 21\,\mu$, and one in the ultra-violet at $\lambda_1 = .1\,\mu$. There are in addition other bands further down in the ultra-violet, for which λ_h is small in comparison to λ. These give us the term $\frac{k'}{\lambda_2}$, and we write our formula

$$\delta = \underset{\text{(Ultra-violet)}}{\frac{k_1}{\lambda^2 - \lambda_1^2}} + \underset{\text{(Inf.-red)}}{\frac{k_2}{\lambda^2 - \lambda_2^2}} + \underset{\text{(Inf.-red)}}{\frac{k_3}{\lambda^2 - \lambda_3^2}} + \underset{\text{(Remote ult.-violet)}}{\frac{k'}{\lambda^2}}.$$

If now we calculate the constants from the observations of the rotations for various wave-lengths, we find that both k_2 and k_3 are equal to zero, which shows us that the infra-red electrons do not contribute to the rotatory dispersion, *i.e.* they are inactive. We can therefore write the formula

$$\delta = \frac{k_1^2}{\lambda^2 - \lambda_1^2} + \frac{k'}{\lambda^2},$$

in which $k_1 = 12.2$, $k' = -5.046$.

The accuracy with which this formula represents the rotatory dispersion of quartz can be seen from the following table:

λ		δ (OBS.)	δ (CALCULATED)
	2.14 μ	1.60	1.57
	1.45	3.43	3.43
Red	.67	16.54	16.56
Green	.51	29.72	29.67
Violet	.40	48.93	48.85
	.27	121.06	121.34
	.22	220.07	220.57

Rotatory Dispersion of Absorbing Media. — In the case of absorbing media, *i.e.* media which have absorption bands in the

region under investigation, we cannot neglect the friction coefficient a_h, and both ϵ and f of equations (4) become complex:

$$\epsilon = 1 + \sum \frac{\theta_h N_h}{1 + i\frac{a_h}{\tau} - \frac{b_h}{\tau^2}}, \qquad f = \sum \frac{\theta_h f_h' N_h}{1 + i\frac{a_h}{\tau} - \frac{b_h}{\tau^2}}.$$

In the Chapter on Absorption (equation (4)) we have seen that if ϵ is complex, then p in the equation $X = Me^{\frac{i}{\tau}(t - pz)}$ (eq. (11), of this chapter), must be complex as well, which, as we have seen, means absorption.

Bearing in mind that p is the reciprocal of the velocity in the medium, and writing as before $p = \frac{1 - i\kappa}{V}$, it becomes at once evident that, since we have two different values for p, corresponding to right and left-handed circularly polarized rays, we must also have two values of κ, the extinction coefficient. In other words, for a given wave-length of circularly polarized light, the absorbing power of the medium will depend on the direction of revolution of the luminous vibration.

This effect has been observed by Cotton (*Comptes Rendus*, 120, pp. 989, 1044) in the case of solutions of copper tartrate and chromium tartrate in potash. The chromium salt has an absorption band in the yellow, transmitting red and green. Cotton found that if circularly polarized sodium light of unit intensity was passed through 1 cm. of his solution, the emergent light had an intensity of .0077 if the vibration was left-handed, while in the case of a right-handed vibration it was .0059. The difference is very marked, though the strong absorption indicates that the original light must be very intense. The effect can be very easily shown by preparing a quarter wave-plate of mica and cutting it in two along one of the directions of vibration; the two halves are to be mounted with their edges in contact, one plate being turned through an angle of 90° with respect to the other. If plane-polarized light is passed through this plate (the plane making an angle of 45° with the directions of vibration in the mica), we shall have two emergent beams of oppositely polarized circular light. The polarizing system is illuminated with a brilliant sodium flame, and the field examined through the absorbing solution, when one half will be found to be much darker than the other. If white light is employed in place of the sodium flame the two halves of the field appear differently colored.

This difference of absorbing power for oppositely circularly polarized vibrations leads us to a somewhat startling conclusion, which was foreseen by Cotton and verified by experiment. Ordinary unpolarized light can be regarded as containing equal amounts of oppositely polarized circular vibrations. (See Chapter on Natural Light.) It ought therefore by mere passage through the solution to exhibit traces of circular polarization. This was found to be the case.

If a suitable medium could be found it might be possible to obtain in this way circularly polarized light just as plane-polarized light is obtained by means of a tourmaline plate. The plane vibrations in the natural light would give no trouble, for, as we know, they are decomposed into circular vibrations, which traverse the medium with different velocities.

Elliptical Polarization produced by Absorbing Active Media.— In the case of transparent active media, the plane-polarized light remains plane-polarized during transmission, emerging with its plane rotated through a certain angle. The emergent plane vibration is the resultant of two equal circular vibrations. As we have seen, in absorbing media, one of these may be reduced in intensity more than the other, and the resultant of two circular vibrations of different amplitude is not a plane vibration but an elliptical one. Cotton found that the tartrate solutions above mentioned transformed plane into elliptically polarized light, it being impossible to completely extinguish the emergent light with a Nicol. The ellipticity was found to be greatest in the region of the spectrum where the difference of absorbing power was greatest.

Possible Production of an Optically Active Substance from an Inactive, by Circularly Polarized Light. — Since the absorption of light is often accompanied by chemical change, i is possible that a solution containing equal numbers of dextro- and laevo-rotatory molecules, and consequently inactive, might acquire rotatory power by the action of circular light. If the molecules were unstable and easily decomposed by light, the effect of a circular vibration would be to break down one set of molecules and leave the others unaffected. Both Le Bel and Cotton have pointed out the possibility of effecting unique chemical transformations by the action of circularly polarized light.

Rotatory Dispersion in Absorbing Media. — One formula for the rotatory dispersion shows us that on crossing an absorption band the sign of the rotation may change, or we may have a high positive value on one side and a low positive value on the other. This amounts to saying that anomalous dispersion of the rotation is to be expected in absorbing media. Cotton found that this was the case.

In the case of the chromium tartrate the rotations were as follows:

λ	δ
657	$+1^\circ 26'$
589	$+2^\circ 30'$
581	$+1^\circ 46'$
562	$-1^\circ 21'$
522	$-2^\circ 50'$
473	$-1^\circ 52'$

CHAPTER XVIII

MAGNETO-OPTICS

The Faraday Effect : Magnetic Rotation of the Plane of Polarization. — The discovery was made by Faraday that a transparent isotropic medium, when placed in a powerful magnetic field, acquires the property of rotating the plane of polarization, when the light traverses the medium in the direction of the lines of magnetic force. The phenomenon differs, however, from natural rotation, in that the direction in which the plane of the vibration turns depends upon whether the light rays are passing through the medium from the north pole of the magnet towards the south, or in the reverse direction. The rotation is therefore doubled if the light is reflected back through the medium, instead of being annihilated as in the case of quartz and other active substances. The effect is most pronounced with media having a high refractive index, such as bisulphide of carbon or dense flint glass. With a powerful Ruhmkorff magnet, the poles and cores of which are bored out to allow of the passage of light rays along the lines of force, the rotation can be easily observed with a thick piece of ordinary plate glass. Sun or arc light is passed through a Nicol prism, the hollow magnet cores, and the glass block between the poles. A second Nicol is placed in such a position as to extinguish the emergent beam. On throwing the current into the magnet, the field immediately becomes brilliantly illuminated, and by turning the analyzing Nicol until darkness is again produced the amount and direction of the rotation can be determined.

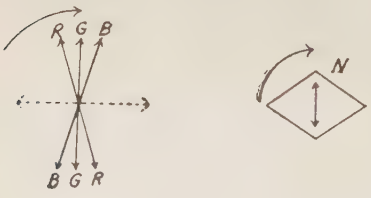


Fig. 313.

Owing to the rotatory dispersion this position will vary with the color, and the field will appear blue, purple, and red in succession, as will be readily understood from Fig. 313, in which the dotted arrow represents the original direction of the vibration, and the arrows the rotated red, green, and blue vibrations. The analyzing Nicol in its original position is indicated by *N*. The rotation in this case is clockwise, and all of the colors are transmitted with more or less freedom, consequently the field appears nearly white. On turning the Nicol in the same direction it will extinguish the red first, leaving an outstanding color of a bluish green ; the green goes next, leaving a purple field made up of the transmitted red and blue, and finally the blue disappears, leaving the field orange-red. If we examine the light through a spectro-

scope, as we turn the Nicol, we shall see a dark band enter the spectrum on the red side and leave it on the violet side.

Explanation of the Magnetic Rotation. — The explanation of the natural rotation in active substances which Fresnel gave, was that the plane vibration was decomposed into two oppositely polarized circular vibrations, which were propagated with different velocities. The same explanation will do for the magnetic rotation, provided that it can be shown that the refractive index of a medium in a magnetic field for circularly polarized light depends upon the direction of revolution. The matter was attacked experimentally by Righi and Becquerel independently, and both investigators found that the interference fringes, formed by two streams of circularly polarized light, one of which had traversed a block of glass placed between the poles of a magnet, were displaced when the magnetic field was formed. The direction of the displacement depended on whether right or left-handed circular light was used, which showed that the effect of the field was to increase the refractive index for one type of vibration and diminish it for the other. It remained only to show that the actual decomposition of the plane vibrations into circular ones actually occurred by some experiment analogous to the one which Fresnel made with his battery of quartz prisms built of right and left-handed crystals in alternation.

Resolution into Circular Components. — The experimental resolution of the light into its two circular components in the Faraday effect is a much more difficult problem than the one which confronted Fresnel, since we do not have at our disposal two liquids of the same index of refraction and of opposite magnetic rotation, with which hollow prisms might be filled, in the manner adopted by Fleischl in the case of natural rotation. The problem has, however, been attacked and solved in a very beautiful manner by Brace.[1]

It will be remembered that in Fresnel's arrangement of right and left-handed quartz prisms, the clockwise circular component which travelled at the higher velocity in one prism, travelled at the slower velocity in the following prism. The very ingenious idea occurred to Brace to reverse the direction of revolution of the circular vibrations at the boundary surface between the two prisms, which can be done with a half-wave plate of mica. By this artifice the same thing is accomplished as by employing prisms of dextro- and laevo-rotatory media in succession. A double prism of extra dense flint glass was employed with a half-wave plate cemented between the two components. The experimental difficulties were, however, found to be too great and no conclusive results were obtained. It subsequently occurred to Brace to make use of reflection instead of refraction, and look for evidences of a division of the ray into two circularly polarized rays. This at first sight seems to antagonize the law of reflection, but it must be remembered that the law of equality between the angles of incidence and reflection is based upon the fact that the velocity is the same before and after

[1] *Wied. Ann.*, xxvi., p. 576, 1885; *Phil. Mag.* (6), i., p. 464, 1901.

reflection. If we apply the Huygens construction for reflection to a case in which the velocity is less after reflection than before, we shall find that the angle of reflection *is less than* the angle of incidence. Suppose now that our two circular components in the magnetized medium encounter a reflecting surface which reverses the direction of revolution of each. The fast component now becomes the slow, and *vice versa*, and we have a division of the ray. The reversal of the circular vibrations was accomplished by a reflection at 72° at a glass-silver surface, but the separation of the two rays was too small to be detected.

Under these conditions a division of the ray into three rays is to be expected, for the silver-surface transforms the circular vibrations into plane ones, *i.e.* it acts like a $\frac{\lambda}{4}$ plate. Each of the resulting plane vibrations is again broken up into two circular vibrations by the medium. Of these one pair have the same direction of revolution as the components from which they were originally derived, and these will be reflected under the condition $i=r$. Of the other two, one is a right-handed revolution derived from the original left-handed component, which we will assume was the "fast" one, and the reflection is under the condition $i>r$. The opposite is true for the other component, or $r>i$. For the block of glass used the angular separation between the central ray and one of the deviated rays was calculated to be less than 3″ of arc, which was less than the instrument was capable of resolving.

The difficulty was finally overcome by making use of multiple reflections. A rectangular block of glass was made by cementing two right-angled prisms ($n=1.903$) together, with a half-wave plate of mica between them to reverse the direction of the circular vibrations (Fig. 314). The incident light enters the prism normally through a small auxiliary prism A, traverses the $\frac{\lambda}{2}$ plate, which turns its plane of polarization through 90°. The light is travelling perpendicular to the lines of force, consequently the magnetic field does not affect it. As soon, however, as it suffers total reflection, it travels along the lines of force, and is consequently broken up into circular components, one of which travels faster, the other slower, than the original disturbance before reflection. A division therefore results, and we have two reflected rays. Total reflection transforms a plane vibration into an elliptical one, while two total reflections give approximately a circular vibration. This makes it a little harder to study out just what happens than in the previous case. The arrangement figured so alters the vibrations that the change is always from fast to slow and from slow to fast, the angular separation increasing each time the rays are reflected along the lines of

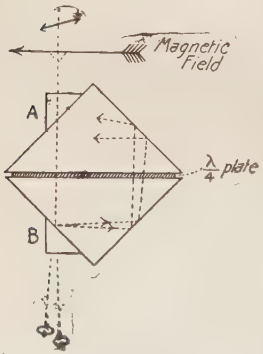


Fig. 314.

force. The light travelled around the prism five times, undergoing twenty internal reflections, and emerged through a second auxiliary prism B at the top of the block. The source was a vertical slit powerfully illuminated with an oxy-hydrogen flame fed with sodium. On turning on the magnetic field the image in the telescope was seen distinctly doubled, and on examination with a Nicol prism the two lines were found to be nearly plane-polarized, due to the fact that the last reflection, combined with passage through the $\frac{\lambda}{2}$ plate, introduced a phase-difference of $\frac{3}{8}\lambda$ between the components of the circular vibrations, by which they were changed into ellipses of considerable eccentricity. The positions of the major axes of these elliptical vibrations (they were at 90° to one another) showed that the circular vibration, in which the direction of revolution was the same as that of the Amperian currents, was accelerated, while the other was retarded by the magnetic field.

The velocity of right- and left-handed circularly polarized light in a magnetized medium was measured by Mills (*Phys. Rev.*, February 1904) by means of a Michelson interferometer. By means of a Brevais double plate, one half of the field was illuminated with right- the other half with left-handed circularly polarized light, the fringes crossing the field in a direction perpendicular to the dividing line between the two halves of the plate. On exciting the magnet the fringes on one side moved up, while those on the other side moved down. The accelerated ray was found to be the one in which the direction of the circular vibration was the same as that of the Amperian currents.

Direction of the Magnetic Rotation. — The results of the earlier experiments upon magnetic rotation appeared to indicate that all substances rotated the plane of polarization in the same direction, when placed in a magnetic field, but continued investigations showed that this was not the case, many diamagnetic substances being found which gave a rotation in the opposite direction. A generalization cannot, however, be made, for dextro- and laevo-rotation are to be found in both paramagnetic and diamagnetic substances, as is shown by the following table:

Diamagnetic		Paramagnetic	
Rotation Positive	Rotation Negative	Rotation Positive	Rotation Negative
Ferrocyanide of Potassium. Water. Lead Borate (Faraday glass). Almost all other solids, liquids, and gases.	Chloride of Titanium.	Iron, cobalt, and nickel. Oxygen. Salts of cobalt, nickel, manganese, and copper.	Ferro and ferri salts of iron. Ferricyanide of potassium. Chromate and bichromate of potassium. Salts of cerium, lanthanum, and didymium.

Relation between the Field Strength and Rotation. — The angular rotation increases in general in proportion to the strength of the field, but this rule is not strictly followed, the most marked exceptions being shown by iron, nickel, and cobalt. In the case of iron the relation between the field strength H and the rotation δ is shown in the following table. If the rotation was proportional to H, the figures of the last column would be approximately the same:

H	δ	$\frac{10^5\delta}{H}$
4420	1.72°	39
8060	3.47°	43
14100	4.41°	31
18500	4.45°	24
30100	4.36°	14

If instead of H we take the *magnetization*, we shall find that the rotation is proportional to this quantity. In the case of iron the magnetization increases with the field strength, but not at the same rate, finally becoming "saturated," beyond which point a further increase of field produces no increase in the magnetization. The rotation also attains a maximum value at the same point.

Time required for the Faraday Effect: Rotation by Oscillatory Discharge. — An experiment was performed by Villari which was interpreted as showing that a certain amount of time was required for the development of the Faraday effect. A block of heavy glass was spun between the poles of an electro-magnet, and the rotation of the plane of polarization was observed to diminish when the speed of rotation exceeded 100 turns per second, becoming practically zero at 200 revolutions. Subsequent experiments made by Bichat, Blondlot,[1] and Lodge [2] showed that some other cause must be found for the effect observed. If a Leyden jar was discharged through a helix of insulated wire surrounding a tube filled with carbon bisulphide, the plane of polarization was found to turn with each impulsive rush of the oscillatory discharge, being waved to and fro at the rate of some 70,000 times per second.

To find out whether any time was required for the development of the effect, Bichat and Blondlot illuminated the upper portion of a slit with the light of the spark and the lower portion with the light restored by the CS_2 tube, and examined it in a revolving mirror. The illuminated slit was seen spread out into a serrated band, but no discontinuity was found between the two illuminated halves, showing that the effect is practically instantaneous.

A tentative explanation of the effect of spinning the glass block in Villari's experiment is given by Lodge in his paper, based upon the assumption that it is due to centrifugal strains induced in the glass by the high speed.

Magnetic Rotation and the Second Law of Thermodynamics. — Lord Rayleigh has called attention to a curious disposition of apparatus which permits the passage of light in one direction, while

[1] *Comptes Rendus*, xciv., p. 1590, 1882.

[2] *Phil. Mag.*, April 1899.

refusing absolutely to transmit it in the reverse direction. Two Nicol prisms are mounted with their directions of vibration at an angle of 45°. Between them consider a medium in a magnetic field of strength sufficient to produce a rotation of the plane of polarization of 45°. The polarized light passing through one Nicol will be rotated into such a plane that the second Nicol will stop it. If we reverse the direction of the light the polarized vibration will be brought into the plane of vibration of the second Nicol and be freely transmitted, since the magnetized medium rotates the plane of polarization in the same absolute direction (*i.e.* considered without regard to the direction of propagation) whichever way the light be travelling.

It is thus possible to construct an apparatus through which we can see without being seen, provided we limit ourselves to monochromatic light. It might be supposed, at first sight, that, by making use of this contrivance, an exception to the second law of thermodynamics might be realized, a radiating body throwing more energy into a second body than it received in exchange and thus elevating its temperature. Wien considered this case in his treatment of the theory of radiation presented at the International Congress of Physics, held in Paris (1900), and offered certain ways of escape out of the difficulty. Lord Rayleigh shows, however (*Nature*, lxiv.), that, after all, no difficulty really exists.

The Kerr Effect. — The important discovery was made by Kerr [1] that plane-polarized light becomes elliptically polarized when reflected from the polished pole of an electro-magnet. The incident light must be polarized either in, or perpendicular to, the plane of incidence, otherwise elliptical polarization results from the metallic reflection. On setting a Nicol prism in such a position as to completely extinguish the reflected light, and exciting the magnet, the light instantly reappeared, and could not be extinguished by further rotation of the Nicol, except by the introduction of a quarter-wave plate. The ellipticity is not very great, and we can regard the effect as a rotation of the plane of polarization.

Intimately connected with the Kerr effect is the rotation of the plane of polarization by thin films of iron in a magnetic field, which was studied by Kundt. Films with a thickness of about one-third of the wave-length of the light were employed, which when placed in a strong magnetic field gave a rotation of over 4°. An iron plate 1 mm. thick, if it were possible to get any light through it, would rotate the plane through 20,000° or 66 complete turns, while a quartz plate of equal thickness gives a rotation of about 20°. If now we consider that the light penetrates the surface of the magnet pole in the act of reflection, the rotation is at once accounted for. A more complete explanation will be given when we come to the Theory of Magnetic Rotation, and the simple explanation just given is quite inadequate, for it has been shown that the effects are considerably modified by surface-films.

Magnetic Rotatory Dispersion. — The angular rotation of the

[1] *Phil. Mag.*, May 1877 and March 1888.

plane of polarization in a magnetic field of given strength increases as the wave-length decreases, provided we limit ourselves to a region of the spectrum within which the substance shows no absorption bands. In the case of absorbing media anomalous rotatory dispersion may occur when we cross the absorption band, provided that the absorption and rotation are both due to the same electron, as in the case of substances which show natural rotation.

The behavior of alcoholic solutions of the aniline dyes in a magnetic field has been studied by Schmauss,[1] who claimed that the effect of the dye was to increase the rotation of the alcohol on the red side of the absorption band and decrease it on the violet side. If this is the case we should expect the solid dye to rotate the plane of polarization for waves on opposite sides of the absorption band in opposite directions. The author has, however, been unable to detect any trace whatever of rotary polarization in a film of solid cyanine so thick that nothing but red light was transmitted. Saturated solutions of cyanine between plates of very thin glass also showed no trace of the phenomenon, and it seems probable that the results obtained by Schmauss were due to experimental errors.

Bates (*Ann. der Physik*, 12, page 1901 (1903)), making use of a much more sensitive method, failed to detect any anomalies, and came to the same conclusion as just stated.

A solution of a didymium salt, which has a strong absorption band in the yellow, has been examined by the author (*Phil. Mag.*, May 1904), and found to exhibit apparent evidences of anomalous rotary dispersion at the edge of the absorption band. The curve obtained is shown in Fig. 315 *a*.

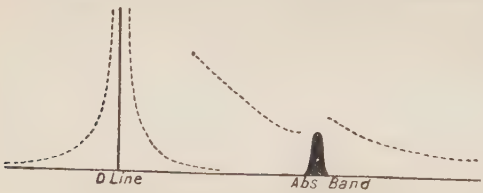


Fig. 315.

Contrary results have been obtained in the case of sodium vapor, in which the rotation has a high value on both sides of the *D* lines, the *direction* of rotation, however, being the same, as shown in Fig. 315 *b*. These cases will be more fully considered in the section on the Theory of Magnetic Rotation.

The question of the manner in which absorption bands affect the magnetic rotation for wave-lengths in their vicinity is of great importance in connection with the theory of rotation, as we shall see presently. A study of the rotation in the infra-red should also add valuable data, as will be apparent when we stop to consider the part played by the infra-red investigations of Rubens and his collaborators in the development of the ordinary dispersion theory.

Ingersoll[2] has carried out a difficult and important investigation of the Faraday and Kerr effects in the infra-red region of the spec-

[1] *Ann. der Physik*, 1902–1903.

[2] L. R. Ingersoll, on the "Faraday and Kerr Effects in the Infra-red Spectrum." *Phil. Mag.*, January 1906.

trum, using piles of very thin glass plates as polarizer and analyzer, which permitted of observations as far as 4.3 μ. The values which he found are given in the following table, observations being taken in the visible spectrum in the yellow and red. They are for a tube of carbon bisulphide 4 cms. in length in a magnetic field of 6000 c. g. s. units. The values calculated from the Drude formula (Hall effect hypothesis), which we shall take up later, are given in the third column. The curve is practically flat beyond 3 μ. The dispersion in the Kerr effect was found to be quite different. In the case of steel it rose to a maximum at 1 μ, and then fell gradually with increasing wave-length. Dubois had shown that the Kerr effect in the visible spectrum increases with increase of wave-length.

Wave-length	Observed	Calculated
μ		
.589	1.00	1.00
.700	.660	.670
.945	.332	.348
1.076	.272	.265
1.126	.248	.242
1.170	.231	.225
1.419	.158	.152
1.80	.099	.095
2.00	.079	.078
2.50	.052	.053
3.00	.041	.041
3.50	.036	.037
4.00	.037	.038
4.30	.042	.041

The Zeeman Effect. — A search for a possible effect of a powerful magnetic field upon a source of monochromatic vibrations was first made by Faraday. Placing a sodium flame between the pole pieces of a large electro-magnet, he examined the appearance of the D lines when the field was "on" and "off." He was unable, however, to convince himself that any change resulted in the appearance of the lines, a circumstance which we now know resulted from the insufficient power of his spectroscope. This was in 1862. Twenty-three years later the same experiment was tried by Fievez, who observed changes which appeared to be the result of the magnetic field. If the lines were single before the current was turned into the magnet, they appeared with dark reversals down their centres as soon as the magnet was excited. These observations do not appear to have attracted much attention, and they were not followed up. It is questionable even whether the phenomenon observed by Fievez was in reality the effect that we are about to study.

In 1896 the fact was established beyond all doubt by Zeeman, that the periodic time of vibration of a source of monochromatic radiations is altered when the source is placed in a magnetic field.

In his first announcement he described the effect as a simple broadening of the lines, but his knowledge of the probable cause of the effect that he was searching for led him to believe that the edges of the line should appear circularly polarized when the light was emitted in the direction of the lines of force. To test this he employed a magnet with perforated pole pieces, and passed the light through a quarter-wave plate, which as we have seen transforms a circular vibration into a plane one. He found that the lines now shifted their position in the spectrum when the direction of the current in the magnet was reversed, which indicated that the edges of the line were circularly polarized in opposite directions, exactly as theory showed that they should be.

When the light was examined in a direction perpendicular to the lines of force, the edges were found to be plane-polarized, from which Zeeman drew the conclusion that, with a sufficiently strong field, the line would appear triple, a prediction which was speedily verified by experiment, the two outer components being polarized with their vibration directions perpendicular to the lines of force, while the central component vibrated along the lines of force.

Soon after Zeeman's announcement Michelson made a very complete study of the influence of magnetism upon radiations by means of the interferometer. From the visibility curves obtained with this instrument, he concluded that the lines were doubled when the light was examined in a direction perpendicular to the lines of force. This was subsequently found to be due to the fact that the light of the central component was polarized in such a direction that the oblique interferometer plate refused to reflect it.

The phenomena of radiation in a magnetic field are easily explained on the electron theory, at least the simple cases just described. A further study has shown, however, that many lines are broken up into quadruplets and sextuplets, the D lines of sodium belonging to this class. In a powerful field D_1 is seen to be a quadruple line, the inner components being polarized with their vibrations along the lines of force, the outer components perpendicular to them. D_2 is a sextuplet, with its four outer components vibrating perpendicular to the field. It was formerly supposed that certain iron lines appeared as triplets with the directions of polarization the exact reverse of the usual order of things. Recent work in Zeeman's laboratory has shown however that this is probably not the case.

No simple explanation can be given for these more complicated cases. They will be considered in the section on the Theory of Magneto-Optical Phenomena.

We will now consider the formation of the circularly polarized doublet seen in the direction of the field, and the plane-polarized triplet seen perpendicularly to it.

Consider the source of monochromatic vibrations as a swarm of atoms, the charged electrons of which rotate in circular or elliptical orbits, or vibrate back and forth along straight lines. On either hypothesis we can account for the altered appearance of the line in a magnetic field, by considering the force which a charged particle in rapid motion experiences in a magnetic field. This force acts in a direction perpendicular both to the direction of motion of the electron, and to the direction of the lines of force, and is equal to zero only when the direction of motion coincides with the direction of the field.

We will first consider electrons revolving in circular orbits, the planes of which are oriented in all possible positions. Those electrons which are moving in orbits perpendicular to the lines of force are subjected to a force which is directed towards or away from the centre of rotation, according to the direction of revolution. The ones which are revolving in such a direction that the force acts towards the centre are drawn in and have their period of revolution

accelerated, while those rotating in opposite directions have their centripetal force diminished and their periods slowed down. The force which acts on the electron as a result of the field, either increases or diminishes the centripetal force which holds the electron in its orbit. An apple whirled around on a string is a useful mechanical analogy. If we increase the tension on the string, we pull in the apple and decrease the time of its revolution. If we relax the string, the orbit opens out and the time of revolution becomes greater.

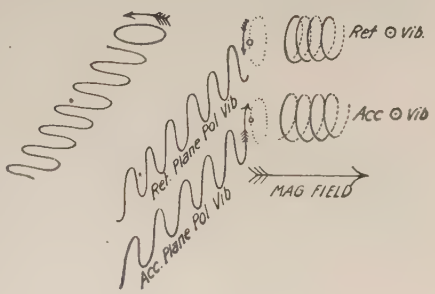


Fig. 316.

Consider now the types of waves sent out by the two types of revolving electrons which we have considered. In a direction parallel to the lines of force, the whirling electrons will radiate circularly polarized rays, one ray polarized clockwise, the other counterclockwise. The periodic time of one is accelerated, while that of the other is retarded; consequently we get two lines in the spectrum, circularly polarized in opposite directions.

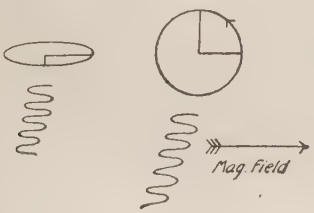


Fig. 317.

In a direction perpendicular to the lines of force the two electrons give off vibrations plane-polarized in the plane of their orbits (Fig. 316), of periods identical with those of the circular vibrations. These rays give us the outer components of the normal Zeeman triplet.

In the case of an electron revolving in an orbit the plane of which is parallel to the lines of force, the force exerted on the electron will be perpendicular to the plane of the orbit. The circular vibration can be resolved into two rectilinear vibrations at right angles to one another. Consider two orbits oriented as shown in Fig. 317. The component parallel to field will be unaffected, and we shall have plane-polarized vibrations of unchanged period emitted in a direction perpendicular to the field. This component sends out no radiation in the direction of the lines of force, however. The other component must be regarded as affected in the manner about to be described. After considering the effect of the field upon an electron moving in a straight line, it will be found profitable to apply the same reasoning to the present case. It will then appear that right- and left-handed circular vibrations are given out along the lines of force, and plane-polarized vibrations of accelerated and retarded frequencies perpendicular to them.

We will now examine the effect of a magnetic field on electrons which normally vibrate back and forth along straight lines, instead of moving in circular orbits. They will be subjected to a force

which acts in a direction perpendicular to the field and their direction of motion. Let AB (Fig. 318) represent the normal path of the electron, and consider the lines of force as perpendicular to the paper. If the electron is at O, and moving down at the moment when the field is thrown on, it will be deflected to C, moving in a curve, the convex side of which is towards the right. On its return trip it will move along a curve, the convex side of which is to the left, since the force acting on it is now in the opposite direction, owing to the reversal of its direction of motion. It will thus pass in succession through the points D, E, F, G, the curve reminding us of the type of vibration resulting from the passage of a plane-polarized ray through a rapidly revolving Nicol prism. (See Light-Beats.) As we have seen, a vibration of this type is the equivalent of two oppositely polarized circular vibrations of different periods; consequently the rays sent out in the direction of the field, *i.e.* perpendicular to the paper, are resolved by the spectroscope into two rays circularly polarized in opposite directions.

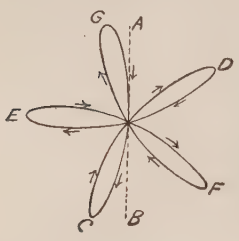


FIG. 318.

The rays sent out by the electron perpendicular to the lines of force, *i.e.* in the plane of the paper, will be plane-polarized, and at first sight would appear to be monochromatic. If we remember, however, that the electron cannot give out transverse waves in the direction of its motion, we shall see at once that the *amplitude* of the waves sent out in any given direction, say AB, will depend upon the position of the electron in its star-shaped orbit, attaining its maximum value when the loop E is being traversed, and its minimum (practically zero) when the vibration is along GC. At any given point in the plane of the paper, the illumination due to this electron will fluctuate between zero and, say, unity. Now a precisely similar condition of affairs results when two periodic disturbances of slightly different periods pass through the same point. At a given instant the phases will be the same, and we shall have increased illumination, the next instant the phases will be opposed and we shall have darkness. We have the same phenomenon in acoustics, two tuning-forks of nearly the same pitch producing beats, or periodic fluctuations of intensity. We shall thus have a type of radiation emitted perpendicular to the lines of force, peculiar in that the intensity suffers extremely rapid fluctuations, which the spectroscope separates into two lines, plane-polarized in a direction perpendicular to the field. The period of the beats will be the period of revolution of the orbit, which we shall presently see is about $\frac{1}{20,000,000,000}$ of a second. As Lord Rayleigh has pointed out in his *Theory of Sound*, if we should interrupt a train of sound-waves of frequency N, n times per second, the resultant disturbance would contain three frequencies $(N+n)$ N and $(N-n)$. It would be extremely interesting if we could interrupt a beam of monochromatic light rapidly enough to cause the line to become triple in the spectroscope. The thing might possibly be done by means of a revolv-

ing diffraction grating, fine lines parallel to the radii being ruled through a silver film deposited around the edge of a mica disk. The elements of the beam would in this case be interrupted in succession by the bars of the grating. If 20,000 lines to the inch were ruled on a 10-inch disk, driven at a speed of 1000 revolutions per second (which is possible), a modern interferometer, *e.g.* the one of Fabry and Perot, should easily separate the components corresponding to the frequencies $(N+n)$ and $(N-n)$.

An apparent difficulty may occur to the reader in connection with the manner in which we have considered the spectroscopic analysis of the "beats" coming from the electron moving in the star-shaped orbit. Beats result when two continuous trains of uniform amplitude, and slightly different frequencies, pass through a point simultaneously. The components, therefore, into which the spectroscope analyzes the disturbance must be continuous in time, *i.e.* must show no fluctuations in intensity. How now is it possible to have continuous illumination in the spectroscope when there are moments at which the slit is in darkness? This question has nothing to do with the persistence of vision: the illumination must be continuous, regardless of any physiological peculiarities of the eye. The difficulty is only apparent, as can be seen by the following considerations. An effect at the slit occupying an infinitesimal of time is by the action of the grating or prism spread out over a finite interval of time when it reaches the eye. This will be better understood after reading the Chapter on the Nature of White Light. In the case of the grating there is no difficulty, since the disturbances from the different grating elements, resulting from a *single* disturbance at the slit, reach the eye in succession. This being true it is obvious that what occurs at the eye at a given *instant* is the result of disturbances lasting for a finite time at the slit. There is thus no trouble about having a continuous disturbance at the eye, where there is a discontinuous disturbance at the slit.

From our knowledge of the magnitude of the separation of the components of the Zeeman doublet, we can calculate how many to-and-fro excursions the electron makes while the path makes one complete revolution. In the strongest fields known the separation amounts to $\frac{1}{6}$ of the distance between D_1 and D_2, which means that each component moves $\frac{1}{12}$ of this distance from the original position of the line. Taking the frequency differences for the D lines, and dividing this by 12, gives us the frequency difference corresponding to the shift. If n is the normal frequency of the light, and the plane of polarization turns N times per second, the two component circular vibrations into which it can be decomposed have frequencies $(n+N)$ and $(n-N)$ (compare Righi's Experiment). The frequency difference in the above case turns out to be roughly 40,000,000,000, and if we divide this number into the original frequency, we shall obtain a number representing the number of to-and-fro excursions made by the electron while its path turns through one complete revolution. This number turns out to be 15,000, which gives us the number of loops in the star-shaped orbit previously figured.

Complicated Types of the Zeeman Effect. — The elementary theory shows us that the middle line of the triplet should have an intensity equal to the sum of the intensities of the outer components. As a matter of fact, many triplets have been observed for which this relation does not hold, the outer lines being actually stronger than the middle lines in some cases. Zeeman has shown that many of these cases can be explained by the polarizing power of the grating, which reduces the intensity of the vibration perpendicular to the grooves in a greater degree than in the case of one parallel to them. With a horizontal magnetic field and the grating vertical, as is usually the case, the middle line of the triplet will be weakened in intensity to a greater degree than the outer lines. In working with a grating, if we wish to obtain the correct intensity distribution we must rotate the vibrations through an angle of 45°, in which case the grating treats all alike. Cases have been observed, however, in which real abnormalities in the intensity distribution exist. These are best explained by assuming that the magnetic field has a tendency to orient the orbits of the electron, setting them perpendicular to the lines of force. If this orientation were complete, *i.e.* if we had only circular orbits perpendicular to the lines of force, it is clear that the central component of the triplet would vanish entirely. As to the probability of the occurrence of such orientation, more will be said presently.

The circularly polarized doublet seen along the lines of magnetic force, and the plane-polarized triplet seen in a direction perpendicular to the lines of force, may be regarded as the normal types. Many lines, however, behave quite differently. The D lines of sodium, when examined at right angles to the field, were found by Cornu, and independently by Preston, to act in a curious manner. D^1 gave instead of a triplet a quadruplet, the two central components being polarized in the same plane as the central line of the normal triplet, while D_2 gave a sextet, the two central components of which were plane-polarized in a similar manner, while the four outer components were plane-polarized like the outer lines of a triplet.

The quartet results from a symmetrical splitting of the central line of the normal triplet. In some cases this separation may be as great as that of the outer components, as is the case with the iron line 4251, which shows as a doublet in a direction perpendicular to the field. In case of the line 2411 the components of the middle line pass over the outer lines, giving us a quartet with reversed polarization. In some cases the middle line of the triplet splits into three components. In all of these cases we observe a normal doublet along the lines of force, since the vibrations perpendicular to the field (outer components of triplet) are affected in a normal manner. A doubling of all three lines gives rise to a sextet, such as we observe in the case of the sodium line D_2. The mercury line 5461 yields a nonet, formed by a tripling of each line of the normal triplet. Still more complicated types have been observed, in which each line of the triplet is replaced by four or five. The most complicated case thus far observed is furnished by the spectrum of molybdenum, the lines of which show 17 to 19 components.

A number of these interesting cases are shown in Fig. 319, which is taken from Voigt's most interesting book on *Magneto and Electro Optics*. No. 1 is a photograph of the molybdenum line 4269.4 taken in the second order spectrum. On the left, we have the components, eight in number, in a direction perpendicular to the lines of force, and on the right, the five components seen parallel to the field. This photograph was taken in Professor Voigt's laboratory, while the six which follow were made by Dr. Lohmann in Halle with an echelon grating. No. 2 shows the neon line 6600 in a magnetic field of 10,800 Gausses. No. 3 is the 6718 line (field strength 5700). No. 4, $\lambda = 6507$, $H = 9950$. No. 5, $\lambda = 6143$, $H = 9300$. In No. 6 we have a decomposition into 12 components ($\lambda = 6335$, $H = 8420$). No. 7, $\lambda = 6402$, $H = 15{,}350$.

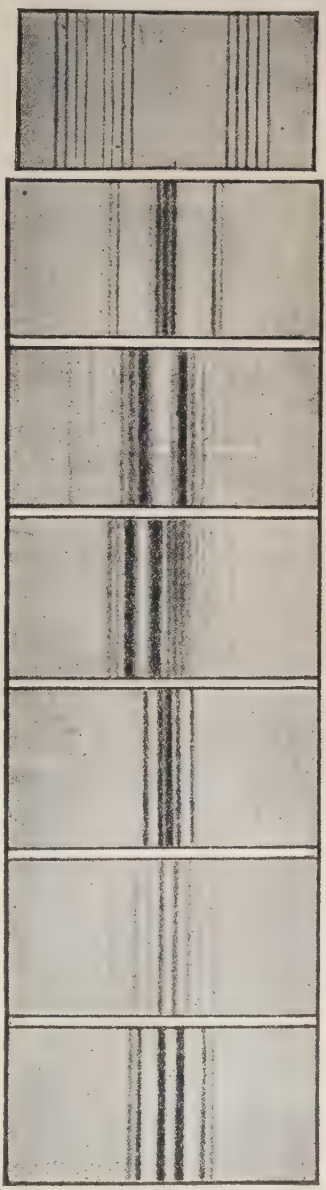


FIG. 319.

Asymmetrical Separation of the Triplet. — Voigt from theoretical considerations came to the conclusion that the outer components of the triplet seen perpendicular to the lines of force should show the following asymmetry: the component on the red side should show the greater intensity, while the one on the violet side should be found at the greater distance from the original line. Zeeman observed an effect of this type in the case of one of the yellow mercury lines. The vacuum tube was placed in the very non-homogeneous field between the conical pole pieces of the magnet. The largest separation of the lines will occur at the centre of the tube where the field is strongest, decreasing gradually to zero both above and below. By projecting an image of the tube on the slit of the spectrograph it was possible to obtain with a single exposure a photograph showing

the separation in magnetic fields of very different intensities. Voigt's prediction was verified in the case of the yellow line 5791, but no trace of the effect was observed in the case of the 5770 line, for which the separation was symmetrical. The difference in the behavior of the two lines is clearly shown in Fig. 320. Effects of a similar but opposite nature were found in Voigt's laboratory in

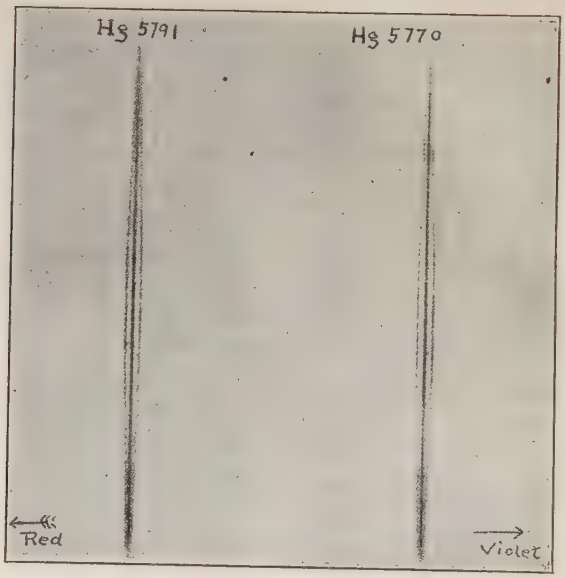


FIG. 320.

the case of the spectrum of tungsten. Voigt explains these anomalies as a result of a shift of the central component from the original position of the line. This shift of the central component is not called for by theory, but it has been found experimentally by Zeeman, and independently by Gmelin.

Shift of the Middle Component. — Zeeman in observing the doublet along the lines of force noticed a faint but very sharp line midway between the two components. In the case of the 5770 mercury line this new line was exactly midway between the components of the doublet, while in the case of the 5791 line it appeared displaced towards the red. These rays were evidently due to light emitted in a direction not strictly parallel to the lines of force and reflected from the inner walls of the tube, for it was plane-polarized. This line evidently corresponds to the central line of the triplet seen in a direction perpendicular to the lines of force. In the case of the 5791 line it was necessary to ascertain whether the asymmetrical position of the faint line between the doublet was due to a shift of the central component of the triplet or to the fact that the circularly polarized components of the doublet were not symmetrically placed with respect to the original line.

Zeeman accordingly investigated the matter with his method of non-uniform fields. As we have seen, a change of wave-length gives us, in this case, a curved line, the outer components of the normal triplet appearing bowed out, the spindle-shaped figure being bisected by the straight middle component. This middle component would appear slightly curved if it had suffered any change of wave-length. Photographs made with the grating showed no evidence of this, and Zeeman accordingly tried the echelon, giving us the following account of his results:

"The shift of the middle line of the triplet may be demonstrated also by our method of the non-uniform field, if an echelon spectroscope is made use of. A curvature of the middle line will be the immediate effect of the shift. If we use Rowland's grating such a curvature would be invisible, nor have I observed it in that case.

"The visibility of the curvature will be much increased by taking care that in the image points corresponding to very different intensities of field lie closely together. In order to obtain this an eleven times reduced image of the vacuum tube, charged with mercury and placed in the field, was projected on the slit of the auxiliary spectroscope. The lens used was a photographico objective of 10 cms. focus.

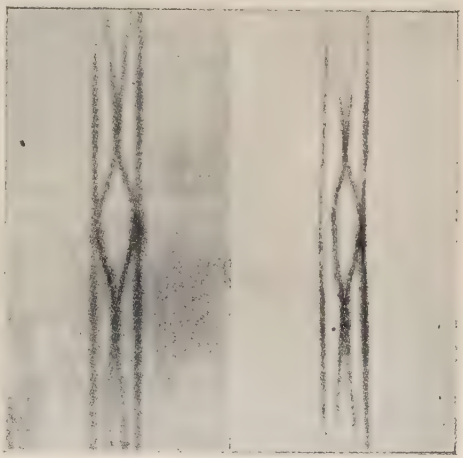


FIG. 321.

"The plate (Fig. 321) gives somewhat enlarged reproductions of negatives relating to line 5791 resp. line 5770. The middle line is given in two succeeding orders. Between these the other components of the triplets are seen. With increasing magnetic force the components deviate further and further from their own middle line. In the central part of the field of view the maximum distance is reached. The component towards the red in the figures is always at the left of its middle line, being concave to it in the central; the second manifestly curved line is the component towards the violet belonging to the other order.

"The curvature of the middle lines, the demonstration of which is the object of our present experiment, is undoubtedly visible in the figure for 5791. It is still more easily seen by comparison with a straight bit of paper.

"In the figure for 5770 this kind of curvature is absent.

"The asymmetry of the magnetic resolution of line 5791 is at once

evident by the fact that one of the middle lines is approached more nearly by the outer component than the other."

These photographs are, at first sight, a little confusing, owing to the circumstance that the echelon gives two orders of spectra. The middle line of the triplet is represented by the two outer lines of the photograph. Their curvature at the centre can be best seen by holding the eye nearly in the plane of the paper and sighting along them. The outer lines of the triplet start at the top and bottom (not reproduced) in coincidence with the outer lines (in reality the middle line of the triplet), swerving away the one to the right, the other to the left, and crossing each other a little above and a little below the centre, where the field intensity has its greatest value.

To test whether the asymmetry was due entirely to the shift of the middle line, Zeeman tried to photograph it with the field on and the field off, and determine whether the shift was of the same magnitude as that calculated on the assumption that the outer lines of the triplet were symmetrically placed with respect to the original line. The shift, however, was too slight to make reliable measurements possible.

The Zeeman Effect and Spectral Series. — As we have seen, spectrum lines are divided in all sorts of ways in a magnetic field, the normal triplet being the exception rather than the rule; the normal triplet occurs however in hundreds of examples in the spectra of zirconium and titanium. A great number of lines of the iron spectrum also become triplets under the action of the magnetic field.

The measurements of Runge and Paschen and others have shown, however, that in the case of lines belonging to the same spectral series, the type of magnetic separation is the same, *i.e.* all lines of a given spectrum that can be represented by a formula are divided in the same way. As Lorentz says:

"In those series which consist of triplets or doublets, the mode of division of the lines is in general different for the lines of one and the same triplet or pair, but, according to the law just mentioned, the same mode of division repeats itself in every triplet or every doublet. Thus, in each triplet belonging to the second subordinate series of mercury, the less refrangible line is split into nine components, the middle line into six, and the most refrangible line into three components.

"Equal modes of division are found not only in the different lines of one and the same series, but also in the corresponding series of different elements. For the lines D_1 and D_2 of sodium, which form the first member of the principal series, are changed into a quartet and a sextet, and the first terms in the principal series of copper and silver present exactly the same division."

As an example of spectral series we may take the case of sodium.

Here we have three series of close double lines, the principal series and two subordinate series, in which the spacing of the lines along the spectrum can be represented by a simple formula.

The D lines form the first observed pair of the principal series,

the other members being in the ultra-violet, the distance between the pairs decreasing in a uniform manner as we proceed down the spectrum. Investigations of the absorption spectrum by the author have raised the number of members of this series from the seven previously known to forty-eight, the largest number that has been observed for any element thus far. A photograph of a portion of this series is shown in Fig. 322 *a*, the number of each unresolved pair being indicated. Just below this, in spectrum *b*, we have the same absorption spectrum, with an iron comparison spectrum. The short wave-lengths are to the left, and the remarkable general absorption which begins at the head of the series and extends to extreme ultra-violet is shown. Spectrum *C* was taken with denser vapor, and shows that each pair of the series is immersed, so to speak, in a band spectrum, analogous to the one accompanying the *D* lines, the magnetic rotation of which we have just considered.

FIG. 322.

The second subordinate series consists also of a series of double lines, and there is undoubtedly some connection between the two series, for the first members of each series, as represented by the

formula, fall at the same wave-length. The first members have not been observed, however, *i.e.* no spectrum lines have been found at this point. That there is some intimate relation between the lines of the two series is also shown by the Zeeman effect, for observations have shown that the more refrangible components of the doublets of one of these series exhibits the same type of magnetic separation as the less refrangible components of the other series.

An account of the various attempts that have been made to explain the more complicated types of magnetic separation would require too much space, and the reader is referred to Lorentz's interesting book, *The Theory of Electrons*, for a very complete presentation of all that is known at the present time about the subject.

Magnitude of the Separation. — Spectroscopic recognition of the Zeeman phenomenon requires an instrument of high resolving power. The most convenient form is undoubtedly the echelon grating devised by Michelson; it can be adjusted in a few minutes, gives a large amount of light and is fairly compact. The author has seen the effect without difficulty with an echelon improvised from four interferometer plates which were mounted on the table of a spectrometer with a step-width of 1 mm. A small screen of cardboard with a rectangular opening 5 mms. wide and 2 cms. high limited the beam coming from the collimator. This was mounted in such a position as to have a strip 1 mm. wide along the edge of the grating, five interfering beams, with high relative retardations, being obtained in this way. The spectrometer was illuminated with the light from a mercury vacuum tube placed between the poles of an electro-magnet and a small direct vision prism placed between the echelon and the telescope to separate the mercury lines. The splitting up of the green line on exciting the magnet was easily seen with this improvised apparatus. A helium tube is equally satisfactory and less troublesome, as it requires no heating. The sodium flame is unsatisfactory, owing to the small distance between the D lines.

The results obtained with the echelon are not, however, easy to interpret, and the proximity of the spectra of other orders gives trouble, when anything more than a qualitative experiment is to be made. For accurate quantitative work the concave grating is undoubtedly the best type of instrument to use.

The magnitude of the separation, even for very intense fields, is very slight. Zeeman concluded from measurements of photographs that for a field-strength of 10,000 c.g.s. units the distance between the outer components of the D_1 quadruplet amounted to $\frac{1}{13}$ of the distance between D_1 and D_2. The same separation will of course be found in the case of the two circular components seen along the lines of force.

The magnetic separation of the different lines in the spectrum of a given substance is by no means the same. This is of course to be expected, for the effect of the field on the motion of the electron will depend upon the ratio of its charge to its mass. Certain lines in the spectrum are, however, supposed to have a common origin, and

theory shows that in this case the magnetic separation will decrease with the wave-length. If we represent the separation by $d\lambda$ we should have the relation $\frac{d\lambda}{\lambda^2} = \text{const.}$, which has been verified for a number of lines by Preston, Reese, Kent, and others. This relation holds, however, only for lines which belong to the same series. It is interesting to remark that in the case of helium the lines of all the six series of lines split up into normal triplets.

We should expect the magnitude of the separation to vary directly with the field-strength H. Any departure would be difficult to reconcile with the theory of the phenomenon. Kent and Reese were of the opinion that the separation did not increase in proportion to the field-strength, the linear relation only holding up to values of H in the neighborhood of 15,000 c. g. s. units. Above this point the separation becomes less than the required amount. The departure is very small, however, and more recent work by Runge and Paschen makes it appear probable that if high enough resolving powers are used, the separation is strictly proportional to the strength of the field.

Study of the Zeeman Effect without a Spectroscope. — A very convenient and simple method of showing the change in the wave-length is that employed by Cotton [1] and Koenig.[2] The spectroscope is dispensed with entirely, and an absorbing flame put in its place. As is well known, the sodium flame has the power of absorbing, strongly, radiations of the same wave-length as those which it emits. If we place a bright sodium flame between the poles of an electro-magnet and in front of it a second sodium flame, preferably a less luminous flame, such as can be obtained by burning a jet of illuminating gas at the tip of a piece of soft glass tubing drawn down to a point, the light which is emitted by the first flame will be partially absorbed by the second, which will appear dark in consequence. If the second flame contains much sodium, it may happen that its edges only appear dark.

If now the magnet is excited, the wave-lengths emitted by the first flame are changed, and the second flame, no longer able to absorb them, brightens up in consequence, or, to be more exact, it no longer appears darker than the background. It is best to try the experiment first along the lines of force, using perforated pole-pieces, and placing the second flame close to one end of the magnet. The phenomenon is less marked in a direction perpendicular to the lines of force, since in this case the emission line breaks up into a triplet, and the second flame is capable of absorbing completely the middle component, the period of which is unchanged. This central component is, however, plane-polarized and can be cut out by means of a Nicol prism, under which conditions the brightening up of the flame is quite as conspicuous as in the direction of the lines of force. We may modify the experiment by placing the absorbing flame in the magnetic field. In this case the vertical vibrations will predominate in the beam coming from the two flames, for the flame

[1] *Comptes Rendus*, cxxv., p. 865. [2] *Wied. Ann.*, 63, p. 268.

in the field absorbs only the horizontal vibrations coming from the other flame. The presence of polarized light can be easily recognized by means of a Savart plate. This experiment was performed by Lorentz in connection with a study of an effect found by Egoroff and Georgiewsky, which will be presently considered.

The Inverse Zeeman Effect. — The division of absorption lines into triplets and doublets when the absorbing medium is placed in a magnetic field is known as the inverse Zeeman effect. It results from the circumstance that the vibrating mechanism which gives rise to emission is also responsible for absorption, and the natural frequencies are affected by the magnetic field in the same way in the two cases. The amount of absorption produced by either component of a doublet or triplet, however, depends upon the state of polarization of the white light from which certain wave-lengths are abstracted by absorption. If the white light is circularly polarized, and traverses the magnetized absorbing medium parallel to the lines of force, one component of the doublet will be absolutely black, while the other will be invisible. If the direction of rotation be reversed, the other component of the doublet appears and the first fades away. The same is true with the triplet and plane-polarized light. If the light is unpolarized to start with, all of the components appear, but they are not black, since only a portion (namely, that polarized in the proper way) is absorbed by the medium. In this case the components can be caused to appear black by employing a Nicol prism or a circular analyzer ($\lambda/4$ plate and Nicol) for viewing the spectrum. Zeeman and Winawer have made an extensive study [1] of the phenomenon on account of its importance in connection with G. E. Hale's remarkable discovery of the Zeeman effect in sun-spots, which will be discussed presently. They employed a doubly refracting rhomb of Iceland spar so oriented that the two plane-polarized images overlapped in a narrow strip along the centre of the field, this strip showing the appearance of the phenomenon with unpolarized light. Their paper is illustrated with a large number of photographs, and gives a very complete and interesting account of the nature of the absorption, not only when the light traverses the medium parallel and perpendicular to the field, but also when the direction of the waves is oblique, in which case we have elliptical polarization. Two of their photographs are reproduced in Fig. 323 (upper two pictures). The left-hand one was made with the Fresnel rhomb, the right-hand one with the circular analyzer.

Their observations are too numerous to be recorded in full, but their recent discovery of *unpolarized* components in the case of absorption lines is of too great interest and importance to be omitted. They passed the light of the arc through the absorbing flame (in the magnetic field at a small angle with the lines of force) and obtained results which are best given in their own words.

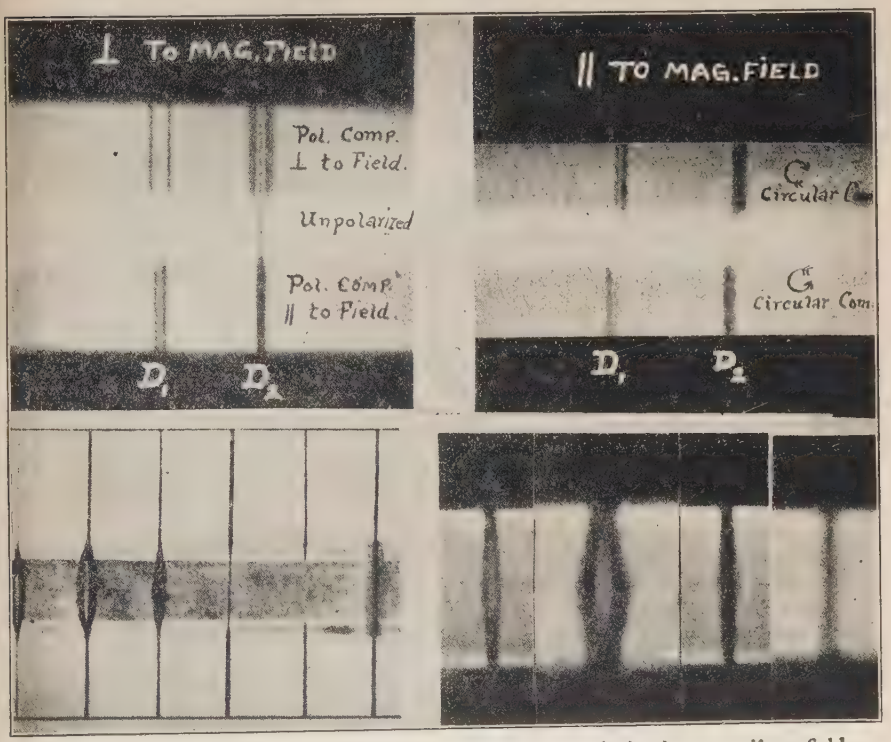
"A still smaller angle between the directions of the beam and of the field may be employed by looking through axial holes and

[1] *Proc. Amst. Acad.*, April and June, 1910.

deviating the beam in the field by means of two small prisms. A remark of Professor Wertheim Salomonson induced us to give prisms a trial."

The arrangement for $\vartheta = 16°$ is shown in Fig. 323, *a*.

The prisms are fixed to copper tubes, which are put into the bored cones of a du Bois electro-magnet and may be turned about their axes. It is therefore possible to adjust the parallelism of the planes of prisms and to arrange vertically the edges.



Sun-spot lines (Mitchell). Magnetic resolution in non-uniform field.

FIG. 323.

Even with very dense vapor, the field being of the order of 20,000 Gauss, the phenomenon closely resembles the pure longitudinal one. No trace of the middle components is visible.

After an increase, however, of the vapor density to the limit obtainable by the introduction of a glass rod charged with melted salt into the gas-oxygen flame, two new black lines appeared in the vicinity of D_1; they were clearly visible against the rather dark background formed by the broadened outer components.

These new lines, which have the same period as the middle components, are unpolarized.

We have come to this conclusion after trying in vain to detect any trace of polarization phenomena of the new components.

In the first place, rotation of a Nicol, placed before the slit of the spectroscope, gave no change of intensity of the lines; only the background formed by the nearly, but not accurately, circularly polarized outer components was slightly changed.

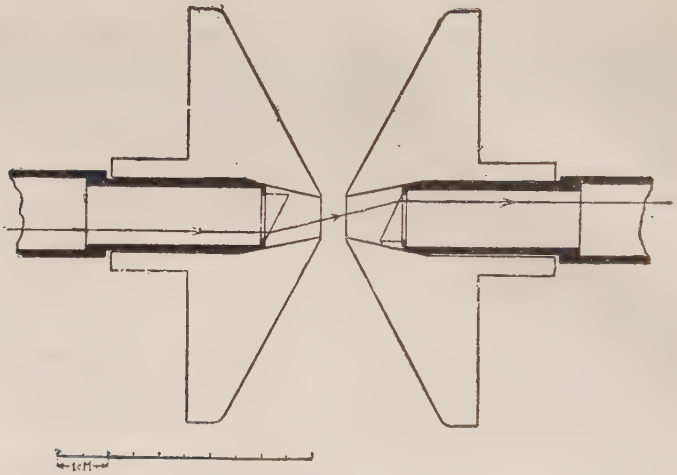


Fig. 324.

After removal of the Nicol a quarter-wave plate with its principal direction under 45° was inserted in the beam and a broad horizontal slit placed near the field. By means of a calc-spar rhomb two stripes are obtained, separating the oppositely polarized circular vibrations.

With vapor of intermediate density, A in Fig. 325 gives the appearance for D_1. The vertical line represents the reversed line due to the arc-light.

With very dense vapor, we get the phenomenon represented in B. New components appear in the initially bright parts of the field of view.

The positions of the new components correspond to those of the inner components of the quartet, at least as far can be judged by eye observation. This observation is confirmed by measurements made on a photograph of, it must be said, only moderate quality.

As to the polarization of the new lines a few remarks may be made. From an inspection of B alone, one might infer a circular polarization of the inner components of a sign opposite to that of the outer ones.

One might be tempted to infer that, under the circumstances of the experiments, the inner components are due to the motion of positive charges.

There is no need of discussing the probability of such a conclusion, as it is refuted by the next observation.

If the quarter-wave plate be rotated in its own plane so that the principal direction more and more nearly approaches the horizontal position, the intensity of the outer components decreases. The inner components, which at first are invisible in two of the quadrants, being entirely hidden by the black, broad, outer components, are presently seen as continuous bands crossing at right angles the horizontal line of separation.

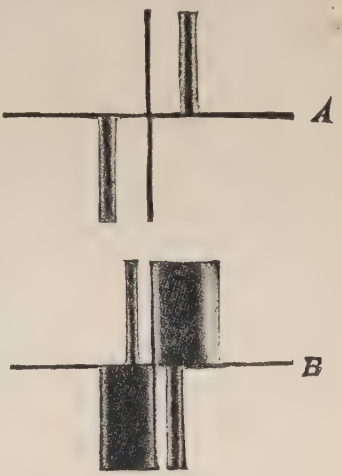


Fig. 325.

Finally, when the principal direction of the quarter-wave plate has become horizontal, there is, as far as concerns the inner components, no difference at all between the upper and lower fields, and only a slight one as far as concerns the outer components.

From the observations recorded we cannot but conclude that under the circumstances of the experiment the inner components of the *new* quartet are unpolarized.

This result seems paradoxical, because one now has become accustomed to expect polarization of all magnetically separated and displaced lines.

The result, however, seems to be in perfect accordance with theory, at least if it be permitted to apply to the middle components of the quartet the theoretical inference drawn for the central component of the triplet.

Lorentz has proved that in the case of a triplet for a frequency $n = n_0$ and $\vartheta < \vartheta_1$, two oppositely elliptically polarized beams may be transmitted, having the same index of absorption, but unequal velocities of propagation. The characteristic vibration ellipses for the two beams are the same, but described in opposite directions.

Since the indices of absorption of the two beams are equal, we may expect that, under the circumstances mentioned, a magnetized vapor can produce in a continuous, unpolarized spectrum only unpolarized absorption lines.

The Zeeman Effect on the Sun. — One of the most brilliant discoveries ever made in Astrophysics was G. E. Hale's observation of the Zeeman effect in the spectrum of sun spots. His photographs, made with the spectro-heliograph, by means of which an image of the solar surface is made solely by light of a single wavelength, showed that in many cases the luminous matter surrounding the spot was rotating at high speed in the form of a vortex or gigantic maelström. If the solar gases contained free electrons, their rotation should produce a magnetic field, and the light emitted by the spot should be circularly polarized, or in the case of absorption lines

we should have the polarized doublet described in the previous section. Observation of the spectrum through a circular analyzer showed that the absorption line was displaced by the rotation of the $\lambda/4$ plate, proving the existence of the Zeeman doublet. The triplet was discovered with a Nicol prism in the case of spots near the edge of the sun, in which case the lines of magnetic force were nearly perpendicular to the direction in which the light was emitted. This discovery of the existence of magnetic fields of vast size on the surface of the sun, produced by whirling vortices of electrically charged matter, comparable in intensity with those which we can produce in the laboratory by the aid of every instrumental refinement over an area of only a few millimeters, is one of the most sensational discoveries ever made.

Zeeman has published in his paper two photographs which illustrate in a striking manner the resemblance between photographs of the absorption lines in the spectra of sun-spots and the magnetically divided lines obtained when the absorbing vapor is placed in a non-uniform field. The intensity of the magnetic field is greatest at the centre of the spot, for it is here that the velocity of the whirl of charged electrons is greatest. These photographs are reproduced in Fig. 323 (lower figures). The left-hand one represents a portion of the solar spectrum, the dark band across the centre representing the spectrum of the spot. The widening of the Fraunhofer lines as they cut across the spot is strikingly analogous to the appearance of the lines obtained by Zeeman in a non-uniform magnetic field.

Partial Polarization of the Light emitted by a Flame in a Magnetic Field. — We have seen that the light emitted in a direction perpendicular to the lines of force consists of three different sets of vibrations, one polarized parallel, the other perpendicular, to the field. If the total amount of light in the two outer components of the triplet is greater or less than the amount in the central component, the light should exhibit traces of plane-polarization. We may regard the revolving electrons as circular or elliptical convection currents, each one accompanied by its own magnetic field, and it would seem therefore as if the external field might well exercise a directive force upon the orbits, orienting them in the same manner as the hypothetical molecular currents are supposed to be oriented in Ampère's theory of magnetism. If this were the case we should expect the light emitted in a direction perpendicular to the lines of force to be more or less completely plane-polarized, as will be readily understood by reference to Fig. 326, in which the electronic orbits have all been brought into the same plane. Though traces of polarization have been detected by Egoroff and Georgiewsky it is very doubtful if the phenomenon is to be referred to this action. In the first place an orientation such as we have assumed would result in an emission of circularly polarized light of a single type and wave-length along the lines of force, instead of the two equal components of different

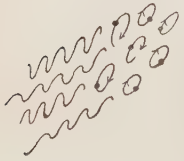


Fig. 326.

periodicities and opposite directions of revolution. How then are we to explain the presence of plane-polarized light in the beam emitted perpendicularly to the field? The experiments above referred to showed that fully 11 % of the emitted light was plane-polarized, with its vibration direction perpendicular to the field, just as it should be if a partial orientation, as assumed above, had taken place. But it was found that the phenomenon only appeared in the case of luminous vapors which showed strong absorption, in other words it only occurred in the case of lines easily reversed. This appeared to indicate that absorption played some part in the production of the polarization.

As we have seen, we may have plane-polarization produced when we have two flames, one in the magnetic field, the other outside. The same thing can take place if we have a single flame in a non-homogeneous field. Cotton, however, found that even in homogeneous fields the emitted light was partially plane-polarized. The complete explanation was given by Lorentz,[1] who showed that if the intensity of the outer components of the triplet is only one-half that of the inner (which is to be expected if no orientation has taken place), then the absorption coefficient for the vertical vibrations is only half as great as for the horizontal. If the original intensity of the central component is I_1 and the intensities of the outer components are I_2 and I_3, we have

$$I_2+I_3=I_1.$$

The outer absorbing mantle of the flame is in the magnetic field also, and as the absorbing power is proportional to the emissive power, it will absorb the central component (horizontal vibrations) more powerfully than the outer components (vertical vibrations.) We have, therefore, after absorption

$$I_2+I_3>I_1,$$

or, since I_2 and I_3 consist of vertical vibrations, we are able to detect traces of polarization.

Egoroff's experiment therefore cannot be regarded as evidence of an orientation of the orbits. If polarization could be found in the case of some gas which does not exhibit the phenomenon of reversal, for example helium or hydrogen, it would be pretty certain evidence that the orbits of the electrons were brought to a greater or less extent into a plane perpendicular to the lines of force.

Abnormal Zeeman Effect in Band Spectra. – Repeated investigations failed to show the existence of the Zeeman effect in the case of the individual lines of which band spectra are formed. Quite recently, however, Dufour has found that the edges and in some cases the individual lines in the band spectra of the fluorides and chlorides of calcium, barium, and strontium, obtained with the electric arc, show the effect. He made his observations in a direction parallel to the field, converting the circular vibrations into plane ones with a quarter-wave plate and observing them with a Nicol prism.

[1] *Rapp. prés au Congres Intern. de Phys.* (Paris, 1900), vol. iii., p. 29.

This enabled him to extinguish either component of the doublet at will, and he found the head of the band displaced to the right or left of its normal position, depending upon orientation of the quarter-wave plate. Fig. 327 is a drawing made from Dufour's photograph of the spectrum of calcium fluoride in the orange region; the upper and lower pictures show the bands under normal conditions, the two middle ones when the source of light is in a magnetic field, and the quarter-wave plate first in one position and then in the other. The shift in the position of the edge of the band is seen to be differ-

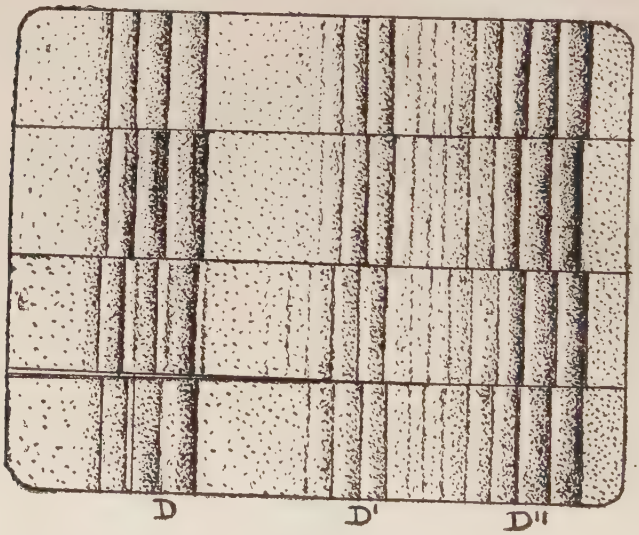


Fig. 327.

ent in the two cases and to be different as well in the case of the different bands. In the case of the band D'', the shift of the two middle spectra, with reference to the upper and lower, is precisely opposite to that which obtains in the case of the other two bands, D and D'. This method of observation will be recognized as the one originally employed by Zeeman in proving the existence of the two circularly polarized components of the doublet before he was able to effect their complete separation. Dufour's experiment showed that the direction of rotation of the two circular components of the doublet seen along the lines of force in the case of the band D'' was the same as in the normal type, and that the band was therefore due to the vibration of negative electrons. The opposite states of polarization found in the other two bands would on the elementary theory indicate that they had their origin in the vibrations of positive electrons. The magnitude of the displacement was about 0.3 of an Ångström unit, with a field strength of 10,000. In addition to the anomalously displaced component in the bands D and D', there were fainter components displaced in the normal direction, as indicated in the figure, which could be explained by assuming a

slight ellipticity in the vibration. In a direction perpendicular to the field, Dufour observed a quartet in the case of each band. In the case of the band D'' the separation of the component in which the vibrations were parallel to the field was greater than that of the perpendicular components. In the case of the other two bands the separation was the same in each case, that is, the middle line of the triplet was divided into two, which fell in coincidence with the two outer lines.

Still more recently Dufour has found (*Compt. Rend.* 1908) that in the band spectrum of hydrogen (sometimes called the second spectrum) we have lines which show the normal Zeeman effect, others which show no effect at all, and still others which exhibit the abnormal effect, *i.e.* reversed circular polarization in the doublet.

These observations, in connection with the work of Becquerel on the abnormal effect in certain absorbing crystals and the author's discovery of positive and negative rotation by magnetized sodium vapor, suggest — but by no means prove — the existence of positive electrons. Voigt, who has probably given more attention to the theoretical side of magneto-optics than any other worker, is strongly opposed to the conception of positive electrons, and it is probable that, in time, all of these anomalies will be satisfactorily explained, either by Voigt's assumed change of field inside the molecule, or by the action of negative electrons moving under constraint or under the influence of their neighbors.

Theory of the Zeeman Effect. — The very elementary and non-mathematical treatment of the Zeeman effect which has been given shows us why the doublets and triplets appear, but tells us nothing regarding the magnitude of the effect. The treatment given by H. A. Lorentz in his *Theory of Electrons* enables us to determine the nature of the charge on the electrons, and the ratio of the charge to the mass, from the magnitude of the separation. To the components of the elastic force which causes the vibration of the electron are added the components of the force due to the magnetic field, which is proportional to the velocity of the electron. The equations of motion, for a magnetic field parallel to the Z axis, are thus of the form

$$m\frac{d^2\xi}{dt^2} = -f\xi + \frac{eH_z}{c}\frac{d\eta}{dt};$$

$$m\frac{d^2\eta}{dt^2} = -f\eta - cH_z\frac{d\xi}{dt};$$

$$m\frac{d^2\zeta}{dt^2} = -f\zeta;$$

in which m = mass, e = charge, H = field strength and f = the elastic force. The last equation shows that the vibration parallel to Z is not affected.

The first two equations admit of two particular solutions:

$$\xi = a_1 \cos(n_1 t + p_1),\ \eta = -a_1 \sin(n_1 t + p_1);$$

$$\xi = a_2 \cos(n_2 t + p_2),\ \eta = a_2 \sin(n_2 t + p_2);$$

in which the frequencies n_1 and n_2 are determined by

$$n_1^2 - \frac{eH_z}{mc} n_1 = n_0^2$$

$$n_2^2 + \frac{eH_z}{mc} n_2 = n_0^2$$

in which n_0 is the original frequency.

The two solutions given above represent circular vibrations of opposite direction and perpendicular to the field. The frequency n_1 is higher than n_0 if eH_z is positive.

Experiment now shows us that the change in frequency is very small in comparison to the frequency, consequently we can write,

$$n_1 = n_0 + \frac{eH_z}{2mc} \text{ and } n_2 = n_0 - \frac{eH_z}{2mc}.$$

Zeeman found that, in the case of the circularly polarized doublets the light emitted in a direction coinciding with that of the magnetic force showed right-handed polarization for the low frequency line of the doublet, which proves that, for a positive value of H_z, the first of the two frequencies given above is the smaller, and that the charge e of the electron must be negative, which is in accord with other experiments. If we measure the distance between the lines, we can determine $n_1 - n_0$, and, knowing the strength of field, calculate the value of the ratio $\frac{e}{m}$. Zeeman calculated this ratio for the D lines of sodium and obtained a number of the same order of magnitude as that obtained for the electrons constituting the cathode rays of vacuum tubes and the β rays of radium.

Theory of Magnetic Rotation. — The theory of magnetic-rotatory dispersion has not been as completely worked out as the theory of ordinary dispersion, and very few experimental verifications of the formulae have been made.

Drude in his *Lehrbuch der Optik* has deduced two different formulae expressing the relation between wave-length and magnetic rotation in terms of the wave-length corresponding to the free period of the electron and certain constants. Two different hypotheses are made to account for the rotation. The first assumes the existence of molecular currents, as conceived by Ampère and Weber, to explain magnetism and diamagnetism. In paramagnetic substances these currents are already in existence, the action of a magnetic field merely orienting them so that their lines of magnetic force are superposed on the exciting field. In diamagnetic substances the currents are induced within the molecule (Weber's theory) as soon as the substance is brought into a magnetic field. These currents will persist as long as the body remains in the field, for the molecular circuits are assumed to be devoid of resistance, and they will be in such a direction as to be repelled by the pole from which spring the lines of force which have brought them into existence.

We can best think of these currents as revolving electrons, which

have, in the case of diamagnetic substances, been set in motion by the inductive action of the field, and which will continue to rotate until the field is destroyed, when the opposite inductive action brings them to rest. When now these revolving electrons are acted upon by the periodic electric forces of the light-waves, the points around which they rotate will suffer periodic displacements, and the lines of magnetic force resulting from the whirling electrons will be moved back and forth with the molecular currents.

The result of this is that the moving electron not only contributes to the electric current density j_x in the fundamental equation (as is the case in ordinary dispersion), but also to the density of the magnetic current s_x. The difference is seen to lie in the fact that ordinarily the electron is not accompanied by a magnetic field, whereas in the present case it is.

The assumption of these molecular currents is a natural hypothesis to make as a basis for a theory of magnetic rotation, since it has been found useful in explaining the phenomena of magnetism and diamagnetism of substances, and the magneto-optical properties of matter are without doubt closely related to their magnetic properties. The hypothesis, however, leads us to equations which, while they account for the rotation of the plane of polarization, call for rotations of opposite sign on opposite sides of an absorption band, that is, the magnetic dispersion curve expressed by the final formula has the same general form as the ordinary dispersion curve, with oppositely directed branches at the edge of the absorption band. As a matter of fact, we have no experimental evidence of magnetic rotatory dispersion of this type. The results, which have appeared to indicate it, have been shown by Becquerel, as we shall see, to depend upon other complications. As we shall see presently, the vapor of sodium in a magnetic field rotates the plane of polarization in the same direction on opposite sides of the absorption bands (D lines); and the same has been shown to be true of the numerous absorption bands exhibited by crystals containing erbium, praseodymium, and neodymium. We must therefore make some hypothesis which leads us to equations calling for rotations of the same sign on opposite sides of the absorption band.

The second hypothesis is that of the " Hall effect."

An electric current or a moving electron is subjected in a magnetic field to a deflecting force which is at right angles to the direction of the current and the lines of force. In a magnetic field, then, an electron which is thrown into vibration by light-waves will experience a force which will be proportional to the velocity with which the electron is moving, and this force must be added to the forces which we have already considered in forming the differential equation which expresses its motion. As we shall see, this method of attacking the problem leads us to a rotatory-dispersion formula which agrees with the results found in the case of sodium vapor and other absorbing media.

Hypothesis of the Hall Effect. — If an electron is set in motion by light-waves in a magnetic field it will experience a force which acts at right angles to its direction of motion and the direction of the

magnetic lines of force. This force will be proportional to the velocity with which the electron is moving, and to the strength of the field. It will be zero when the electron reaches its turning point, and will attain its maximum value at the moment when the position of equilibrium is passed. If the charge of the electron is e, and if it moves a distance $d\eta$ in time dt along the y axis, the force acting on it in the direction of the x axis will be represented by

$$K_x = \frac{e}{c}\frac{\partial \eta}{\partial t} h_z,$$

in which h_z is the strength of the magnetic field, which we will assume parallel to the z axis. If the magnetic field is parallel to the y axis, and the electron moves along the z axis, the force will be

$$K_z = -\frac{e}{c}\frac{\partial \zeta}{\partial t} h_y.$$

These forces are to be added to the right-hand member of the equation of motion of the electron,

$$(42) \qquad m\frac{\partial^2 \xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta}\xi - re^2\frac{\partial \xi}{\partial t} + \frac{e}{c}\left(\frac{\partial \eta}{\partial t} h_z - \frac{\partial \zeta}{\partial t} h_y\right).$$

The equations for the electric and magnetic current-densities are as before (Dispersion Theory):

$$4\pi j_x = \frac{\partial}{\partial t}(X + 4\pi \Sigma e N \xi), \text{ etc.},$$

$$4\pi s_x = \frac{\partial \alpha}{\partial t}, \text{ etc.}$$

For periodic disturbances we write as before

$$e\xi\left(1 + i\frac{a}{\tau} - \frac{b}{\tau^2}\right) - \frac{i\theta}{4\pi c\tau}(\eta h_z - \zeta h_y) = \frac{\theta}{4\pi} X.$$

If we take the z axis parallel to the lines of force, $h_z = h_y = 0$, and write

$$1 + i\frac{a}{\tau} - \frac{b}{\tau^2} = \Theta, \quad \frac{\theta}{4\pi c\tau e} h = \Phi,$$

we have

$$e\xi\Theta - ie\eta\Phi = \frac{\theta}{4\pi} X,$$

$$e\eta\Theta + ie\xi\Phi = \frac{\theta}{4\pi} Y,$$

$$e\zeta\Theta = \frac{\theta}{4\pi} Z.$$

Multiplying the first two equations first by Θ and then by Φ, and then multiplying the second pair of equations thus obtained by i, and solving for ξ, η, and ζ, gives us

$$4\pi e\xi(\Theta^2-\Phi^2)=\theta(\Theta X+i\Phi Y),$$
$$4\pi e\eta(\Theta^2-\Phi^2)=\theta(\Theta Y-i\Phi X),$$
$$4\pi e\zeta\Theta=\theta Z,$$

which by differentiation and substitution give

$$4\pi j_x=\frac{\partial X}{\partial t}\left(1+\sum\frac{\theta N\Theta}{\Theta^2-\Phi^2}\right)+i\frac{\partial Y}{\partial t}\sum\frac{\theta N\Phi}{\Theta^2-\Phi^2},$$
$$4\pi j_y=\frac{\partial Y}{\partial t}\left(1+\sum\frac{\theta N\Theta}{\Theta^2-\Phi^2}\right)-i\frac{\partial X}{\partial t}\sum\frac{\theta N\Phi}{\Theta^2-\Phi^2},$$
$$4\pi j_z=\frac{\partial Z}{\partial t}\left(1+\sum\frac{\theta N}{\Theta}\right),$$

which can be abbreviated by writing ϵ'' for the term in the parenthesis and ν for the second summation:

$$4\pi j_x=\epsilon''\frac{\partial X}{\partial t}+i\nu\frac{\partial Y}{\partial t},$$
$$4\pi j_y=\epsilon''\frac{\partial Y}{\partial t}-i\nu\frac{\partial X}{\partial t},$$
$$4\pi j_z=\epsilon'\frac{\partial Z}{\partial t}.$$

Light-Rays Parallel to the Magnetic Field. — In this case, α, β, X, and Y depend only on z and t, and substitution of the above values in the fundamental equations gives us

$$\frac{1}{c}\left(\epsilon''\frac{\partial X}{\partial t}+i\nu\frac{\partial Y}{\partial t}\right)=-\frac{\partial\beta}{\partial z},\quad \frac{1}{c}\left(\epsilon''\frac{\partial Y}{\partial t}-i\nu\frac{\partial X}{\partial t}\right)=\frac{\partial\alpha}{\partial z},$$
$$\frac{1}{c}\frac{\partial\alpha}{\partial t}=\frac{\partial Y}{\partial z},\quad \frac{1}{c}\frac{\partial\beta}{\partial t}=-\frac{\partial X}{\partial z},\quad Y=Z=0.$$

Eliminating α and β by differentiating the first pair of equations with respect to t and the second pair with respect to z,

$$\frac{\epsilon''}{c}\frac{\partial^2X}{\partial t^2}=\frac{\partial^2X}{\partial z^2}-\frac{i\nu}{c^2}\frac{\partial^2Y}{\partial t^2},$$
$$\frac{\epsilon''}{c}\frac{\partial^2Y}{\partial t^2}=\frac{\partial^2Y}{\partial z^2}+\frac{i\nu}{c^2}\frac{\partial^2X}{\partial t^2}.$$

Integrating as before by writing

$$X=Me^{\frac{i}{\tau}(t-pz)},\quad Y=Ne^{\frac{i}{\tau}(t-pz)}$$

gives us $\epsilon''M=p^2c^2M-i\nu N,\ \epsilon''N=p^2c^2N+i\nu M.$

Multiplying the first equation by N and the second by M gives us $M = \pm iN$, and by substitution of these values in the equations gives

$$p^2c^2 = \epsilon'' + \nu \text{ and } p^2c^2 = \epsilon'' - \nu.$$

We have seen in the Chapter on Optics of Metals that when p is complex, we can write $p = \frac{1 - i\kappa}{V^2}$, in which V is the velocity in the medium. In the present case we can therefore write

$$p^2c^2 = \frac{(1-i\kappa)^2c^2}{V^2} = n'^2(1-i\kappa')^2 = \epsilon'' + \nu,$$

$$p^2c^2 = n''^2(1-i\kappa'')^2 = \epsilon'' - \nu,$$

in which n' and κ' represent the refractive index and extinction coefficient for left-handed circular vibrations, n'' and κ'' for right-handed vibrations.

Substituting for ϵ'' and ν their equivalents

$$\sum\left(1 + \frac{\theta N \Theta}{\Theta^2 - \Phi^2}\right) \text{ and } \sum \frac{\theta N \Phi}{\Theta^2 - \Phi^2},$$

$$n'^2(1-i\kappa')^2 = 1 + \sum \frac{\theta N}{\Theta - \Phi}, \quad n''^2(1-i\kappa'')^2 = 1 + \sum \frac{\theta N}{\Theta + \Phi}.$$

If we limit ourselves to a region of the spectrum outside of the absorption band, we can neglect $i\frac{a}{\tau}$ and write $\kappa' = \kappa'' = 0$, and since Φ is small in comparison to Θ,

$$n'^2 = 1 + \sum \frac{\theta N}{\Theta}\left(1 + \frac{\Phi}{\Theta}\right), \quad n''^2 = 1 + \sum \frac{\theta N}{\Theta}\left(1 - \frac{\Phi}{\Theta}\right).$$

In the Chapter on Natural Rotation we have seen that the rotation of the plane of polarization in terms of the refractive indices for the two circular vibrations is given by

$$\delta = z\frac{\pi}{\lambda}(n'' - n') = z\frac{\pi}{\lambda}\frac{n''^2 - n'^2}{n'' + n'},$$

in which we can write $2n$ for $n'' + n,'$ n being the mean refractive index for the circular vibrations.

Substituting in $\delta = z\frac{\pi}{\lambda}\frac{n''^2 - n'^2}{2n}$ the values for n''^2 and n'^2

gives us $\delta = -\frac{\pi}{n}\frac{z}{\lambda}\sum \frac{\theta N \Phi}{\Theta^2}$ for the rotation

and $n^2 = 1 + \sum \frac{\theta N}{\Theta}$ for the refractive index.

Magnetic Rotatory Dispersion.—Substitution of the values of Θ and Φ in the above equation gives us

$$\delta = -\frac{\pi}{2}\frac{z}{n\lambda^2}h\sum\frac{\theta N}{\left(1-\frac{b}{\tau^2}\right)^2}\cdot\frac{\theta}{e}.$$

Let $A = -\frac{\pi z}{2}h$, then $\delta = \frac{A}{n\lambda^2}\sum\frac{\theta^2N^2}{\left(1-\frac{b}{\tau^2}\right)^2}\cdot\frac{1}{eN}$,

and remembering that $b = \frac{m\theta}{4\pi\epsilon^2} = \tau_h^2$ (τ_h being the free period of the electron), and that $\theta N = \theta_h'$,

$$\delta = \frac{A}{n\lambda^2}\sum\left[\frac{\theta_h'}{1-\left(\frac{\tau_h}{\tau}\right)^2}\right]^2\cdot\frac{1}{eN}.$$

Consider two types of electrons to be present, then

$$\delta = \frac{A}{n\lambda^2}\left[\frac{\theta_h^1}{1-\left(\frac{\tau_h}{\tau}\right)^2}\right]^2\cdot\frac{1}{e_1N_1} + \left[\frac{\theta_h''}{1-\left(\frac{\tau_h'}{\tau}\right)^2}\right]^2\cdot\frac{1}{e^2N^2},$$

and since $\frac{1}{\epsilon_1 N_1} = -\frac{1}{\epsilon^2 N_2}$, there being no free charge, we can combine $\frac{1}{\epsilon N}$ with our constant, if we change the sign of the second parenthesis,

$$\delta = \frac{A_1}{n\lambda^2}\left\{\left[\frac{\theta_h'}{1-\left(\frac{\tau_h}{\tau}\right)^2}\right]^2 - \left[\frac{\theta_h''}{1-\left(\frac{\tau_h'}{\tau}\right)^2}\right]^2\right\}.$$

If we now consider the second parenthesis as representing the effect of remote ultra-violet electrons, for which τ_h' is very small in comparison to τ, the term reduces to $(\theta_h'')^2$, as we showed in the Chapter on Dispersion;

$$\therefore \delta = \frac{A_1}{n\lambda^2}\left\{\left[\frac{\theta_h'}{1-\left(\frac{\tau_h}{\tau}\right)^2}\right]^2(-\theta_h'')^2\right\},$$

and substituting λ for τ,

$$\delta = \frac{A_1}{n\lambda^2}\left\{\left(\frac{\theta_h'-\lambda^2}{\lambda^2-\lambda_1^2}\right)^2 - (\theta_h'')^2\right\}.$$

Squaring the parentheses, dividing λ^2, and writing $B = A_1(\theta_h')^2$ and $C = A_1(\theta_h'')^2$, we get

$$\delta = \frac{1}{n}\left(\frac{-C}{\lambda^2} + \frac{B\lambda^2}{(\lambda^2-\lambda_1^2)^2}\right).$$

In this formula the first term in the parenthesis represents the effect of an absorption band so far down in the ultra-violet that it can be regarded as contributing a certain rotatory power which varies inversely with the square of the wave-length. In the dispersion formula the corresponding term contributes a certain fixed amount to the refractive index, independent of wave-length, while in the present case we have λ^2 occurring in the denominator. The second term is the more interesting, for we have $(\lambda^2-\lambda_1^2)^2$ in the denominator. The term will have large values for values of λ very near λ_1, *but* the sign will not change when we cross the absorption band, since the square of the minus quantity, which we have when $\lambda_1>\lambda$, is a positive quantity. This shows us that the sign of the rotation is the same on opposite sides of the absorption band, the rotation decreasing, however, as we recede from the band in either direction.

Proof of the Rotatory Dispersion Formula.—The experiments of Macaluso and Corbino [1] have shown that the rotation is in the same direction on opposite sides of the absorption band in the case of sodium vapor. A small sodium flame, placed between the poles of an electro-magnet, was traversed by a beam of polarized white light, in the direction of the lines of magnetic force. A Nicol prism was oriented so as to completely extinguish the light when the current was not traversing the coils. On forming the magnetic field a brilliant yellow light was found to be transmitted by the Nicol, which the spectroscope showed to consist of narrow bands symmetrically placed on each side of the D lines. By turning the Nicol first in one direction and then in the other it was easy to see that the rotation was of the same sign on opposite sides of the band. The formula thus applies qualitatively to the magnetic rotation exhibited by a sodium flame. To test it quantitatively the rotation must be observed over a wide range of wave-lengths on opposite sides of the absorption band. In the case of sodium vapor n varies but little from unity, except within a small fraction of an Ångström unit of the D lines. Moreover, we are concerned only with the effect of the electrons which cause the D lines, for since those of shorter period exert no appreciable influence on the refraction, as we have seen, it is justifiable to assume their effect on the rotatory power as negligible. We can consequently neglect the term $-\frac{C}{\lambda^2}$, and write the formula

$$\delta=\frac{B\lambda^2}{(\lambda^2-\lambda_m^2)^2}.$$

If the formula is to be used in the immediate vicinity of, or between, the D lines, we must make use of two terms and write,

$$\delta=\frac{A\lambda^2}{(\lambda^2-\lambda_{D_1}^2)^2}+\frac{B\lambda^2}{(\lambda^2-\lambda_{D_2}^2)^2}.$$

[1] *Compt. Rend.*, cxxvii., p. 548.

This formula has been verified by the experiments of the author,[1] at least for the region of the spectrum not comprised between the *D* lines. Preliminary experiments in collaboration with H. W. Springsteen [2] showed the feasibility of testing the formula by measuring the rotation of the vapor of metallic sodium, formed in exhausted tubes in a powerful magnetic field, and established the fact that the numerous absorption lines of the vapor in the red and green-blue region exercised powerful rotatory effects.

In this preliminary work glass tubes were used, which were exhausted and sealed off from the pump. It was subsequently found that the hydrogen liberated from the sodium interfered greatly with the rotatory effects, and in the subsequent work the tubes were kept in connection with the pump. As the phenomena exhibited by the vapor are extremely beautiful, and very easily shown, the apparatus in its final form will be described in detail.

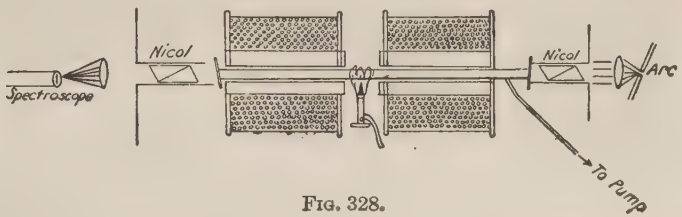


FIG. 328.

A piece of thin, seamless steel tubing (bicycle tubing) of such diameter as to permit of its being slipped easily through the hollow cores of the electro-magnet is procured. A short piece of small brass tubing is brazed into one end, through which the tube is exhausted. The ends are closed with small pieces of plate glass cemented with sealing wax. The arrangement of the apparatus is shown in Fig. 328. A piece of sodium the size of a walnut is rolled out into a cylinder and inserted into the tube just before the second end-plate is cemented on. The tube is at once placed in position in the magnet and exhausted. If a piston pump is used for the exhaustion, a glass stop-cock should be put between the pump and the tube to prevent back leakage of air. Care must be taken to have the lump of sodium midway between the poles of the magnet. The steel tube is now heated by means of a Bunsen burner, and the pump worked to remove the hydrogen liberated from the sodium, after which the burner is removed and the tube allowed to cool.

Light from a heliostat, or an arc lamp, is now passed in succession through a Nicol prism, the steel tube, a second Nicol, and then concentrated on the slit of a spectroscope. If the instrument has a large dispersion (a 14-foot concave grating was used in the present case) all of the phenomena now to be described can be seen.

[1] "Magneto Optics of Sodium Vapor and the Rotatory Dispersion Formula," *Phil. Mag.*, Oct. 1895.
[2] *Phys. Rev.*, June 1905.

The Nicols are set for complete extinction and a small flame placed beneath the tube. As soon as the vapor begins to form, two very bright lines will appear in the position of the D lines the moment the magnet is excited. These lines represent the constituents of the white light, which are rotated through 90° by vapor and

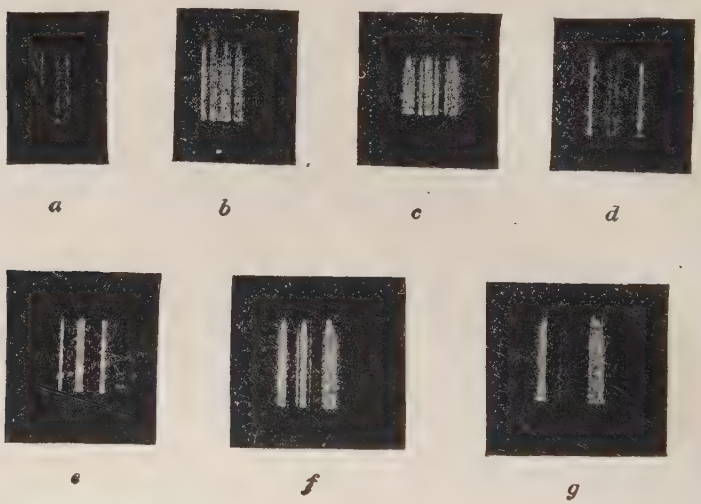


Fig. 329.

passed by the analyzing Nicol. The lines are in reality double, though their duplicity cannot be made out when they first appear. As the density of the vapor increases the components separate, four lines being distinctly visible. This condition is shown in Fig. 329 *a*, which is from a photograph. The lines continue to separate, and presently a second pair appears between them for which the rotation is 270°, the dark regions between representing rotations of 180°. This stage is shown in Fig. 329 *b* and *c*.

In the former the two inner 90° lines are beginning to fuse together, the centre being partially dark, however; in the latter the fusion is complete and the centre of the system of lines is bright. With a further increase in the vapor density the outer lines (90°) separate still further, and widen out into broad flares of light, other lines appearing between them corresponding to larger rotations, the system resembling a set of diffraction fringes, as shown in Fig. 330, which is from a photograph made with a large plane grating and a lens of 3 metres focal long. These bright lines represent rotations of 270°, 450°, 630°, etc., and by measuring their positions with an eye-piece micrometer, the wave-lengths corresponding to these rotations were determined. The centre of the system, as we may designate a point midway between D_1 and D_2, becomes bright and dark in succession, as many as eight complete alternations having been observed in some instances. This corresponds to a rotation of 1440°. If the burner is removed the changes take place very

rapidly, the centre " winking " bright and dark almost as rapidly as one can count.

The results obtained from measurements made with the micrometer are shown in the form of a curve, Fig. 331, observed values being represented by circles. Values of the constants A and B in the formula were calculated from two observed values of δ, and the values of δ for various wave-lengths calculated. These calculated values are represented by crosses on the plate, and will be found to fall almost exactly upon the experimental curve. The value of the constant B, which is associated with the absorption line D_2, was about double that of A, which belongs to D_1. Tables of rotations for

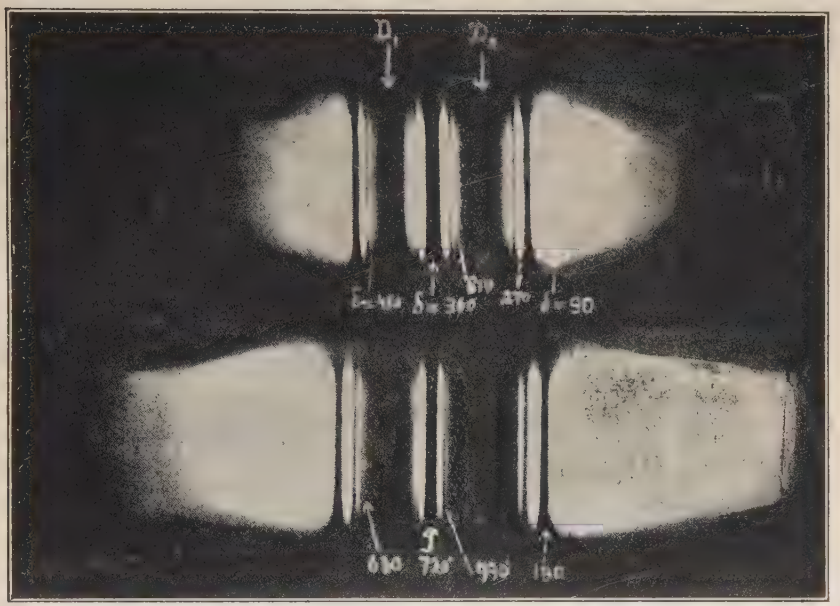
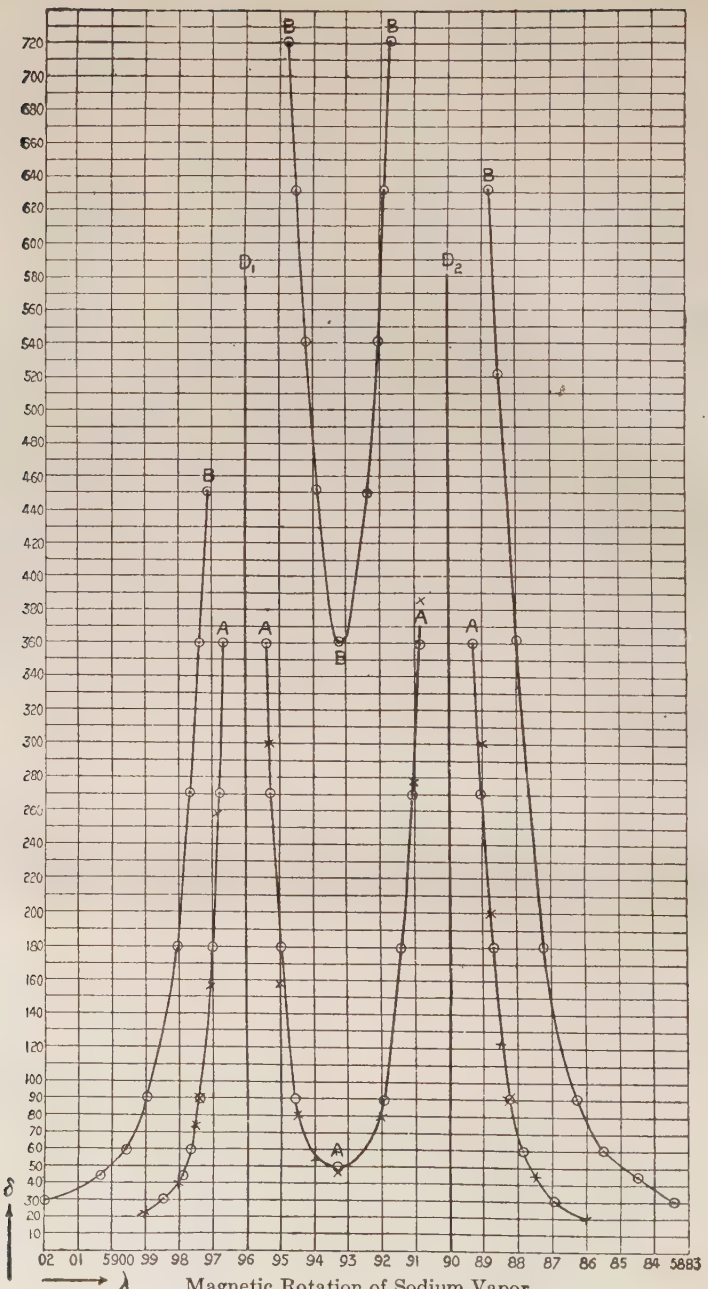


FIG. 330.

various vapor densities were made, and the formula tested under various conditions. With fairly dense vapors the observed value of δ midway between D_1 and D_2 was usually larger than the calculated. A fuller discussion of the results will be found in the original paper.

With vapor of considerable density the rotation is measured in a different manner. The Nicols are set in a parallel position, and the entire spectrum appears with the exception of the broad absorption band at the D lines. On each side of this absorption band a dark 90° rotation band appears. As we turn the Nicol these dark bands move, the one up, the other down the spectrum. By noting their positions we determine the values of λ corresponding to the rotation of the Nicol in degrees measured from the position of extinction.

The spectrum of the light transmitted, with the Nicols in various positions, was photographed, and the positions of the dark bands



Magnetic Rotation of Sodium Vapor.
Curve A, rare vapor. Curve B, dense vapor.

FIG. 331.

measured subsequently. Owing to the great density of the vapor it was found that D_1 and D_2 could be considered as forming a single absorption band, and λ_m was given an intermediate value 5893. For a particular density and length of vapor column, the constant B was found from a single observation of δ. The observed and calculated values are given in the following table:

VALUE OF CONSTANT $B = 135600$

λ	δ (obs.)	δ (cal.)
5980	5°	4°.47
5950	10°	10°.4
5933	20°	23°.2
5923	40°	38°
5917	66°	58°.9
5912.5	90°	89°.2
5874	90°	93°.1
5869	43°	43°
5864	40°	40°.6
5852	20°	20°.2
5833	10°	9°.2
5814	5°	5°.2

This table shows that with very dense vapor the rotatory dispersion is well represented by a single term formula, the observations being limited to a region not very near D_1 or D_2.

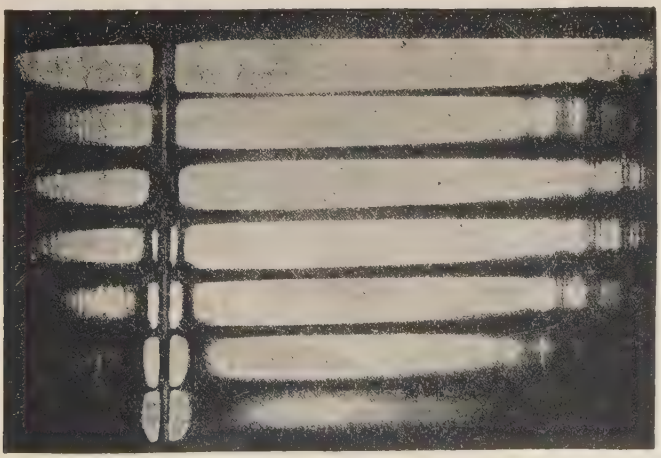


FIG. 332.

A series of photographic records obtained in this manner is shown in Fig. 332, which, taken collectively, exhibit the general form of the rotatory dispersion curve.

In a subsequent paper [1] on the magnetic rotation of sodium

[1] Wood, *Phil. Mag.*, (6) 14, p. 145, 1907.

vapor in the vicinity of the D lines the complete curve is discussed. Photographs were made on a much larger scale which permitted of more accurate measurement. A porcelain tube was used in this work, heated by a small electric oven; the current in the magnet was kept constant by means of a rheostat and ammeter, and other precautions taken to exclude sources of error. In some of the photographs a pair of faint bands of considerable width was observed within the region of absorption, *i.e.* nearer the centre of the absorption line than the narrow bands due to large rotations. The cause of these has not been ascertained and should be investigated. Both absorption lines were taken into account, and it was found that the rotation of wave-lengths between the D lines, which obviously results from the combined effects of the two lines, is greater than the sum of the separate rotations due to each line as calculated from the rotation of wave-lengths below D_2 and above D_1. Thus the formula does not hold if we try to apply it to the region between the lines. There is room for further investigations on this point.

The Bright-Line Rotation Spectrum of Absorbing Vapors. — With the apparatus described in the previous section, a very remarkable phenomenon appears when the vapor has considerable density. With the Nicols crossed and the magnet excited, the transmitted light, when examined with a prism spectroscope, will be found to form a most beautiful bright line spectrum, the general appearance of which is shown in Fig. 3 of the colored frontispiece. Over two hundred lines can be counted in the red, and about one hundred and thirty in the blue-green, region. A photograph of a portion of

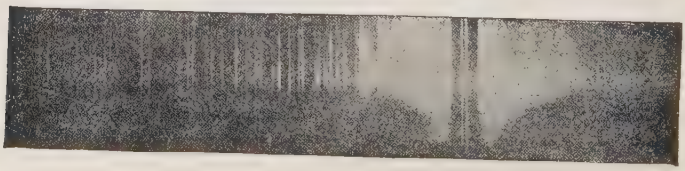


Fig. 333.

the spectrum is shown in Fig. 333, while Fig. 3 of the colored frontispiece gives a good idea of the general appearance of the entire spectrum. These lines have been photographed with a 14-foot concave grating, and found to coincide with absorption lines, the significant fact being, however, that comparatively few of the absorption lines are represented in the rotation spectrum. Just why this is so is not apparent. The rotatory power of an electron is probably inversely proportional to its mass. The lighter the electron in proportion to its change, the greater will be the perturbations in its orbit produced by the magnetic field. It is possible that the absorption lines which exercise rotatory power result from the negative electrons of small mass, while the other absorption lines are due

to heavier corpuscles, perhaps carrying positive charges. The fact that the bright lines of the fluorescent spectrum appear to coincide with those of the magnetic rotation spectrum, favors this hypothesis, for we should expect the lighter electrons to be set in more violent vibration by the light-waves than the heavier ones. The lines were found to form groups, which could be identified by comparison with the fluorescent spectrum which we shall take up later. An extended study of the magnetic rotation spectrum has been made by F. E. Hackett in collaboration with the author (*Astrophysical Journal*, XXX, No. 5, December 1909). A photograph of the spectrum in the blue-green region, in coincidence with the iron comparison spectrum, is shown on Plate 7, opposite page 577, Fig. 2. Just above (Fig. 1) is a photograph of the fluorescent spectrum which will be discussed in a subsequent chapter. The similarity between the two is obvious. Figs. 3 and 4 show small portions of the magnetic rotation and fluorescent spectra in the green region on a larger scale. Fig. 15 shows a portion (about one-half) of the rotation spectrum in the red. These photographs were made with a Rowland concave grating. All trace of the rotation disappears if hydrogen or nitrogen at a pressure of more than a few centimeters is present in the tube; this is also true for the fluorescence. Iodine vapor also gives a very beautiful bright-line spectrum. A few crystals are introduced into a small glass bulb which is highly exhausted and sealed off from the pump. This bulb, when placed between the perforated conical pole pieces of a Ruhmkorff magnet (Nicols crossed) and gently warmed, restores light of a most beautiful emerald green color, which the spectroscope shows to be discontinuous. It is instructive to prepare two bulbs, one exhausted, the other at atmospheric pressure. The latter shows no effect whatever. A photograph is reproduced on page 581 in the Chapter on Fluorescence.

Positive and Negative Rotation by Absorption Lines. — The greater part of the evidence which we have obtained thus far regarding the structure of the atom indicates that the centres of vibration which emit the spectral lines are negatively charged corpuscles. The positive charges appear to be associated with the atom as a whole, and the assumption is often made that the positive electrification is of uniform distribution.

The Zeeman effect shows us that the *D* lines of sodium are due to vibrators carrying negative charges, a fact which is true of all other lines which show the effect. That a negative charge is associated with the centres of vibration which emit the *D* lines is also shown by the direction (positive) of the magnetic rotation of the plane of polarization, for waves of very nearly the same frequency as that of the *D* lines. As is well known, most band spectra do not show the Zeeman effect at all, consequently we are unable to apply this test to the investigation of the nature of the charge associated with the centres of emission of the lines of which the bands are made up. Quite recently Dufour has observed the effect in some band spectra, as we have seen.

Some of the lines which make up the complicated channelled

absorption spectrum of sodium vapor have been found by the author to have the power of rotating the plane of polarization when the light is passed through the magnetized vapor in the direction of the lines of force.

It is of the utmost importance to determine the nature of the rotation which gives rise to this multitude of bright lines, *i.e.* whether it is positive or negative.

In the case of the rotation for wave-lengths in the vicinity of the *D* lines, there is no difficulty in determining the direction, for the broad bands of rotated light which border the absorption lines can be moved from side to side by slight rotations of the analyzing Nicol; or we may employ the device so frequently used, the Fresnel double prism of right- and left-handed quartz, which tells us at a glance the direction of the rotation. In the case of the narrow lines of the channelled spectra, no information can be gathered as to whether the rotation is positive or negative by rotating the analyzing Nicol, for the smallest possible turn from the position of extinction causes the continuous spectrum to brighten up, obliterating the rotation lines. An attempt was first made to employ metallic arcs in place of the white-hot crater, as the source of the light, on the chance that some of the lines might be of the right wave-length to suffer rotation in the region of the channelled spectra. If any of the lines were found to be rotated by the vapor, the direction of the rotation could be easily determined by rotating the analyzing Nicol until they were extinguished. No lines were found, however, which had just the right wave-length. It seemed possible, however, that the selective rotatory power of the vapor could be utilized to furnish a source of light made up of just the right wave-lengths; in other words, magnetized sodium vapor between crossed Nicols could be used as a light filter. The light passed by the crossed Nicols when the magnetic field was excited was accordingly sent through another magnetized tube of vapor and examined with a third Nicol and spectroscope. It was hoped that by setting the third Nicol for extinction, and causing the bright-line spectrum to appear again by excitation of the second magnet, it would be possible to determine the direction of rotation of the lines by observing in which direction it was necessary to rotate the third Nicol in order to blot them out. The first magnet, with its sodium tube and polarizing prisms, delivers plane-polarized light of exactly the wave-lengths of the bright lines of the magnetic rotation spectrum. This light is then passed through a second magnetized tube of sodium vapor, a Nicol prism, and a spectroscope. The Nicol having been set for extinction, the bright-line spectrum disappeared, reappearing again as soon as the magnet was excited. It was found, however, that rotation of the third Nicol was wholly without effect on the appearance of the lines, notwithstanding the fact that the light was originally plane-polarized. *The magnetized sodium vapor appeared to have completely depolarized the light.* The cause of this phenomenon is not difficult to explain. The lines which make up the magnetic rotation spectrum, though they appeared as narrow as the iron arc-lines in a photograph made two years ago with a

concave grating of 12 feet radius, are not in reality monochromatic. The action of an absorption line is to rotate the plane of polarization of waves of nearly the same wave-length through various angles depending on their proximity to the absorption line. It is these waves which are transmitted by the Nicol. The line therefore has a finite, though narrow, width, and the second tube of magnetized vapor rotates the monochromatic constituents, of which the line is made up, through various angles. Some of the light in the line is therefore passed by the third Nicol in every position.

From their analogy to the bright rotated lines which border the *D* lines when examined under similar conditions, we should expect all of the lines of the magnetic rotation spectrum to be double, and a good deal of time was spent in attempts to show their duplicity, by means of an échelon grating. No very definite results were obtained, however, and more recent experiments have shown that the rotatory power of most of the absorption lines is confined to wave-lengths on one side of the line only. This same action is observed in an exaggerated degree by the ultra-violet absorption line of mercury ($\lambda = 2536$), which broadens very unsymmetrically. The form of the absorption curve and the magnetic rotation as shown with the Fresnel rotating quartz prisms are shown in Fig. 334, *a* and *b*. The spectrum obtained by passing white light through the vapor placed between crossed polarizing prisms is shown in Fig. 334, *c*, the fainter line being rotated 270°.

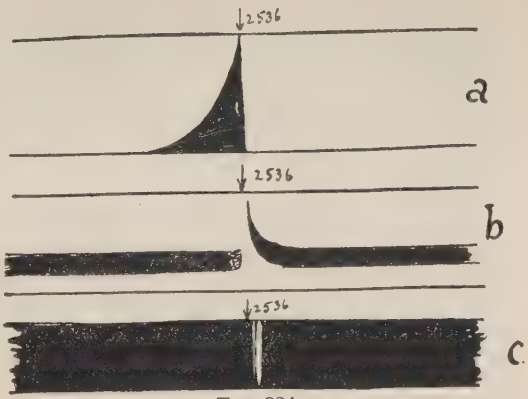


Fig. 334.

This shows us that the lines of the magnetic rotation spectrum would not necessarily appear double, even with the highest resolving powers (neglecting rotations larger than 90°). Though the lines appear as narrow as arc-lines even with a large grating, the magnetized sodium vapor and polarizing prism show us that in reality each line embraces a narrow range of the spectrum, the individual components of which are rotated through very different angles by the vapor.

The experiment which finally showed clearly the nature of the rotation was made with a pair of Fresnel quartz prisms. This combination consists of two wedges, one of dextro- the other of laevo-rotary quartz. Its action is described in the next section. They were much thinner than those usually employed, as it was felt that it would be better to work with a single broad band of extinction

than a large number of parallel bands. The magnetically rotated lines are faint in comparison with the continuous spectrum from which they are derived, and it is consequently important to have the background upon which they are to show up as dark as possible. With a thick Fresnel prism we have the continuous spectrum at its full intensity traversed by a number of parallel dark bands, which correspond to the points on the slit at which the plane of polarization is parallel to the plane of extinction (long diagonal) of the analyzing Nicol, which is placed immediately behind the slit. There is in consequence more or less diffused light from the grating, which renders the background (the dark bands), upon which the rotated lines are to appear, much too luminous. To get rid of this effect, the best method is to use a thin prism, and cover the slit except for a small portion immediately above and below the single dark band of extinction.

With this arrangement of the apparatus the magnetically rotated line should penetrate the dark band from above or below, according to whether the rotation is positive or negative. If we excite the magnet and gradually heat the sodium tube, we see sharp needles of light shoot down from the continuous spectrum into the dark region immediately to the right and left of the *D* lines, as has been described by Macaluso and Corbino, Zeeman, and others. If we reverse the magnetic field the needles of light shoot up from below. The direction in which the plane of polarization is rotated by the *D* lines indicates that they are caused by vibrations of negative electrons. The important question to be answered is whether the absorption lines of the band spectra rotate the plane of polarization in the same or in the opposite direction, and whether they all behave alike.

The magnetic rotation spectrum being much brighter in the red and orange than in the green and blue region, the first observations were made in this part of the spectrum. The spectroscope was a medium-sized instrument, consisting of a telescope and collimator of about 180 cms. focus, furnished with a plane grating.

The sodium tube was heated until the fine black absorption lines in the red appeared distinctly in the continuous spectrum above and below the horizontal dark band due to the Fresnel prism. The current was then thrown into the magnet, the self-induction of which is so great that the field does not rise to its full intensity for several seconds, so that there was plenty of time to see exactly what happened. As soon as the switch was closed numerous needles of light commenced to penetrate the dark region, *some of them shooting down from above, others shooting up from below,* as shown in Fig. 6 of the colored frontispiece. (Owing to a misunderstanding the engraver has represented the absorption lines at an angle: they are of course vertical.) Of these, some only extended halfway or less across the dark band, while others crossed it completely. On opening the switch the luminous needles slowly withdrew from the dark background into the bright region from which they came, reminding one of the tentacles of an alarmed hydroid. The phenomenon is most beautiful and shows us at once that some

of the absorption lines rotate the plane of polarization in the positive direction, while others rotate it negatively.

Photographs of the phenomenon are reproduced in Fig. 335 and also on Plate 7, Fig. 12. On the elementary theory we should infer from this experiment that positive electrons were present as well as negative. We are hardly justified in this, however, as it is possible to explain the two types of rotation in other ways, *e.g.* by a reversal of the magnetic field within the atom.

An attempt was made to study the direction of rotation by Hackett and Wood in the blue-green rotation spectrum, but there was little or no trace of the bright needles of light shooting up or

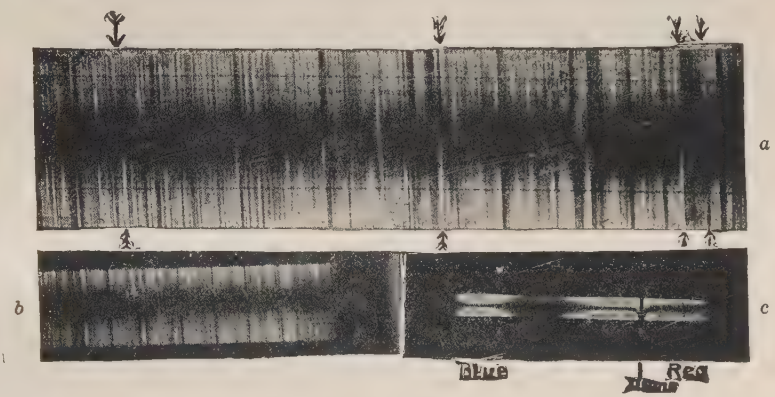


FIG. 335.

down into the dark interference band. Their failure to appear in the green is due to the smaller angular rotation of the plane of polarization. The problem was finally solved in a different manner for the green spectrum. These wave-lengths in the plane-polarized light passing through sodium vapor along the lines of force in a magnetic field are rotated through a few degrees so that they can pass through a Nicol set at extinction for this region of the spectrum. A small rotation of the Nicol in either direction will now brighten up the whole field. In one direction some of the bright lines will still be seen, though less distinctly, against the bright background of the rest of the spectrum. Careful observation, however, will show that if the Nicol is rotated about four degrees in the opposite direction, these lines disappear. These lines have therefore been rotated about four degrees in this direction by passing through the sodium vapor in the magnetic field. It was observed that the rotation in the brightest lines belonging to the magnetic series was in the same direction as that due to the *D* lines. The rotation of these lines is therefore presumably due to negative electrons.

To extend this process to the whole spectrum it was only necessary to take two photographs of the spectrum on the same plate, one in which the Nicol was rotated four degrees to the left, and the other in which it was rotated four degrees to the right. These two photo-

graphs were totally different, as the bright lines of one were absent in the other. By careful comparison of this plate with the magnetic rotation spectrum the direction of rotation of all the strong lines could be ascertained at leisure.

Magnetic Rotation within an Absorption Band: Experiments of Zeeman. — It was established theoretically by Voigt (*Wied. Ann.*, 67, 359, 1899) that, in the case of an absorption line separated into a magnetic doublet, the rotation of the plane of polarization was positive for all periods lying outside of the components of the doublet and negative for all periods between the components, the light traversing the medium in the direction of the lines of force.

This was verified by Zeeman (*Proc. Amsterdam Acad.*, June 1902), who made use of a method similar to the one which had been previously employed by Voigt in demonstrating magnetic double refraction.

The light of an arc lamp, after passage through a Nicol, was focussed upon the slit of a grating spectrometer, in front of the slit of which was placed a Fresnel bi-quartz prism, similar to the one employed in the experiment on the division of a plane-polarized ray into two circular components by rotatory media. Between the slit and the grating a second Nicol was mounted which cut off the vibrations which had been rotated into its plane of extinction by the quartz wedges. The amount of rotation at each point of the slit depended on the difference between the thicknesses of the right- and left-handed quartz elements at the point in question, and the spectrum was found to be traversed by a system of dark bands

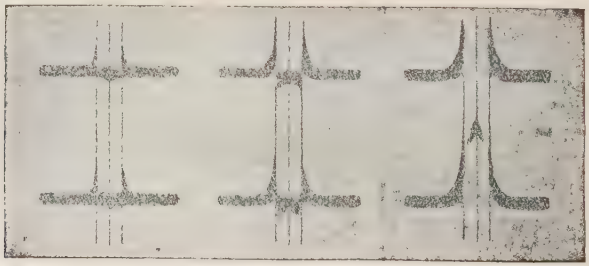


Fig. 336.

parallel to its length. Between the first Nicol and the spectrometer the absorbing flame of sodium was mounted in a magnetic field, any rotation produced by it adding itself to that produced by the quartz wedges. A rotation impressed upon any wave-length by the flame thus caused a vertical deviation of the dark band at the corresponding point of the spectrum, a shift equal to the width of a complete fringe corresponding to a rotation of 180°.

With a field of 15,000 c.g.s. units the dark absorption line was distinctly resolved into a doublet, and on increasing the amount of sodium in the flame the dark bands outside of the components

curved upwards, while the portion of the band between them slid down in the opposite direction, as shown in Fig. 336, in which the appearances of the bands for different densities of the absorbing flame are shown. Photographs of the phenomenon are reproduced in Fig. 337.

Increasing the strength of the field caused the portion of the band between the components to move back towards its original position, which was in agreement with Voigt's prediction that the negative rotation within the band decreased with increasing field strength. This is, of course, true only with fairly strong fields; in other

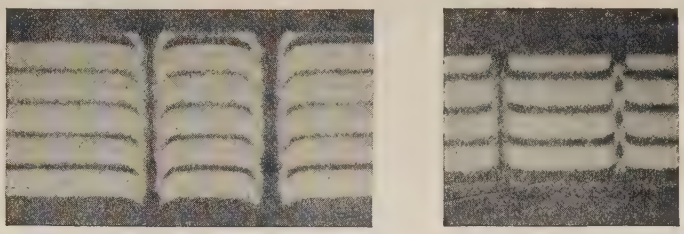


Fig. 337.

words, for a given density of vapor the negative rotation between the components reaches a maximum value for a certain strength of field. In the case of the positive rotation which occurs outside of the lines, no such turning-point is found.

Magnetic Rotation at Low Temperatures. — Some very remarkable experiments have been made by Jean Becquerel, and by H. du Bois and J. Elias (*Ann. der Physik*, Vol. 27, p. 233, 1908), with crystals at low temperatures, which throw a great deal of light upon the much disputed question of the nature of the magnetic rotatory dispersion in the vicinity of absorption bands.

The effects of low temperatures on the absorption bands of tysonite and xenotime have been mentioned in the Chapter on Absorption. Becquerel placed the crystals at the temperature of liquid air in a strong magnetic field and found that the position of the bands changed, the phenomenon being of the same nature as the Zeeman effect, only on a much greater scale, the separation of the components of some doublets amounting to a distance greater than that between the *D* lines. Still more remarkable was the discovery of the fact that, in the direction of the magnetic field, the bands corresponding to the absorption of the circular components of given sense are not all displaced in the same direction in the spectrum. On the elementary theory this would indicate that some of the bands are due to negative and others to positive electrons, and this is the interpretation which Becquerel gives. The effect is very clearly indicated in Fig. 338 for the two absorption bands 5221 and 5252 of xenotime. This photograph was made by Becquerel with a Rowland concave grating in the second order, and the scale is 1 mm. = 1.6 Ångström units.

The upper spectrum represents the absorption when we employ

circularly polarized light of a certain sense, passing it through a crystal plate 8 mms. thick, immersed in liquid air in a strong magnetic field. On reversing the direction of rotation of the circular light we obtain the lower spectrum. As will be seen, the 522 band moves to the *right*, the 525 one to the *left*. The latter shows dissymmetry clearly. This effect was found by Becquerel only at very low temperatures (−259°). The component which the magnetic field displaces towards the blue increases at the expense of the other component.

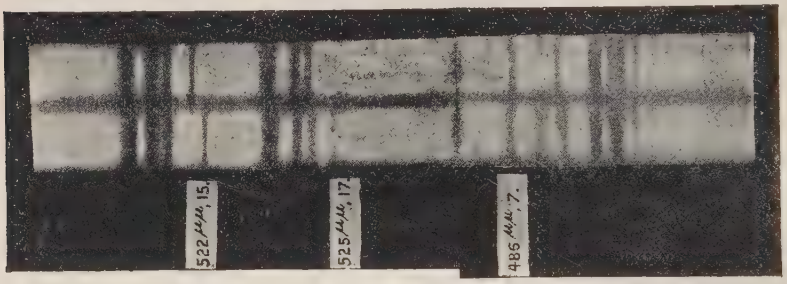


FIG. 338.

One of the most important points brought out in this very important piece of work was that, in every case, the magnetic rotation was in the same direction on opposite sides of the double band into which the magnetic field splits the original band, and in the opposite direction within the band, precisely the same as with absorbing vapors such as sodium.

Cases in which we appear to have opposite rotations, such as the praseodymium band studied by the author, are shown to result from superpositions of bands, which can be separated by lowering the temperature. Moreover, if we have dissymmetry in the intensity of the circularly polarized bands to such extent that one practically disappears, we find what appears to be a band showing opposite rotations on its two edges. In reality one of these rotations is the rotation within the band, and we cannot offer such cases in support of the theory of molecular currents.

In conclusion Becquerel states that in no case has any evidence been found in support of this theory.

Rays Perpendicular to the Magnetic Field. — On the hypothesis of molecular currents we should expect no effect to be produced by the magnetization of the medium when the rays of light are perpendicular to the lines of force. The Hall effect hypothesis, however, calls for an effect in this case, which, though small, has been detected. Consider the waves as propagated along the x axis instead of along the lines of force (z axis). In this case we have the relations

$$\epsilon'' \frac{\partial X}{\partial t} + i\nu \frac{\partial Y}{\partial t} = 0, \text{ since } j_x = 0,$$

$$\frac{1}{c}\left(\epsilon''\frac{\partial Y}{\partial t}-i\nu\frac{\partial X}{\partial t}\right)=-\frac{\partial \gamma}{\partial x},\qquad \frac{\epsilon'}{c}\frac{\partial Z}{\partial t}=\frac{\partial \beta}{\partial x},$$

$$\alpha=0,\ \frac{1}{c}\frac{\partial \beta}{\partial t}=\frac{\partial Z}{\partial x},\qquad \frac{1}{c}\frac{\partial \gamma}{\partial t}=-\frac{\partial Y}{\partial x}.$$

Eliminating β and γ gives

$$\epsilon''X+i\nu Y=0,$$

$$\frac{\epsilon'}{c^2}\frac{\partial^2 Y}{\partial t^2}=c^2\frac{\partial^2 Y}{\partial x^2}+i\frac{\nu}{c^2}\frac{\partial^2 X}{\partial t^2},\quad \frac{\epsilon'}{c^2}\frac{\partial^2 Z}{\partial t^2}=\frac{\partial^2 Z}{\partial x^2}.$$

Elimination of X from the first two equations gives

$$\left(\epsilon''-\frac{\nu^2}{\epsilon''}\right)\frac{\partial^2 Y}{\partial t^2}=c^2\frac{\partial^2 Y}{\partial x^2}.$$

Writing as before, $X=Me^{\frac{i}{\tau}(t-p'x)}$, $Y=Ne^{\frac{i}{\tau}(t-p'x)}$, $Z=\Omega e^{\frac{i}{\tau}(t-p'x,)}$.

The velocities of the x and y components are obviously the same, since they are symmetrical about the lines of force. The z component may, however, be propagated with a different velocity, hence we are obliged to distinguish between p' and p.

By differentiation and substitution we get

$$\epsilon''-\frac{\nu^2}{\epsilon''}=p'^2c^2,\ \epsilon'=p^2c^2,\ M=-\frac{i\nu}{\epsilon''}N.$$

This equation shows us that the z and y components are propagated with different velocities, and that the vibration, if originally a plane one, making an angle of 45° with the lines of force, becomes transformed into an elliptical one, in other words the medium is doubly refracting. The velocity of the z and y components is given by substituting the values of ϵ' and ϵ'' in the above equations.

$$p^2c^2=n^2(1-i\kappa)^2=\epsilon'=1+\sum\frac{\theta N}{\Theta}.$$

gives the velocity of a vibration parallel to the field, while

$$n'^2(1-i\kappa')^2=1+\sum\frac{\theta N\Theta}{\Theta^2-\Phi^2}-\frac{\left(\sum\frac{\theta N\Phi}{\Theta^2-\Phi^2}\right)^2}{1+\sum\frac{\theta N\Theta}{\Theta^2-\Phi^2}}$$

represents the velocity of the component perpendicular to it.

The difference between n and n' will be very small unless Θ is very small, which only occurs in the immediate vicinity of an absorption band.

The above formulae were derived by Voigt, who writes $\left(\frac{v}{O_y}\right)^2$ for $n'^2(1-i\kappa')^2$, v representing the velocity of light in space and O_y the

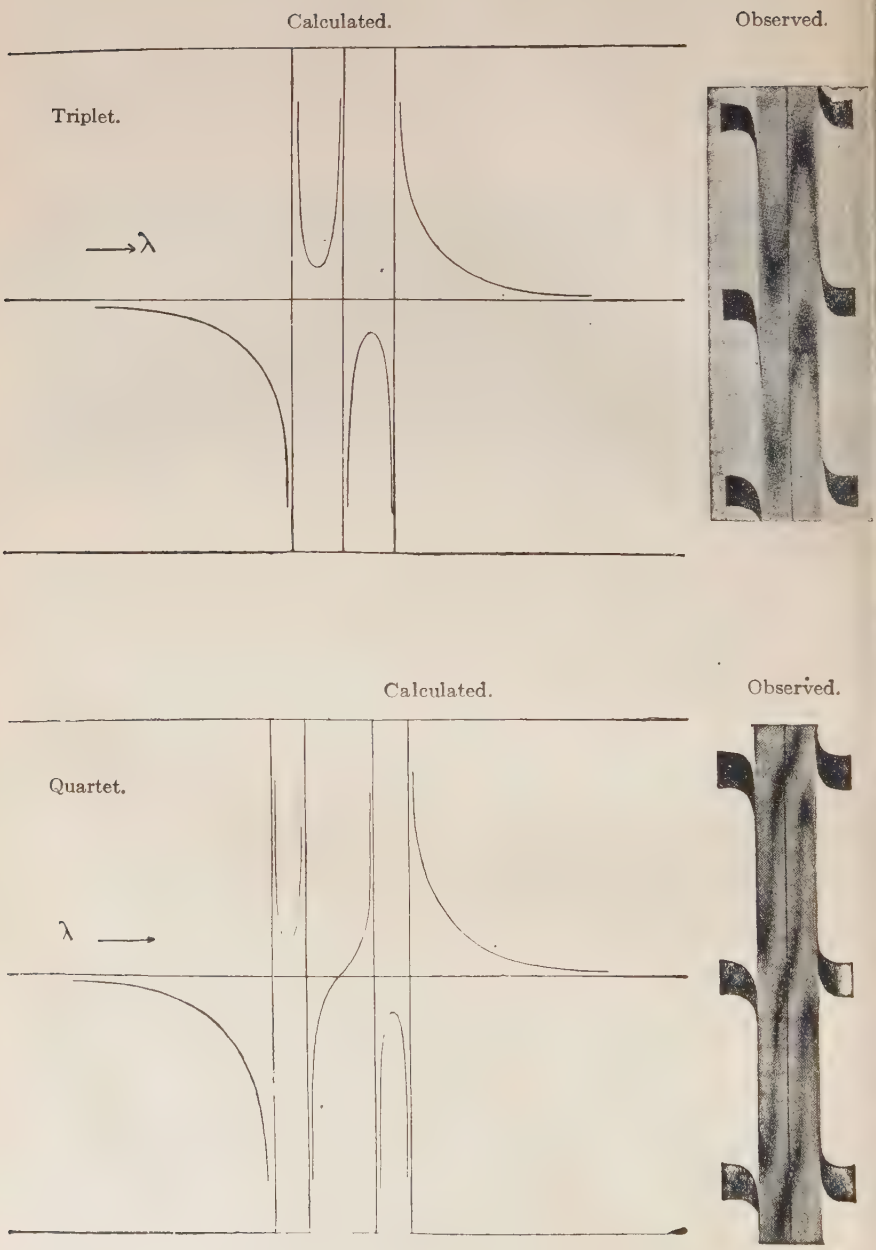

Calculated.
Observed.
Triplet.
λ
Calculated.
Observed.
Quartet.
λ

FIG. 339.

complex amplitude of the y component. The field strength h occurs in Φ only, consequently the velocity of the z component is uninfluenced by the magnetic field.

An attempt was made by Voigt to detect the double refraction by placing a block of glass between the poles of a powerful electromagnet, and causing the light to traverse it a large number of times in a direction perpendicular to the lines of force. The ends of the block were silvered, with the exception of two narrow strips for the entrance and exit of the beam. A Babinet compensator was used to detect a possible change of phase between the two components, and though a very slight shift of the fringes was noted when the field was thrown on, it was too small to be regarded as established.

The effect has recently been found by Cotton and Mouton[1] in the case of some organic fluids. Nitro-benzol shows it to good advantage, a column 4.2 cms. in length in a field of 18,500 Gausses giving a retardation to the vibration parallel to the field amounting to a phase-difference of 36′. Bisulphide of carbon gave a retardation to the perpendicular component of 6′.

The double refraction was, however, found by Voigt and Weichert in the case of sodium vapor. They placed a small sodium flame between the poles of the magnet and passed through it a beam of white light, polarized at an angle of 45° to the lines of force. The interference bands seen in the compensator were found to be curved in opposite directions, on opposite sides of the absorption band, the appearance being similar to the anomalous dispersion curve exhibited by the vapor. This experiment was subsequently enlarged upon and improved by Zeeman and Geest (*Proceed. Amsterdam Acad.*, Jan. 25, 1905). It would be well worth while investigating the behavior of dense sodium vapor in vacuo with respect to its magnetic double refraction.

Double Refraction near the Components of Absorption Lines magnetically split into Several Components. — Zeeman and Geest obtained some extremely interesting results by applying the same method of study to light which had traversed an absorbing flame in a direction perpendicular to the magnetic field. In this case D_1 is split up into a quartet and D_2 into a sextet. A formula was deduced which expressed the phase-difference between the vibrations parallel and normal to the field, and the deformations of the horizontal fringes, which resulted from the phase-differences, were calculated and recorded graphically. These curves were then verified by experiment, the agreement being most excellent. The calculated curves and observed fringe contortions are shown in Fig. 339. The calculated curves are for a single dark band. The observed show two adjacent bands, it being impossible to limit the observation to a single one. A field strength insufficient to give the sextet was used, the D_2 line appearing as a triplet. The reader is referred to Voigt's *Magneto- and Electro-Optics* for a fuller account of these experiments and their mathematical treatment.

[1] Cotton and Mouton, *Compt. Rend.*, 144, p. 229, 1907.

CHAPTER XIX

ELECTRO-OPTICS

In the previous chapter we have studied the optical properties of bodies when subjected to a powerful magnetic field, and the effect of a magnetic field upon the emission of light. In the present chapter we shall study not only the analogous phenomena produced by powerful electrostatic fields, in so far as they are known at the present time, but shall investigate certain other relations between light and electricity, such as the discharge of electrified bodies by the impact of ultra-violet light-waves. Faraday sought in vain, immediately after his discovery of the rotation of the plane of polarization by bodies placed in a magnetic field, for some analogous effect resulting from an electric field. Similar efforts were made by Quincke and by Roentgen, the former measuring, by an interference method, the velocity of light in the dielectric of a charged condenser, and the latter searching for a possible double refraction induced by an electric field. The first positive results were obtained by Kerr, who imbedded the wire terminals of an induction coil in a block of glass placed between crossed Nicols and found that there was a restoration of light when the coil was in operation. The restoration was not instantaneous, however, first appearing about two seconds after the coil was set in operation, and not reaching its maximum value for thirty seconds. The restored light could not be extinguished by rotation of the analyzing Nicol, which showed that the phenomenon was not a simple rotation of the plane of polarization, but an elliptical polarization resulting from the fact that the medium had become doubly refracting. The phenomenon was observed at its best when the plane of polarization made an angle of 45° with the direction of the wire electrodes, and vanished when the plane was either parallel or perpendicular to this direction. In the case of the resin colophonium a similar effect but of opposite sense was observed, the two substances acting like positive and negative crystals. The slowness with which the phenomenon appeared and disappeared indicated that, in all probability, it was to be referred to a strain induced in the glass resulting either from the continued action of the field, or possibly from temperature changes due to traces of conductivity. Kerr himself was of the opinion that an orientation of the molecules occurred as a result of the action of the electric field, and that it was hindered in a measure by frictional forces.

The Kerr Electro-Optic Effect in Liquids. — If the phenomena discovered by Kerr were merely of mechanical or thermal origin they would have very little theoretical interest. His discovery that similar effects were produced by liquids, in which any considerable strain could not be set up, indicated that the effect was, in

all probability, a true electrical one. He examined a large number of fluids and determined the nature of the double refraction, *i.e.* whether positive or negative, by means of a strip of glass, which when compressed or expanded, as the case might be, compensated the effect produced by the electric field. It was found that positive double refraction, similar to that exhibited by quartz in the absence of an electric field, was exhibited by bisulphide of carbon, bromine, molten sulphur, and phosphorus. Negative refraction analogous to that of Iceland spar was found in the case of many oils.

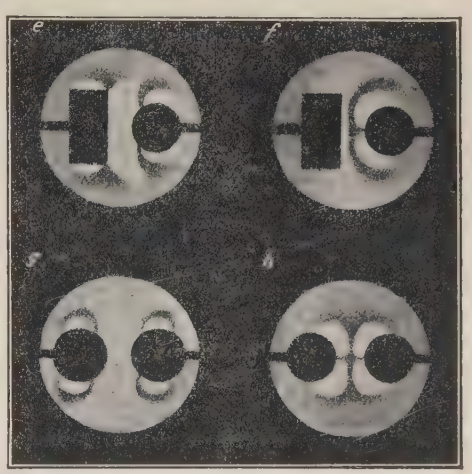


Fig. 340.

In the case of small electrodes in close proximity in the liquid the curvature of the lines of force gives a very complicated state of double refraction, and very curious distributions of light and shade may be observed with crossed Nicols. Fig. 340 is from a paper by Brongersma.[1] Roentgen investigated the effects of internal motions of the fluid upon the appearances presented in similar experiments and made the observation that the horizontal dark strip between the ball and cylinder electrodes in the fourth figure was carried up by an upward motion of the current, as if a certain time was required for the effect to occur.

The Method of Abraham and Lemoine. — The method originated by Abraham and Lemoine in their investigations of the time required for the development of the electro-optic effect is one of great interest and one that can doubtless be applied in modified form in many fields of investigation in which the measurement of very minute intervals of time is required.

By this method it has been found possible to measure a time interval of one *two hundred millionth* of a second. A rifle bullet, moving with a velocity of 500 metres per second, would travel about $\frac{1}{400}$ of a millimetre, and a light-wave about 1.5 metres, during this interval of time. They illuminated the field of their instrument with the light of a spark produced so to speak by the discharge of the fluid condenser the double refraction of which was being studied. If the light from this spark was made to travel over a distance of 400 cms. before passing between the plates of the condenser all trace of the double refraction disappeared, while a retardation due to the introduction of a path of only 80 cms. reduced the double refraction to one-half of its maximum value.

[1] M. H. Brongersma, *Wied. Ann.*, 14, p. 222, 1882. Roentgen, *Wied. Ann.*, 10, 77.

Electro-Optics of Nitro-Benzol. — An investigation was made by Schmidt[1] of the value of the Kerr constant for a large number of substances. This constant j occurs in the empirical formula for the path-difference measured in wave-lengths of the two components of the vibration.

$$D = jlK,$$

in which l is the length of the fluid column traversed by the light and K the strength of the electric field. In the case of nitro-benzol he found that the double refraction was 60 times as great as that of bisulphide of carbon. This substance is therefore especially adapted to investigations or demonstration of the Kerr phenomenon, and was used in a subsequent investigation by Aeckerlein[2] of the effect of the electric field upon the two components of the vibration. By employing rapid electric oscillations and an alternating electric field he reduced the troublesome effects due to conduction currents, temperature changes, etc., to a minimum, and found that the vibration perpendicular to the lines of electric force was retarded, while the parallel component was accelerated, the acceleration being about one-half of the value of the retardation.

The Electro-Optic Analogy of the Zeeman Effect. — As Voigt has pointed out in his book, an electric effect is to be expected, analogous to the Zeeman effect, when a source of radiation is placed in a powerful electrostatic field. In a direction parallel to the field we should expect a simple displacement, while in a perpendicular direction there should be two lines, one polarized with its electric vector perpendicular to the lines of force displaced three times as much as the other oppositely polarized component. The calculated effect is very small, and it has never been observed. Powerful electric fields cannot be applied very well, owing to the conducting power of ionized or radiating gases. It is possible that something might be found by observing the curious type of discharge which occurs when the electrodes of a highly exhausted tube are brought within a millimetre of each other. A very high potential is required, with a spark gap in the circuit. We then obtain a small arc-like discharge between the electrodes, and it is possible that an examination of the metallic lines which it emits with a powerful échelon might show something of interest. A paper by the author describing this discharge, and the method of obtaining it, will be found in the *Physical Review* for 1897.

The Photo-Electric Effect. — The history of the photo-electric effect begins with the discovery by Hertz in 1887 of the influence of the ultra-violet light from one spark gap upon the discharge of a neighboring gap. If the distance between the electrodes was made just sufficient to prevent the discharge, illumination by ultra-violet light from another spark caused the passage of sparks. Hertz found that it was the action of the light upon the negative electrode only which occasioned the discharge. In 1888 Hallwachs discovered

[1] W. Schmidt, *Ann. d. Physik*, 7, p. 142, 1902.
[2] Aeckerlein, *Phys. Zeit.*, 7, p. 594, 1907.

that certain substances, when negatively charged, lost their charge when illuminated with ultra-violet light, and that if originally uncharged, they acquired a positive charge as a result of the illumination. It has since been ascertained, as a result of investigations by Lenard, Thomson, Merritt, and others, that the action results from the expulsion of negative electrons brought about by the absorption of the light, in other words cathode rays leave the surface of the plate. Two distinct methods of investigation have been used. In the first, the metal surface is illuminated in a vacuum, and the positive charge which it acquires when illuminated is measured with an electrometer. This method gives us the maximum velocity with which the electrons leave the surface, for the potential continues to rise until it becomes great enough to hold back the electrons which have the highest velocity.

In the second method the illuminated plate is placed in opposition to and at a short distance from a second plate, which is connected to earth through a galvanometer, which measures the current resulting from the discharge of the electrons. This method gives us the number of negative electrons which leave the surface of the metal for a given intensity of illumination. The number has been found to be proportional to the intensity of the light, while the velocity, which is between 10^7 and 10^8 cms. per second, is independent of it.

In the case of the normal photo-electric effect, the orientation of the electric light vector is of influence only in so far as it affects the amount of energy absorbed, as has been shown by Pohl for solid (Zn and Pt) and liquid (Hg, Na, and K) metals, for ultra-violet light. If, however, we are dealing with an alkali metal in the region of spectrum for which the selective effect is found, for equal amounts of absorbed light we have a far larger number of electrons emitted when there is a component of the electric vector perpendicular to the surface. According to Pohl and Pringsheim this is the reason of the phenomena observed by Elster and Geitel with fluid amalgams of sodium and potassium in the visible region of the spectrum. The photo-current due to the electric vector vibrating in the plane of incidence exceeded that due to the perpendicular vector fifty fold.

The variation of the angle of incidence gave only for the component $\perp$ to the plane of incidence a proportionality to the absorption, while for the $\parallel$ component, the component of the photo-current perpendicular to the surface was very nearly proportional to the calculated $\perp$ component of the absorbed light. The factor of proportionality was naturally much greater in the latter case.

Photo-Electric Effect and Wave-Length. — The influence of the wave-length of the light upon the emission of the electrons has been studied by E. Ladenburg, Markau, and A. W. Hull (*Phys. Zeit.*, 8, 590; 9, 821; 10, 537). It was found that the maximum potential of the illuminated plate increased as the wave-length of the light decreased, in other words the electrons were driven out at a higher velocity by short waves. As we have seen, their velocity

is independent of the intensity of the light, which influences only the number emitted in unit time. Ladenburg found that the potential of a platinum plate when illuminated by $\lambda=260\ \mu\mu$ was 1.07, the value increasing to 1.86 as the wave-length decreased, $\lambda=201\ \mu\mu$. The number of electrons emitted also increases as the wave-length of the light decreases. In determining this, however, the intensity of the radiation used has to be taken into account, the number emitted is proportional to the intensity of the light. If E_1 represents the saturation current shown by the galvanometer, which is the measure of the number of electrons emitted, and E_2 the energy of the radiation of wave-length λ, we must plot the quantity $\frac{E_1}{E_2}$ against λ in order to obtain a curve showing the relation between the number of electrons thrown off and the wave-length of the light. Ladenburg found that the E_1 curve had a maximum at $\lambda=212$, falling rapidly on each side of this point, E_2 increased steadily with increase of wave-length, while $\frac{E_1}{E_2}$ increased very rapidly with decrease of wave-length, the curve resembling a dispersion curve in the ultra-violet region.

The observations of Ladenburg do not hold, however, for the alkali metals. These, according to the investigations of Pohl and Pringsheim, show in the first place the normal photo-electric effect, for which, with equal amounts of absorbed energy, we have with decreasing wave-length an increasing number of emitted electrons; they show, in addition, a selective photo-electric effect, limited to a narrow spectral range, indicating that we are dealing with a resonance phenomenon. This occurs only when there is present a component of the electric light vector perpendicular to the surface. This selective effect may exceed the normal effect in the same spectral region from twenty to one hundred fold.

Potassium, for example, shows a strong maximum of photo-electric sensibility at $\lambda = 4400$ with the light incident obliquely, while for normal incidence there is no trace of the maxima, the curve being identical with that found by Ladenburg for platinum.

Polarization of the Light. — Elster and Geitel, who have done an immense amount of very valuable work along these lines, made the discovery that the orientation of the plane of polarization of the light influenced the rate at which the electrons were given off, but was without influence on their velocity. If the electric vector is in the plane of incidence the photo-electric effect is much stronger than when the vector is perpendicular to the plane. In the former case there is a component of electric force perpendicular to the surface, and we may perhaps regard this as better adapted to drawing the electrons out of the surface, than an electric vibration parallel to the surface.

It seems probable that the vibrations of the light set up vibrations of the electrons, the amplitude of which increases until it becomes so great that the electron escapes from the attractive force which previously kept it within the metal.

The number of electrons emitted by the surface depends upon the amount of light absorbed, at least for vibrations perpendicular to the plane of incidence. In the case of vibrations parallel to the incidence plane, Elster and Geitel found that the emission of electrons was proportional to the component of the absorbed light which was perpendicular to the metal surface. In the case of the fluid alloy of sodium and potassium, if the electric vector was parallel to the incidence plane the emission of electrons, in proportion to the absorbed energy, was fifty times as great as when the electric vector was parallel to the metal surface, a circumstance which Elster and Geitel referred to the greater penetration of the light into the metal. In the case of solid and highly polished metal surfaces, Pohl has found that the emission is proportional to the absorbed energy regardless of the state of polarization.

The Normal and the Selective Photo-electric Effect. — The extended investigations of Pohl and Pringsheim, which have been referred to in the preceding section, show that the discrepancies between the observations of different observers are due to the fact that there are two distinct effects, which are sometimes observed separately, and sometimes superposed. In the normal effect the number of electrons liberated increases with decreasing wave length, and the orientation of the plane of polarization is of influence only as the absorption of the light depends upon it. In the selective effect, which has thus far been found only in the case of the alkali metals, the orientation of the plane of polarization exerts a marked influence, the emission of electrons in proportion to the absorbed energy being many times greater when there is a component of the electric vector perpendicular to the surface. The two effects are shown graphically in Fig. 340 *a*, which is for the amalgam of sodium and potassium. The selective effect extends from $\lambda = 3500$ to about $\lambda = 4800$ and is only apparent in the $E\,\Pi$ curve made with the incident light polarized with the electric vector E parallel to the plane of incidence.

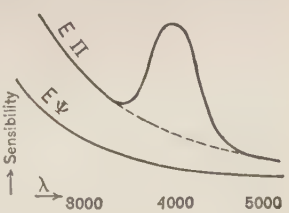


Fig. 340 *a*.

So far as observations go at the present time, the emission of electrons appears to be the same in all directions, that is, the number crossing unit area normal to the surface is equal to the number crossing unit area parallel to the surface. The case is somewhat analogous to the emission of X rays, for which Lambert's cosine law does not hold. It would be interesting to see whether with polarized light the emission has a larger value in the direction in which the electric vibration takes place. This matter has not yet been investigated.

Photo-Electric Effect in Gases. — Some doubt has been felt as to whether gases are ionized by light, the apparent effect being ascribed by Thomson to the action of the light upon suspended dust particles. The recent work of Bloch (*Compt. Rend.*, 146, p.

892) has shown that both effects occur, that due to dust particles being the stronger, and disappearing only after the radiation has traversed a column of the gas half a metre in length. The ionization of air carefully freed from dust, or the true photo-electric effect, was found to be much less, and to disappear entirely after the radiation had traversed a layer 3 mms. in thickness. Thomson has published an interesting paper on the subject,[1] in which, referring to Kaye's work, he shows that the ionization of air is 8 times, of CO_2 16 times, and of ammonia 150 times greater, when illuminated by ultra-violet radiation, than when screened from it.

Theory of the Photo-Electric Effect. — There are at the present time two theories of the photo-electric effect. We may regard it as a simple resonance phenomena, the electron vibrating in unison with the light-waves, until its amplitude becomes so great that it is expelled from the atom. Whether it derives its energy from the absorption of the radiation which frees it, or from the internal energy of the atom, is a question which can be answered only after a careful study of all of the relations which have just been discussed. The effect has been found to be entirely independent of temperature, which shows us that the heat energy of the metal plays no part. It is very difficult to reconcile all of the effects with the assumption of absorption of a light-wave, the amplitude of which is constant over the wave-front, decreasing as the distance from the source increases. There is a very recent "Licht-quanten hypothese" of Planck and Einstein, on which we are to regard the energy of a system of rays, not as spread continuously over an ever increasing surface (wave-front) but as made up of localized centres of energy which move out without subdivision, and can be absorbed only as a whole. These "units" of radiation can be shown to have an energy proportional to the oscillation frequency, which explains the increase in the velocity of the expelled electron, with the decrease of wave-length, if we assume the "licht-quanten" to give up its energy to the electron. It also explains the circumstance that the electron velocity is independent of the intensity, for on Einstein's hypothesis the intensity of the radiation is represented by the proximity of the light units. Planck, however, does not consider that the wave-front is discontinuous, the hypothesis being restricted to the absorption or emission of a resonator, which cannot progressively increase or diminish by infinitesimal steps, but only by finite "units."

The application of the "light units" to the wave-front is radically new, and quite at variance with our older notions regarding ether waves and amplitudes. It is difficult to form a picture of just what the "licht-quanten" is, but Thomson has suggested that light may consist of impulses running along lines of force. The wave-front, on this hypothesis, would not be uniformly illuminated, but consist of bright points, so to speak, and what we usually understand by intensity or amplitude would depend upon the proximity of the bright points.

[1] J. J. Thomson, "Ionization of gases by ultra-violet light, and the structure of light," *Camb. Proc.*, 14, Part IV., 417, 1907.

An attempt has even been made by G. J. Taylor to prove the existence of the "light units" by photographing diffraction fringes with light of such feeble intensity that but few " units " were included in each Huygens zone. The results were negative, however. Thomson was driven to this hypothesis by our inability to explain why only a very small percentage of the molecules of a gas are ionized by a light-wave. The conception appears very fantastic, in view of our preconceived ideas regarding ether waves, but it is very suggestive that something resembling *structure* in light-waves is pointed out by no inconsiderable number of recently discovered phenomena. We may eventually be driven to a compromise between the old corpuscular theory and the wave theory. Some very recent work by Baeyer indicates that the energy of each expelled electron is equal to the energy of one "licht-quanten" of the wave-length utilized in producing the expulsion. If this is true it is a very remarkable verification of Planck's hypothesis.

The following summary of the theory was prepared by R. Pohl:

A satisfactory theory of the normal effect must apply if possible to the electrons liberated by X-rays, as well as by light, and explain the following facts:

1. The uniform velocity of the electrons for light of a given wave-length. 2. The independence of velocity of the intensity. 3. The increase of velocity with decreasing wave-length. 4. The increase of the number of electrons with decreasing wave-length. 5. The independence of the number and velocity of the electrons of the temperature. 6. The circumstance that with increasing wave-length the effect drops to zero.

According to Lenard the emission is due to a resonance phenomenon. Light of a definite period excites electrons of equal period, the vibrations overcoming the forces which hold the electron in the metal. This force must be independent of temperature, and the velocity of the electrons is determined by the nature of the atom and not by the intensity of the light. This requires the questionable assumption that we have in the atom electrons of all possible velocities from 1 to 100,000 (electrons liberated by X-rays).

No. 4 is explained by the assumption of an absorption which, as is the case with cathode-rays, lessens only the *number* but not the velocity, and by the existence of the Volta series.

These assumptions of Lenard and others stand in opposition to the Planck-Einstein light-unit theory. Einstein supposes that the energy element nh, in which n is the frequency and $h = 6.55 \cdot 10^{-27}$ erg-sec., is found again *in the metal* as kinetic energy $\frac{1}{2} mv^2$ (or eV) of the electron. This kinetic energy of the electron we observe *outside of the metal* after the subtraction of the work P of the electron by passage through the surface $eV = hn - P$.

This formula gives for ultra-violet light of wave-length 200 $\mu\mu$, for $P = o$, 6.3 volts independent of the material, which corresponds to the value found by v. Baeyer (*Verh. der deutsch. phys. Ges.*, 12, 870, 1910). For X-rays, the wave-lengths of which are probably less than .01 $\mu\mu$, the calculated value is 10^5 volts, while the value observed by Bestelmeyer is $6.3 \cdot 10^4$ volts.

CHAPTER XX

TRANSFORMATION OF ABSORBED RADIATION. FLUORESCENCE, PHOSPHORESCENCE, AND RESONANCE-RADIATION

In the case of all substances which absorb light, *i.e.* in cases when the sum of the transmitted and reflected energy is not equal to the incident energy, we have a transformation of the radiant energy into energy of some other form. The commonest case is that in which the luminous radiations are spent in warming the body, the absorption of the light increasing the kinetic energy of the molecules. Just how the transformation takes place we do not know, though it is not difficult to make assumptions. In the case of selective absorption, where we refer the phenomenon to resonance, or the vibration of electrons of the same free period as the absorbed vibrations, we assume something akin to friction, which impedes the free motion of the electron and produces a rise of temperature. In some way the energy taken up by the electron is spent in increasing the velocity of the molecule, which makes it seem as if the action were an action between the molecules, and not something going on within the molecule, for we cannot speak of a molecule as rising in temperature. As the temperature of the substance increases, it emits more and more energy in the form of long heat-waves, and it is this reëmission of energy which prevents the temperature from rising indefinitely. If the substance absorbs strongly waves of all lengths, it is possible to throw radiant energy into it at such a rate that its temperature rises to the point of incandescence, as was shown by Tyndall. The incident energy may consist wholly of invisible heat-rays, in which case we have a reëmission of energy in the form of waves of shorter wave-length. This phenomenon has received the name of Calorescence.

Tyndall's experiment consisted in the formation of a dark heat focus by means of a large condensing lens, the visible radiations being filtered out by means of a solution of iodine in bisulphide of carbon, contained in a glass cell. A piece of blackened platinum foil held in the focus was speedily raised to a red heat. In this case the emission of light is a pure temperature effect. Certain substances, however, possess the peculiar property of emitting light when illuminated, without any appreciable rise in temperature. The emitted light is usually of a different color from that of the exciting radiation, and the emission may continue for some time after the illuminating light is cut off.

If the emission ceases as soon as the exciting radiations cease to fall upon the substance, the phenomenon is called Fluorescence:

if it persists for an appreciable time, the term Phosphorescence is applied to it. In general, fluorescence is only exhibited by gases and liquids, phosphorescence by solids, though we sometimes find the terms confused, the term fluorescence being applied to uranium glass and certain crystals. It is, perhaps, best not to attempt to draw a sharp line between the classes of phenomena, for it has been shown recently that we may have a gradual transition of fluorescence into phosphorescence.

Fluorescence. — The name fluorescence is derived from fluor spar, a native fluoride of calcium, a substance which was first observed to exhibit this peculiar emission of light. The subject was first investigated by J. Herschel (*Phil. Trans.*, 1845, p. 143) and Sir David Brewster (*Trans. of Edin.*, 1846, part ii., p. 3), who examined solutions of sulphate of quinine, which emit a brilliant blue light in all directions when illuminated with a beam of sunlight.

Herschel found that light which had traversed the solution was incapable of exciting any further emission, and that the blue luminosity was confined to the surface. He termed the phenomenon epipolic dispersion, believing that he was dealing with a new type of polarization. The light on entering the solution became "epipolarized," a lateral emission or dispersion resulting from the process, and this epipolarized light was incapable of exciting further fluorescence. Brewster found, however, that by employing an intense beam of light, the blue emission marked the entire path of the beam, and he accordingly changed the name to "internal dispersion."

It occurred to Stokes that the blue light "dispersed" by the quinine solution might not be the blue light of the illuminating beam, but a new creation due to the absorption of more refrangible radiations. This would explain the inability of the light to excite further fluorescence after it had already traversed a sufficient thickness of the solution, the rays effective in provoking the emission being removed by absorption. Experiments verified this surmise, establishing the general law that the fluorescent radiations are always of longer wave-lengths than those of the light which excites them. This change in the wave-length can be very easily observed by interposing colored glass in the path of the illuminating beam. A beam of sunlight, from which the orange-yellow and green has been removed by means of a sheet of dense cobalt glass, is concentrated with a lens upon a few crystals of uranium nitrate. Although the illuminating beam is of a deep blue violet color the crystals shine with a brilliant green light. A piece of the ordinary canary glass, which is colored with oxide of uranium, can be substituted for the nitrate crystals. Small vases of this glass, which can now be found in almost any glass-store, are suitable for the experiment. It can be easily recognized by the greenish color which it assumes when held in sunlight, which is in marked contrast to the yellow color of the transmitted light. One of the best substances for the exhibition of fluorescence is an aqueous solution of uranin, an alkaline salt of fluorescene. The solution should be extremely dilute. It is best to begin with pure water contained in a

rectangular glass tank, the light from an arc-lamp or the sun being brought to a focus at the centre of the tank. On adding a drop or two of a fairly strong solution of the dye the path of the beam becomes luminous, shining with a brilliant green light. On adding more of the dye the fluorescence retreats towards the region where the light enters the solution, owing to the increased absorption of the rays which are capable of exciting the fluorescence.

Other solutions can be easily prepared which fluoresce with different colors. An aqueous solution of aesculin, which can be prepared by pouring hot water over some scraps of horse-chestnut bark, shines with a beautiful blue light, while an alcoholic solution of chlorophyl, which can be prepared by soaking green leaves in strong alcohol, exhibits a red fluorescence. Among other substances which exhibit the phenomenon may be mentioned, solutions of sulphate of quinine (acidified with a few drops of sulphuric acid) and paraffin oil, both of which fluoresce with a blue light.

Methods of investigating Fluorescence. — An exceedingly simple and ingenious method was devised by Stokes for detecting fluorescence and phosphorescence, which is applicable to cases where the emitted light is so feeble as to be overpowered by the irregularly reflected light. This method depends on the change of wave-length which accompanies both phenomena. Two screens are prepared, one of which transmits the violet and blue, absorbing the green, yellow, and red, while the other absorbs the violet and blue, transmitting the rest of the spectrum. Dense cobalt glass combined with a thin sheet of signal green glass, or a solution of cuprammonium, makes a suitable blue-violet screen; while yellow glass or a solution of bichromate of potash will answer for the other. The two together should be practically opaque even to a fairly strong light.

If, now, a powerful beam of light is admitted to a dark room or box through the blue screen, objects illuminated by it will be invisible through the yellow screen unless they fluoresce or phosphoresce, that is, give out less refrangible radiations than those which fall upon them. Stokes succeeded in showing that ordinary paper, cotton, bones, ivory, leather, cork, horn, and many other substances exhibit the phenomenon.

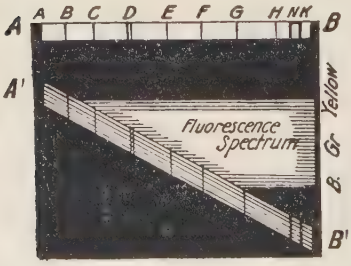


Fig. 341.

This method, while admirably adapted to the detection of fluorescence, is not suited to the study of the relation between the wave-lengths of the fluorescent and incident light.

The method adopted by Stokes was analogous to Newton's method of crossed prisms; a very narrow and intense solar spectrum was thrown upon the surface of the liquid under investigation, the fluorescence resulting in this case from monochromatic light of varying wave-length. This spectrum was then viewed through a prism

held in such a position as to deviate the spectrum in a direction perpendicular to its length, as shown in Fig. 341, in which AB is the undeviated and $A'B'$ the deviated spectrum, as it would appear if the surface of the liquid merely reflected or scattered light without fluorescence. This spectrum will always be visible to a greater or less degree, owing to the fact that even the surface of a transparent liquid scatters a certain amount of light without change of wave-length.

In the case of fluorescent or phosphorescent substances, we have in addition the complete fluorescent spectrum produced by the monochromatic illumination at each point of the spectrum. The fluorescent spectra together form the broad band shown in the figure, from which it is apparent that the wave-length of the fluorescent light is never less than that of the light which causes it. Any exception to Stokes's law would give rise to an extension of the band on the other side of the deviated spectrum.

Stokes's law, that the waves of the fluorescent light are never shorter than those of the exciting rays, was questioned by Lommel, who believed that he could detect the complete fluorescent spectrum of Magdala red, which contains red, yellow, and green rays, when the fluorescence was excited by sodium light alone.

Hagenbach investigated this same substance and came to a different conclusion. The eye-piece of a spectroscope was replaced with a vertical slit, upon which the spectrum was focussed. An image of this slit, illuminated in monochromatic light, was thrown by means of a lens partly upon the surface of the fluorescent solution, and partly upon a small piece of white porcelain placed close to the surface. The porcelain reflected only the monochromatic light, while the liquid emitted the fluorescent light. On viewing the image through a prism, the fluorescent spectrum B, and the monochromatic image of the slit A, reflected from the porcelain, appeared as shown in Fig. 342 (1). The two images were separated by a wide gap, the fluorescent light being much less refrangible than the exciting rays. By turning the prism of the spectroscope, the wave-length of the light from the slit was increased, which caused the image A to approach B, the gap between becoming narrower as shown in (2). No change in the fluorescent spectrum was observed. On still further increasing the wave-length, A came into coincidence with B. Up to this point the fluorescent spectrum remained unchanged, but from now on a further increase in the wave-length resulted in a contraction of the spectrum B, as shown in (4), no radiations appearing in it of shorter wave-length than those in the image A. An exception to Stokes's law would have made itself manifest as a faint illumination in the region to the right of the slit image.

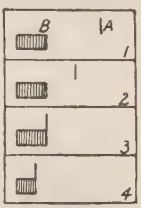


Fig. 342.

Lommel then repeated his experiments, using both sodium light and monochromatic light from a spectroscope, and found as before that not only Magdala red but a number of other substances showed evidences of emitting fluorescent waves shorter than the

exciting ones. He distinguished three distinct classes of fluorescence. Bodies belonging to the first class were capable of giving out their *complete* fluorescent spectrum when excited by any radiations exciting fluorescence. Under the second class were grouped substances of which the fluorescent spectrum contained no shorter radiations than the exciting ones. A third type, which he called composite fluorescence, embraced substances having a fluorescent spectrum consisting of two parts, similar respectively to the spectra shown by substances of the first two classes. These substances behaved as would a mixture of a substance belonging to the first class with one belonging to the second.

A very careful study of the subject has been made recently by Nichols and Merritt (*Phys. Rev.*, June and July 1904), who measured with a spectrophotometer the distribution of intensity in the fluorescence spectrum, when the wave-length of the exciting light was varied. They found that in the case of all the substances which they examined there were marked exceptions to Stokes's law, the position of the maxima of the fluorescence spectrum being independent of the wave-length of the exciting light. They were able to produce powerful fluorescence when the exciting light was of much greater wave-length than that at the centre of the fluorescent band.

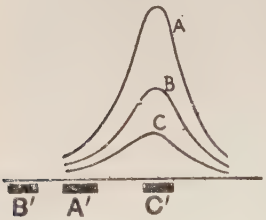


Fig. 343.

An aqueous solution of fluorescein is admirably adapted for the purposes of illustration. In Fig. 343 we have the intensity curves of the fluorescent spectra A, B, and C when excited by approximately monochromatic light cut out of a spectrum in the region A', B', and C'. The fluorescence is seen to be most intense when the exciting wave-lengths lie on the edge of the fluorescent spectrum which is towards the violet, C', but still of considerable intensity when the illuminating light is made up of a band on the red side of the point of most intense fluorescence.

Polarized Fluorescence. — The fluorescent light emitted in an oblique direction from the surfaces of isotropic media was found by Millikan (*Phys. Rev.*, September and November 1895) to be polarized in a manner similar to that of light obliquely emitted by white-hot surfaces. This polarization results from refraction of the light as it passes out of the medium into the air.

Certain crystals possess the remarkable property of emitting a polarized fluorescence. The most interesting case is that of magnesium platino-cyanide, which can be prepared by the addition of a solution of magnesium sulphate to a solution of barium platino-cyanide, until no further precipitation of barium sulphate takes place. The colorless solution is filtered, evaporated and crystallized. The crystals, which are of a deep red color, have most remarkable optical properties, showing a brilliant green surface color on the sides of the prisms, while the ends selectively reflect a deep violet light, which is polarized even at normal incidence. The crystals

should be mounted in small glass bulbs hermetically sealed, as otherwise they crumble into a yellow powder. If the bulb is filled with benzol, the violet metallic reflection, due to absorption, shows to better advantage as the vitreous reflection is abolished. The fluorescence of the crystals can be best observed by exciting them in a concentrated beam of sunlight which has been passed through a piece of dense cobalt glass. We will suppose the crystal to be standing upright, upon one of its bases, and the incident beam horizontal. If the fluorescent light, which emerges from the sides of the prism, be observed through a Nicol prism, it will be found that the color is orange-yellow when the polarizing plane of the Nicol is perpendicular to the axis of the prism, and scarlet when the plane is parallel to the axis. If the exciting light be polarized horizontally, the color of the fluorescent light is yellow, changing to red as the plane of polarization is rotated through 90°. Examination with a second Nicol shows that the orange-yellow fluorescence is polarized perpendicular, the scarlet parallel to the axis of the crystals. If, now, the crystal be placed in a horizontal position, and the beam of unpolarized light directed against an end surface, the fluorescent light will be unpolarized and of a scarlet color. If the incident light be polarized in a vertical plane, and the crystal be turned on a vertical axis so as to vary the angle of incidence, the red color changes to yellow. If, on the other hand, the plane of polarization be horizontal, no change is observed on turning the crystal. This shows that the change from red to yellow takes place as the angle which the direction of vibration makes with the crystal's axis changes from 90° to 0°, the direction of vibration being perpendicular to the plane of polarization.

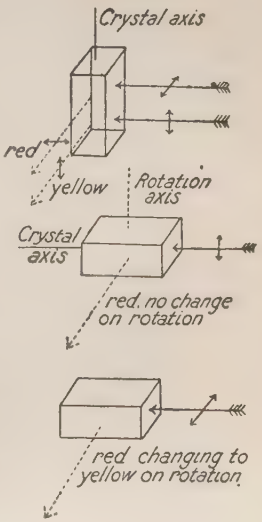


Fig. 344.

The results obtained in the two positions of the crystal are in perfect agreement, as will be seen by reference to Fig. 344, in which the exciting rays are indicated by solid arrows, the fluorescent rays by dotted arrows, the direction of vibration in each case being indicated by double-headed arrows. Polarized fluorescence has recently been observed by the author in the case of certain gases, and will be referred to presently.

Variation of the Intensity with the Angle of Emission. — The intensity of the light emitted by self-luminous solid and liquid substances varies as the cosine of the angle of emission (Lambert's law). On this account the intrinsic intensity is not increased by foreshortening of the source, *i.e.* by viewing it in an oblique direction. This is not, however, the case with fluorescent light, as is indicated by some experiments made by the author.

If a rectangular glass tank, or even a beaker glass, is partly filled

with a solution of uranin (fluorescein) and a condenser discharge passed between cadmium electrodes close to the surface, the phenomenon can be very clearly seen. The surface is powerfully fluorescent, and if it be viewed from below, the intrinsic intensity will be found to increase rapidly as the surface is foreshortened, becoming of dazzling brilliancy at grazing emission. If a glass plate is interposed between the spark and the fluid, the effect of foreshortening becomes less marked or disappears entirely, for in this case the fluorescence is chiefly caused by the radiations which penetrate the body of the fluid, and the powerful surface fluorescence, excited by the ultra-violet rays, disappears. A still better method is to illuminate one face of a right-angle prism of crown glass with the light of the spark, which causes a blue fluorescence of the surface layer. The luminous surface is to be viewed through the other face of the prism. The intensity viewed in the normal direction is very slight, as can be seen by looking at the reflection of the luminous surface in the hypothenuse face of the prism. Seen edgewise the intensity is fully thirty times as great, as was found by measurement at an angle of 5 degrees with the surface.

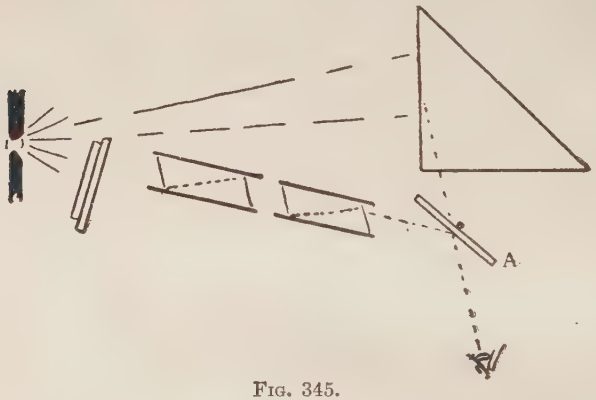


Fig. 345.

Measurements of the change of intensity with the angle of emission were made in the following way.

A portion of the light from the spark fell upon the prism, while another portion, after traversing a sheet of ground glass and a blue screen, was passed through a pair of Nicol prisms and reflected to the eye by means of the narrow strip of silvered glass A, mounted on a pivot in front of the prism (Fig. 345). By turning one of the Nicols the intensity of the light seen reflected in the strip could be balanced against the fluorescent background against which it was seen. The color was very accurately matched by means of a thin sheet of cobalt glass combined with a gelatin film stained with one of the blue aniline dyes. The intensity in the normal direction was measured by matching the reflected light against the image of the fluorescent surface seen by total reflection in the prism.

The fluorescent prism was mounted on the table of a small spectrometer and viewed through the telescope, the lenses of which were previously removed, and a small slit put in place of the eyepiece. In this way the angular direction from which the luminous surface was viewed could be easily determined.

A number of corrections were of course necessary, for the angles in air are greater than the angles within the glass, on account of refraction. Then, too, there is a small loss due to reflection within the prism which becomes greater as the angle is increased.

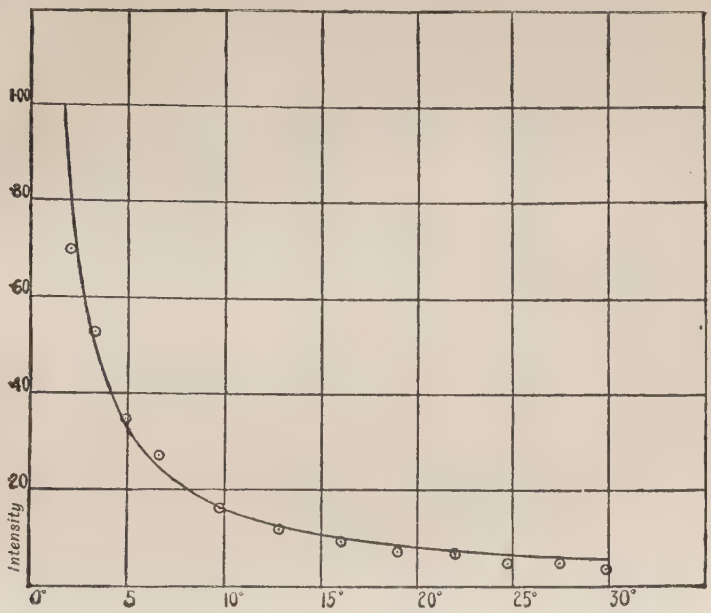


FIG. 346.

The values obtained are shown graphically in Fig. 346, intensities being plotted as ordinates, and angles as abscissae. The observed values, which have been corrected for the small loss due to internal reflection, are represented by circles, while the curve drawn through them is the theoretical curve, calculated on the assumption that the intensity of the radiation from each fluorescent molecule is independent of the direction within the glass. Under such conditions the intensity of the illumination of the surface would double each time its apparent area was halved by foreshortening.

As will be seen, this condition appears to be pretty nearly fulfilled, though accurate measurements between zero and five degrees were difficult. The fluorescent radiations obey therefore within the medium the same law which holds in the case of a transparent radiating gas. Outside of the medium, that is, in air, the case is very different. The intrinsic intensity is greatest when the surface is viewed in the normal direction, owing doubtless to refraction,

for the rays emerging at nearly 90° were incident at nearly the critical angle and there is a large loss by reflection.

Fluorescence Absorption. — The effect of fluorescence upon absorption was investigated by Burke, who claimed that a block of uranium glass absorbed light more strongly when fluorescing than when in darkness. His results were called into question by Camichel and others, but were corroborated by Nichols and Merritt, who made a very careful photometric study of the phenomenon. The effect comes out only when the separately measured intensities of the source of light and the fluorescent cube are added together and compared with the measured intensity of the source as seen through the fluorescing cube. A method was finally devised by the author,[1] which would show the phenomenon directly if it existed. The source and fluorescent body were made intermittent, and an arrangement used which permitted the observer to throw the flashes in-step or out-of-step. In the former case the flashes from the source traversed the medium while it was fluorescing; in the latter, while it was in darkness. The total amount of light received by the eye was found to be the same in each case, showing that the phenomenon was non-existent. Nichols and Merritt have more recently repeated their work, and have come to the same conclusion, so that the subject appears to be disposed of.

Phosphorescence. — The term fluorescence, strictly speaking, should be limited to the cases of the gases and liquids, since, in the case of solids, the emission of light continues for an appreciable time after the exciting radiations are cut off. In some cases the emission of light may continue for several hours, in others it lasts for but a very small fraction of a second after the exciting rays are removed. Most remarkable is the prolonged emission of Balmain's luminous paint, a sulphide of calcium, which glows in the dark for many hours after exposure to a strong light. It appears probable that something analogous to a chemical change is produced by the action of the light, the condition being unstable and the process reversing as soon as the substance is screened from the action of light. This reversing of the process may be prolonged or rapid, and is accompanied by the emission of light. The energy of the absorbed light is stored in the substance in the form of potential energy of the atoms. At very low temperatures the condition brought about by the action of light may be more or less stable, as has recently been shown by Dewar. A fragment of ammonium platino-cyanide was cooled by means of liquid hydrogen and exposed to a strong light. On removing it to a dark room no trace of phosphorescence was perceived, but on removing the crystal from the chilled tube and allowing it to warm up, it presently burst into a brilliant green phosphorescence.

A partial stability can be shown at ordinary temperatures with Balmain's luminous paint. If this be kept in absolute darkness for twenty-four hours it becomes non-luminous. A further emission of light may, however, be produced by concentrating invisible

[1] Wood, "On a Method of showing Fluorescence Absorption directly if it exists," *Phil. Mag.*, December 1908.

infra-red radiations upon it. This at first sight appears to be an exception to Stokes's law, but if the experiment be continued for a few minutes the luminosity ceases. The infra-red radiations have merely carried on the reverse process, originally produced by violet light, farther than it is able to go spontaneously. Not until the powder has again been exposed to light and kept overnight in the dark can the experiment be repeated. The color of the phosphorescent light obtained in this way differs from that of the light by the spontaneous breaking down of the molecular condition produced by the light, being distinctly greenish instead of deep blue.

If the Balmain paint be exposed to infra-red radiations while it is phosphorescing the luminous energy is liberated much more rapidly. This property has been utilized for obtaining records of the infra-red spectrum. A sheet of glass is painted over with luminous paint and exposed to sunlight. If a spectrum is then thrown upon it it will be found that the effect of the red and infra-red region is to render the portions of the surface upon which they fall, at first more luminous than their surroundings; the emission soon ceases, and on examination we find a dark region where the infra-red radiations have destroyed the phosphorescence. Very good records have been obtained in this way of the infra-red solar spectrum by Draper, Becquerel, and Lommel. The phosphorescent plate, after exposure to the spectrum, was placed in contact with a photographic plate, by means of which the record was made permanent.

Duration of the Phosphorescence. Phosphoroscope. — While some phosphorescent substances remain luminous for a considerable time after their exposure to light, the majority cease to give out visible radiations in a few seconds after the exciting radiations cease to fall upon them. An instrument was devised by Becquerel for examining substances in complete darkness, a small fraction of a second after their exposure to a brilliant light. This instrument, which is known as the phosphoroscope, consists of two metal disks side by side on the same axle. The disks are perforated with an equal number of apertures which are arranged out of step, and are driven at high speed by a train of cog-wheels. The substance to be examined is placed between the disks, and a strong beam of light directed upon it through the apertures of one of them. If the eye be brought close to the other disk the object will be seen only at the moments when the light beam is cut off, and it will be visible therefore only by phosphorescent light. The disks are mounted in a cylindrical metal box, to screen the substance from all light except that reaching it through the perforations.

With this instrument it is possible to observe an object one one-thousandth of a second or less after its illumination. Becquerel found that phosphorescence was much more common than had been supposed. The salts of the alkali metals, compounds of aluminium, and nearly all organic compounds were found to be phosphorescent. Compounds of the heavy metals for the most part showed no trace of luminosity, the salts of uranium and platinum being marked exceptions, however.

All solid fluorescent substances were found to be phosphorescent;

fluorescent liquids, on the other hand, showed no trace of the phenomenon. The author has found the same to be true for sodium vapor. A simple phosphoroscope with a single revolving disk can be set up in a very few minutes, and gives excellent results with uranium glass, uranium nitrate, and other phosphorescent substances. The disk, which is 30 or 40 cms. in diameter, can be made of cardboard with holes about half a centimetre in diameter punched at regular intervals around its circumference. The distance between the holes should be about 2 cms., not less. The disk can be mounted on the shaft of a small electric motor, or on a whirling table, or it may even be mounted on a lead pencil held in vertical wooden supports and set in rotation with a top string. A beam of sunlight, reflected from a mirror, is focussed on one of the holes; the diverging cone is received by a second lens on the other side of the disk and again brought to a focus, this time upon the object under examination, *e.g.* a lump of nitrate of uranium. On setting the disk in motion, and viewing the object through the small holes, taking care not to get in the way of light, the phosphorescence can be easily observed. Obviously the eye must be moved about until the position is found in which the incident light is cut off from the object when it is exposed to view. By laying the crystals upon white paper the effect is more striking, for it is then apparent that we are seeing the crystals by their own light alone. The room should be made as dark as possible, of course, though the phosphorescence is apparent even in a room brilliantly illuminated with sunlight.

Influence of Temperature. — It has been shown by Dewar (*Chem. News*, 70, 252, 1894; *Proc. Chem. Soc.*, 10, 171) that many substances at low temperatures exposed to light do not fluoresce until they are warmed, the change produced appearing to be stable at low temperatures. It would be interesting to examine the absorption spectrum of a fluorescent substance at low temperatures before and after its exposure to light. In this way direct evidence of a molecular change might be obtained. It is even possible that prolonged exposure to a brilliant light might produce a change that could be recognized in other ways. It is still apparently an open question as to how much luminous energy can be stored up in a fluorescent body at a low temperature.

Balmain's paint, cooled in solid carbonic acid and ether, and exposed to a powerful light while at a low temperature, emits no light until it is removed from the freezing mixture. The influence of temperature can be well shown by painting a sheet of metal with the paint, exposing it to sunlight and then heating it suddenly with a Bunsen burner in a dark room. The sudden rise of temperature is accompanied by a brilliant emission of light. Dewar found other substances, however, which phosphoresced only at low temperatures. Gelatine, celluloid, paraffin, ivory, and horn, which at ordinary temperatures only exhibited the feeblest traces of phosphorescence, became very luminous at $-180°$. The following substances were found especially brilliant at $-180°$: acetophenon, benzophenon, asparagin, hippuric and uric acids, diphenyl, salicylic acid, and egg

shells. Ammonium platino-cyanide showed no luminosity at − 180° until the liquid air was poured off, when it immediately lighted up like a lamp.

E. Goldstein has recently made some remarkable observations on the phosphorescent spectra excited by cathode rays. If the substances were cooled in liquid air the phosphorescent bands broke up into large numbers of fine lines, resembling the emission lines of gases.

Lenard and Klatt have found that in the discontinuous spectra of many phosphorescent substances, certain bands appear only at low temperatures, others at high. Quite recently H. Becquerel has made some interesting observations on the effects of low temperatures on the phosphorescent emission bands of some of the salts of uranium, broad and diffuse bands breaking up into sharp and narrow bands. Fig. 347 shows this effect in a striking manner. The upper spectrum is of a double sulphate of uranium and ammonia, the middle one of uranium and potassium both at the temperature of liquid air, the lower spectrum is of the latter salt at room temperature. The comparison spectra are of iron.

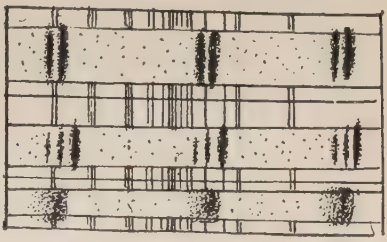


Fig. 347.

Spectra of Phosphorescent Light. — The first extended study of the spectra of phosphorescent substances in relation to the wave-length of the exciting light was made by E. Becquerel from 1843 to 1872. The substance to be examined was formed into a narrow rod, which, when illuminated in the phosphoroscope, was examined with a prism and telescope, the luminous rod taking the place of the slit. He found that the majority of substances showed spectra which remained the same when the wave-length of the exciting light was changed. The spectra were usually portions, more or less complete, of a continuous spectrum. Barium sulphide, however, he found emitted red light when illuminated with blue light, and yellow when the exciting radiations were confined to the ultra-violet. Especially brilliant and striking is the phosphorescent spectrum of the ruby. Artificial or synthetic rubies of large size and wonderful perfection can now be obtained from any jeweller at very small cost, and make beautiful objects for demonstration. The light of the sun or arc, passed through a sheet of very dense blue cobalt glass which removes all of the red and yellow rays, when focussed on the gem, causes it to emit a beautiful deep crimson light, which the spectroscope shows to be made up of several narrow bands. A small plano-convex lens with a circular disk of cobalt glass cemented to it furnishes a very easy and certain test for the ruby, for a paste imitation emits no red light when placed in the focus. It will not, however, distinguish between natural and synthetic rubies, for they are identical in every respect. Chrysogen,

an organic compound, exhibits a very characteristic phosphorescent spectrum, when illuminated with ultra-violet light; it consists of four bright and rather wide bands, extending from the red to the blue.

The most interesting of all the substances investigated by Becquerel are the salts of uranium. Uranyl nitrate, phosphate, and sulphate show each one its own characteristic spectrum of seven or more bands in the orange, yellow, and green regions. The appearance of the bands is quite different in each case. Quite recently (1907) H. Becquerel has found that at the temperature of liquid air these bands break up into groups of much narrower bands, much as the absorption bands of the rare earths investigated by Jean Becquerel. Some very interesting phosphorescent spectra were obtained by Morse [1] by illuminating different samples of fluor-spar with the light of electric sparks passed between different metals. The phosphorescent spectra contained bright lines the positions of which varied with the nature of the metallic electrodes; that is, with the wave-lengths present in the exciting light. It is interesting to compare these results with behavior of sodium vapor when stimulated with monochromatic light.

An immense amount of very important work has been done by Lenard and Klatt [2] upon the phosphorescence of the sulphides of barium, strontium, and calcium, containing slight traces of other metals as impurities. They found that the phosphorescence resulted from the presence of the impurity. Some of the bands persisted much longer than others after the extinction of the exciting, and the duration depended upon the temperature. In general when the substance is at a very low temperature all of the bands disappear almost at the instant that the stimulating light is extinguished. As the temperature rises, the bands begin to persist for an appreciable time, or become " Dauer-bands," each band having its particular temperature of longest duration. A further rise of temperature converts them again into " Momentan-bands." Some bands have their longest duration at $-100°$, others at 0, and still others at $+100°$. This property furnished the investigators with a means of separating the bands and measuring their position.

Theories of Fluorescence. — The fact that fluorescence is always accompanied by absorption shows that the phenomenon is in some way connected with the vibration of the electrons.

At first sight it may appear as if all that is necessary is to assume that the ions set in vibration by the ether waves become themselves sources of radiation. If this is the case all absorbing media should fluoresce, which is not the case. Moreover, we have seen that we can have a gradual transition from fluorescence to phosphorescence, and it is unthinkable that phosphorescence, of even a small fraction of a second's duration, can be simply the radiation of an electron which continues in vibration, after once having been set in motion. Even if the phosphorescence lasted but $\frac{1}{1000}$ of a second the electron would be obliged to perform of itself 600 billion vibra-

[1] *Astro-phys. J.*, 21, p. 83 (1905).
[2] *Drude's Annalen*, 15, p. 225–282, 425–484, 633–672 (1904).

tions. Though the damping of the vibration of an electron by radiation can be shown to be small, it is scarcely possible that a vibration, once started, can continue for such a length of time without receiving energy from some source. Another difficulty is the circumstance that with monochromatic light as the exciting radiation, the fluorescent radiation is distributed over a wide range of wave-lengths chiefly of lower refrangibility.

A satisfactory theory of fluorescence must first of all distinguish between absorbing media which fluoresce and those which do not; furthermore, it must explain the change of wave-length and the increase in the duration of the emission when the substance is in the solid form.

As a matter of fact no satisfactory theory exists, but it is instructive to examine briefly into the attempts which have been made to establish one. The fundamental assumption which has been made in every case is that the fluorescent light is emitted by atoms or electrons which are set in vibration by the light-waves. This explanation was given by Stokes, and it may be in part correct, though it fails to show why all absorption is not accompanied by fluorescence. It can hardly be extended to phosphorescence, and as the two phenomena are so closely related it seems probable that some at least of the facts of fluorescence are not to be accounted for on the simple assumption of a forced vibration, but must be referred to some complicated chemical process.

A theory was developed by Lommel, who built up an equation of motion of an atom vibrating under the impact of light-waves. This equation was similar in form to the one which we have already discussed in the treatment of the Helmholtz dispersion theory.

The molecular condition of the substance has apparently as much to do with its power of fluorescing as its chemical constitution. For example, many substances which in the solid state show no trace of fluorescence, when dissolved in various liquids become powerfully fluorescent. Other substances possess the property both in the solid and dissolved states, some to a greater degree in the former, others in the latter. Still others, such as barium platino-cyanide, which in the crystalline form are powerfully fluorescent, show no trace of the phenomenon in solution. For this behavior we have at the present time no satisfactory explanation, no theory of fluorescence having up to the present time been formulated which is capable of explaining even the simplest facts.

Effect of the Solvent on the Intensity of Fluorescence. — The solvent exercises a very marked influence upon the intensity of the fluorescence. This subject was carefully investigated by Knoblauch.[1] He found that the intensity of the fluorescent light was strictly proportional to the intensity of the exciting light, but that this intensity varied greatly when the substance was dissolved in different liquids, as is shown in the following table:

[1] *Annalen der Physik*, 54, p. 193, 1895.

	Water	Glycerine	Methyl Alcohol	Ethyl Alcohol	Acetone	Isobutyl Alcohol	Amyl Alcohol	Ethyl Ether	Gelatine	Xylol	Toluol	Benzol
Magdala red			4	4			3				1	1
Eosine (sodium)	1	2	6	5			4		3			
Phenosafranine	1	6	7	9	11	9	10			4	3	2
Fluorescein lithium	2	3	5	4			1					
Chrysolin	2	3	3	3		3	1					
Chrysaniline			1	2			3					
Curcumin			1	2		3	4					
Aesculine	3	3	3	3	1	3	2					
B. Phenylnaphthylamin			5	5	3	5	4	2		1	1	1
Phenanthrene				1								2
Anthracene				4	3	5	4	4		5	5	5
Petroleum				5	4			3		6	6	6
Fluorescene			1	2			3					

The figures indicate the order of intensity, 10 indicating very strong fluorescence, 1 very feeble.

This table is of use in the preparation of fluorescent solutions, and shows us that some solvents are better adapted to certain substances, other solvents, however, to others.

Knoblauch explains the effects observed as due partly to the influence of the dielectric constant of the medium upon the intensity of the emitted radiation, and partly to the difference in the degrees of dissociation of the dissolved molecules. It is, however, difficult to bring any satisfactory theory to bear upon observations of this nature, for too little is known about the actual nature of fluorescence.

The color of the fluorescent light often varies with the nature of the solvent. Kaufmann and Beisswenger found that 3-aminophthalimid fluoresced green in water and alcohol, violet in benzol or ligroin. Dimethylnaphteurhodin is green in ligroin, yellow in pyridin, orange in benzylcyanid, and red-orange in methyl alcohol. No satisfactory theory of these changes has, as yet, been formulated.

Fluorescence of Vapors. — The molecular condition being much simpler in the case of gases and vapors than in that of liquids and solids, we should naturally expect that the most valuable data would come from the investigation of their behavior with respect to fluorescence. Unfortunately only a few vapors have been found up to the present time which exhibit the phenomenon. Of these sodium is by far the most interesting.

The fluorescence of vapors has been studied by Wiedemann and Schmidt (*Wied. Ann.*). Anthracene and a number of other organic compounds were vaporized in exhausted glass bulbs and illuminated with a concentrated beam of light. The experiment is not difficult to repeat. A pinch of anthracene is put in a bulb of glass 5 cms. in diameter, exhausted and sealed off from the pump. Light from an arc-lamp or the sun is concentrated by means of a large lens of

short focus, on a point at the centre of the bulb, which is then warmed as uniformly as possible by waving the flame of a Bunsen burner about it. A cone of deep violet fluorescent light will presently appear within the bulb. A similar phenomenon has been observed by the author in a bulb containing a small crystal of iodine. The fluorescence in this case can be observed at room temperature; warming the bulb does not increase the intensity of the fluorescent light. The fluorescence disappears as soon as the vapor becomes dense enough to exhibit much color. This matter will be considered more at length later on, in connection with the action of chemically inert gases upon the fluorescence. It has been found that electro-negative gases exert the most marked influence. The admission of air destroys the fluorescence, which only manifests itself in a high vacuum. It is worthy of remark that the magneto-rotatory power of absorbing vapors disappears almost completely in air at atmospheric pressure; and in the case of sodium vapor, at least, the same electrons seem to be responsible for the magnetic rotation and the fluorescence. We will now consider one of the most remarkable cases of fluorescence known, that of the vapor of metallic sodium, the study of which is throwing a great deal of light upon the mechanics of molecular radiation.

FIG. 348.

The Fluorescence of Sodium Vapor. — Wiedemann and Schmidt were the first to observe the fluorescence of this vapor. The metal was heated in exhausted glass bulbs and the spectrum of the fluorescent light examined with a spectroscope of low dispersion. The spectrum was found to consist of broad bands or flutings in the green, a continuous region in the red, and a hazy yellow band coinciding with the *D* lines. This spectrum was first photographed, by the author, in collaboration with Moore (*Astrophysical J.* and *Phil. Mag.*, 1903), with a concave grating of short focus, and appeared to be the complement of the absorption spectrum taken under similar conditions. The absorption spectrum of the vapor at different densities is shown in Fig. 342.

With vapor of moderate density (spectrum *d*) a channelled or

fluted absorption appears in the red, orange, and yellow, and another in the green-blue region, the yellow-green region being transmitted freely. The red-orange portion spreads over and below the *D* lines as the density increases, while the blue-green one pushes up in the other direction, the two eventually meeting at wave-length 5500, where a broad hazy band appears, which is just appearing in spectrum *b*, and is very distinct in *a* which was taken with very dense vapor in a red-hot tube. With a spectroscope of low dispersion we see merely a close double green line, and the extreme violet, everything else being cut off. The color of the transmitted light is as deep as, and about the color of, that transmitted by the densest cobalt glass. In the study of the fluorescence it was impossible to use glass bulbs owing to the rapid blackening of the glass due to the reduction of silica by the metal. Steel tubes were used, the ends of which were closed with glass plates cemented on with sealing-wax. On illuminating the vapor with monochromatic light furnished by a spectroscope it was found that when the vapor was excited by blue-violet light the yellow end only of the fluorescent spectrum manifested itself, together with an emission of light of the same color as the exciting light. As the wave-length of the exciting light was increased, *i.e.* changed gradually to green, the fluorescent region moved down the spectrum, so to speak: certain changes appeared to take place in the position of the bands, which indicated that the subject was worthy of a most careful investigation. The work was carried on by the author during the following year, the apparatus being gradually improved and the dispersion of the spectroscope increased. Very remarkable relations between the nature of the exciting light and the fluorescent light were found, which we will now consider.

The apparatus in its final form consisted of a seamless tube of thin steel 3 inches in diameter and 30 inches long, with a steel retort at its centre in which a large amount of sodium could be stored. The retort was made by fitting two circular disks of steel to a short piece of tubing, just large enough to slip snugly into the larger tube. The circular ends of the retort were provided with oval apertures as shown in Fig. 349 (Fig. 1). The retort was half filled with sodium, the molten metal being poured in through one of the apertures. It was then introduced into the tube and pushed down to the centre, after which the plate glass windows were cemented to the ends of the tube with sealing wax, as shown in the figure. This arrangement prevented the rapid diffusion of the vapor, and enabled a large supply of metal to be kept at the centre of the tube. The tubes used in the earlier work required recharging after two hours' continuous operation, while the retort tube could be operated for several hundred hours on a single charge.

If a steel tube cannot be procured, a large brass tube will answer the purpose. The retort is not necessary except for long-continued photographic study, the lump of sodium being placed at the centre of the tube. The conical beam of light can be thrown in in an oblique direction, and the fluorescence observed through the opposite end of the tube.

After exhausting the tube to a pressure of a millimeter or two a Bunsen burner is placed under the retort, and a cone of sun or arc light focussed on the oval aperture by means of a lens of long focus placed to one side of the tube, and a small piece of mirror-glass.

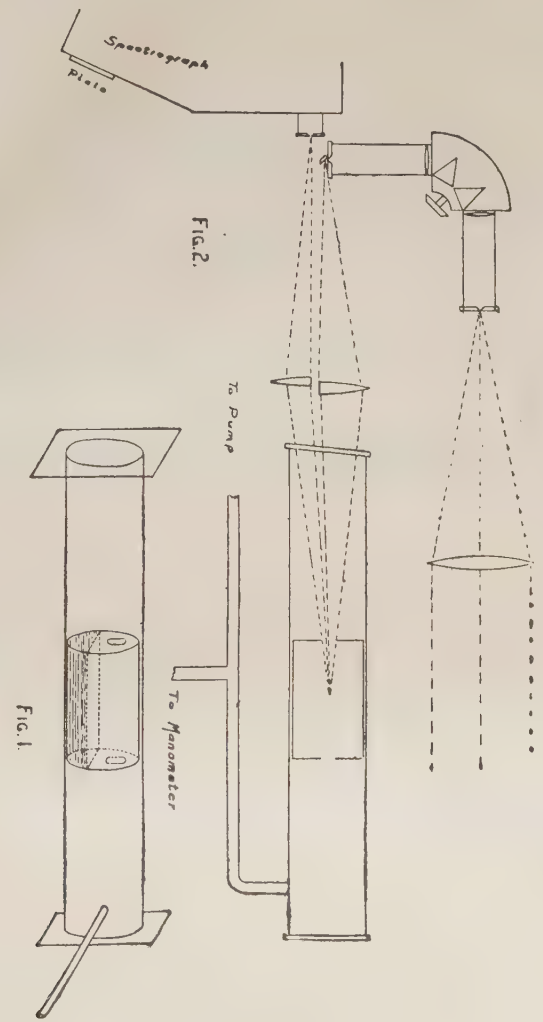
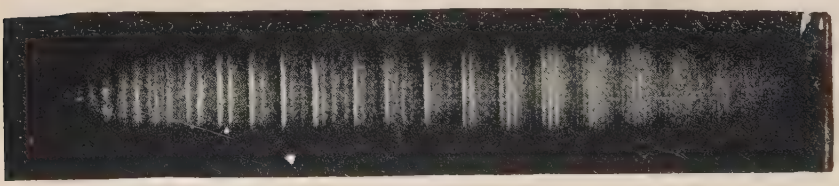


Fig. 349.

The arrangement of the apparatus used in the earlier work for exciting the vapor with monochromatic light is shown in Fig. 349.

As soon as the sodium vapor begins to form, a brilliant spot of green fluorescent light will be seen at the aperture of the retort.

The spectrum of the light is made up of an enormous number of fine lines, which in the yellow and yellow-green regions are arranged in groups or bands, which lie close together in the vicinity of the D lines, widening, however, as the blue region is approached. This spectrum is shown in Fig. 350 and in the colored frontispiece (Fig. 4). Coincident with the D lines there appears a hazy band (the surrounding region being nearly devoid of light), which, if the vapor is not too dense, can be resolved into a double line, the components of which coincide with D_1 and D_2. These lines appear, not only when the vapor is stimulated with light of the wave-length of the sodium lines, but also when a powerful beam of blue-green light is thrown into the vapor. This shows us that the mechanism which gives rise to the D lines is connected in some way with the one which produces the complicated channelled absorption spectrum. It has not yet been determined whether D_2 is emitted when the vapor is excited by the light of D_1.



Blue Green Yellow

Fig. 350.

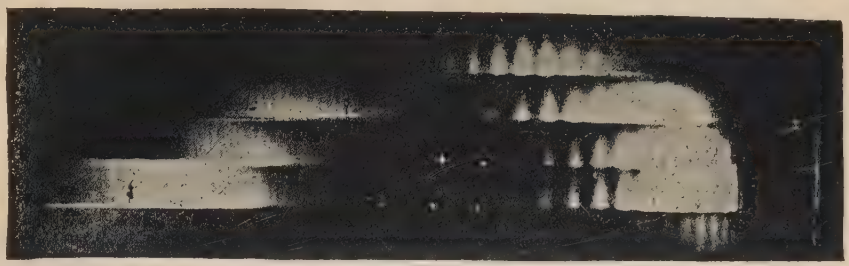
The rest of the fluorescent spectrum behaves in a very different manner. Stimulating the vapor with light of a deep violet color from the spectroscope produces no effect; as the wave-length is gradually increased a yellowish fluorescence appears, which spectroscopic examination shows to be made up of two parts, a reëmission of the same wave-lengths as those absorbed (blue) and the extreme yellow end of the fluorescent spectrum, comprised between wave-length 571 and 505. This portion consists of bands, which a high resolving power resolves into fine lines. As the wave-length of the exciting light is further increased, the point of maximum fluorescence moves down the spectrum, the first bands or groups of lines disappearing. In other words, as the exciting light moves up the spectrum, the fluorescent light moves down the spectrum. There is, however, in all cases an emission of light of the same wave-length as the exciting light. These relations will be better understood by referring to Fig. 351, where a number of photographed spectra are shown one above the other, and to the colored frontispiece (Fig. 8). The region of the spectrum excited is represented by the luminous band on the left-hand side of the spectra.

Stokes's law is violated in a most flagrant manner, especially when the exciting light is near the middle of the fluorescent spectrum. This is well shown in the lower spectrum (b) of Fig. 352, in which the range of the spectrum covered by the illuminating light is indicated

by the white band below the spectrum. The spectrum (*a*) was obtained by illuminating the vapor with white light.

The most remarkable phenomenon of all appears, however, when the slits of the monochromatic illuminator are narrowed to the width of a hair. The stimulating light now is nearly limited in range to the width of one of the sodium absorption lines. In other words,

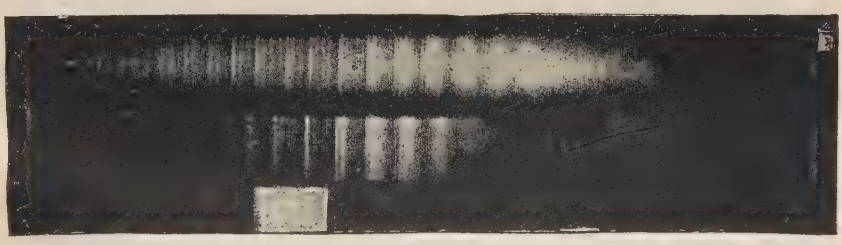
D lines.



a.
b.
c.
d.
e.

FIG. 351.

we take hold of, and shake, so to speak, but one of the many electrons which make up the molecule. The fluorescent spectrum is now very weak, and the eye must be carefully rested if the remarkable changes which accompany a change of wave-length of the exciting light are to be made out. It is at once apparent that the character of the spectrum is much altered, and as the wave-length of the exciting



a.
b.

FIG. 352.

light is slowly altered, the lines of the fluorescent spectrum appear to move about in the liveliest manner. The whole spectrum appears in motion, the luminous bands moving in a rippling manner, like moonlight on water. The motion is soon seen to be an illusion due to the continual disappearance and reappearance of the bright lines, the phenomenon reminding one forcibly of the scintillations produced on a zinc sulphide screen by the radium bombardment. This phenomenon led at once to the discovery of Resonance Spectra.

Resonance Spectra of Sodium Vapor. — When a vapor or gas, illuminated by a powerful beam of monochromatic light, reëmits this light without change of wave-length we may term the emitted

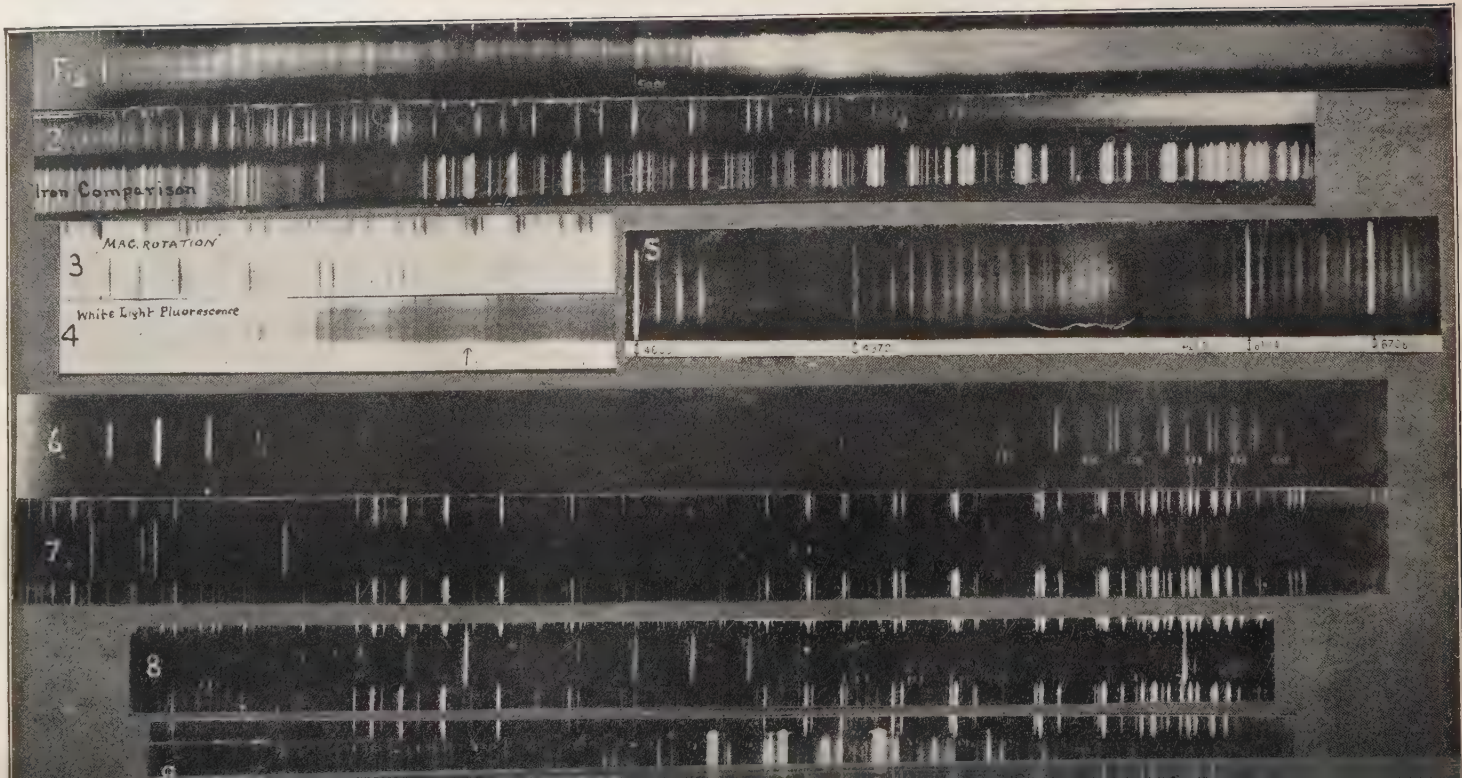
light *Resonance Radiation*. Sodium vapor illuminated by sodium light is an example, as has been shown by the author. Recent investigations still in progress have shown that mercury vapor at room temperature emits resonance radiation when excited by the ultra-violet light of wave-length 2536 emitted by the mercury arc. If, in addition, other monochromatic radiations are given out, we may term the spectra formed in this way *Resonance Spectra*. The best example is the vapor of sodium illuminated by the light of metallic arcs, previously decomposed by a spectroscope.

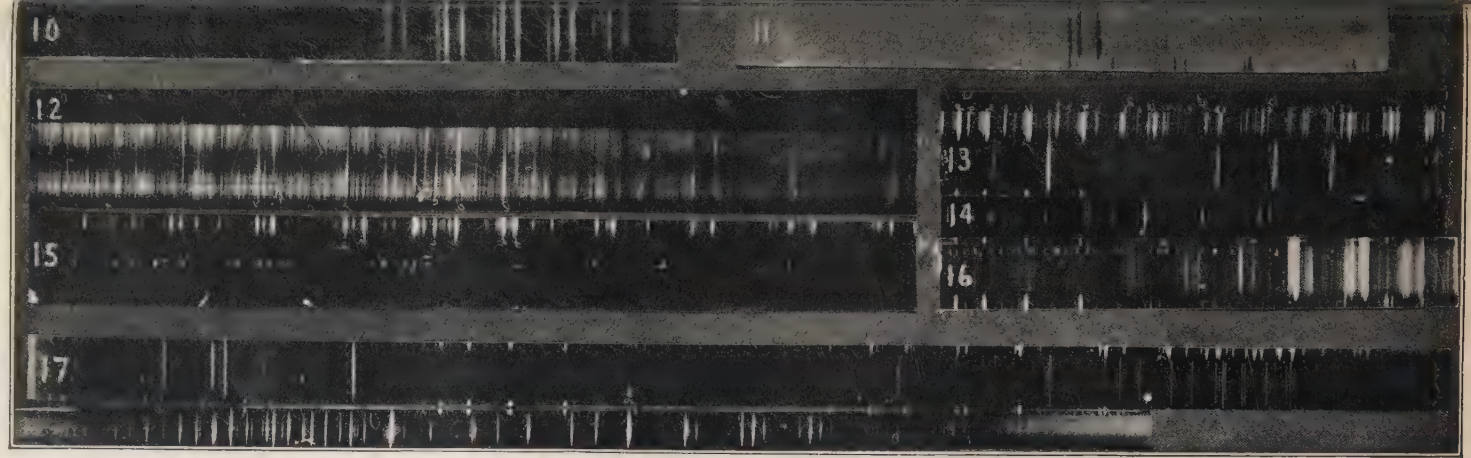
The vapor of sodium, obtained by heating the metal in a highly exhausted steel tube to a temperature of about 400 degrees, yields an absorption spectrum of great complexity. In addition to the *D* lines and the other lines (ultra-violet) of the principal series, which come out reversed, we find the entire visible spectrum, with the exception of a very narrow region in the yellow, filled with fine and sharp absorption lines. This we shall call the channelled absorption spectrum, and we find it divided into two distinct regions, one extending from wave-length 4500 to 5700, and the other extending from about 5800 to the extreme limit of the red. Photographs taken with the 21-foot grating in the second-order spectrum, show that it is even more complicated than was originally believed. We find, on the average, from 60 to 70 absorption-lines within a space only 12 Ångström units in width ; in other words, as many as 30 lines in a region not wider than the distance between the *D* lines. This means that in the blue-green channelled absorption spectrum, which is about 1200 A.E. in width, there are roughly speaking about 6000 absorption lines. Taken collectively these lines form themselves into a number of groups, which resemble the groups seen in the absorption spectra of iodine and bromine, and in certain banded emission spectra.

To attempt to unravel this spectrum, or find any regularities in it by the usual means, is quite out of the question, for the lines are so numerous and so close together that we could pick out series that would conform to any law that we might choose to invent.

As has been shown by the author, however, we possess a very beautiful experimental method of analyzing the spectrum, and of determining just which lines belong together ; a method, moreover, which may in time yield results which will enable the theoretical physicists to tell us the exact nature of the piece of machinery which we call the sodium molecule.

If we illuminate the vapor with a powerful beam of white light, it becomes strongly fluorescent, emitting a spectrum which is probably the exact complement of the absorption spectrum. This fluorescent spectrum is obviously of little help to us, for it is quite as complicated as the absorption spectrum. If, however, we throw monochromatic light into the vapor, instead of white light, we observe a very remarkable phenomenon. We now have a series of bright lines spaced at very nearly equal intervals along a normal spectrum, and separated by a distance equal to about 37 Ångström units. Various series of lines with varying distribution of intensity can be brought out by changing the wave-length


Fe
2
Iron Comparison
MAG. ROTATION
3
White Light Fluorescence
4
5
6
7
8
9



From paper by R. W. Wood and F. Hackett: Astro-Phys. J., 1909.

PLATE 7. — RESONANCE AND MAGNETIC ROTATION SPECTRA OF SODIUM VAPOR

1. Fluorescence excited by white light.
2. Magnetic rotation spectrum.
3 and 4. Small portion of same at $\lambda = 5000$.
5. Fluorescence excited by lithium arc.
6. Fluorescence excited by Bi arc.
7. Fluorescence excited by zinc arc.
8. Fluorescence excited by barium arc.
9. Fluorescence excited by copper arc.
10 and 11. Fluorescence excited by magnesium arc.
12. Positive and negative magnetic rotation in red.
15. Magnetic rotation spectrum in coincidence with above.
13. Barium arc fluorescence, 14. Magnetic rotation,
16. Copper arc fluorescence, all in coincidence.
17. Zinc arc fluorescence and magnetic rotation spectrum, in coincidence.

of the exciting light. In every case, light of the same wave-length as that of the exciting light is emitted by the vapor, and in addition a large number of other frequencies, which bear a definite relation to each other.

A very large amount of experimental work has been done already, and as much or more remains to be done. It is impossible to give in a limited space a very satisfactory account of the work, and the reader is referred to the original papers, which will be found in the *Philosophical Magazine* and *Astrophysical Journal* from 1906 to 1909.

The wave-lengths of the lines in the resonance spectra excited by the monochromatic radiations from a cadmium-vapor lamp have been determined to within one or two tenths of an Ångström unit. It was found that stimulation of the vapor with the radiation of wave-length 4800 caused it to emit a series of bright lines spaced at very nearly equal intervals along a normal spectrum.

Whether the lines are in reality spaced at equal distances is a matter of considerable importance, since theoretical treatments of the radiation emitted by electron systems disturbed in various ways can only be verified by the test of experiment. Professor Larmor has pointed out to me in a letter that a non-radiating system of electrons in steady orbital motion, when disturbed by the absorption of a radiation corresponding in frequency to the frequency of one of the electrons, should then emit radiations giving us a series of lines equally spaced along a normal spectrum. Excitation by the green line of cadmium gives a resonance spectrum consisting of a series of doublets, the components being a little closer together than the *D* lines.

The bismuth arc is by far the best exciter of the vapor which we have, and it furnishes the most typical resonance spectrum. It has but a single line which is operative, and this line is located in the remote blue. The resonance spectrum consists of a series in the blue, which with long exposures can be traced well up into the green, and a host of lines at the upper end of the spectrum in the yellow-green region. A photograph of the spectrum is shown on Plate 7, Fig. 6, and on the colored frontispiece, Fig. 5.

The resonance spectrum excited by the green triplet (*b* group) of the magnesium arc is very remarkable, for we find the triplet reproduced more or less perfectly at regular intervals. If the spacing of the individual lines forming the series was the same for each series, it is clear that we should have the triplet reproduced over and over again along the spectrum. That this is so only to an imperfect degree is due to the difference in the spacing of the lines of the different series. This spectrum is reproduced on Plate 7, Fig. 10. Fig. 7 shows the spectrum excited by the light from the zinc arc, Fig. 8 from the barium arc, and Fig. 9 from the copper arc.

The light of the lithium arc stimulates a most remarkable group of resonance spectra. It is the only means that has been found of exciting definite series of lines in the red region of the fluorescence spectrum. There are four exciting lines in all, but they are so widely separated that the resonance spectra excited by them only

overlap to a very slight degree. A photograph taken on a panchromatic plate of the complete spectrum excited by the lithium arc is reproduced on Plate 7, Fig. 5. The points of excitation are indicated by arrows. Its general appearance is shown in Fig. 7 of the colored frontispiece.

The blue line ($\lambda = 4603$) excites a resonance spectrum consisting of a group of nearly equidistant lines in the violet and another group in the yellow-green.

The blue-green line ($\lambda = 4972$) excites a remarkable series of lines midway between the groups. Taken as a whole, this plate is the most beautiful illustration of resonance spectra excited by monochromatic stimulations that has been obtained up to the present time.

A chart (Plate 8) has been made of the principal resonance spectra that have been obtained thus far, and a study of this will give an idea of the complexity of the phenomena. The positions of the exciting lines are indicated by arrows. Channelled spectra in the ultra-violet region, which accompany the other lines of the principal series, also exhibit fluorescence. The fluorescent light is of course invisible, which makes its detection difficult. The sodium tube was closed with a large quartz lens, and powerful fluorescence excited with a Heraeus zinc arc in quartz. The image of the green fluorescent spot was focussed upon the slit of a quartz spectrograph by means of a quartz lens. Under these conditions ultra-violet fluorescent light, if present, should appear in the spectrum. Such was found to be the case, as is shown by Fig. 353; the

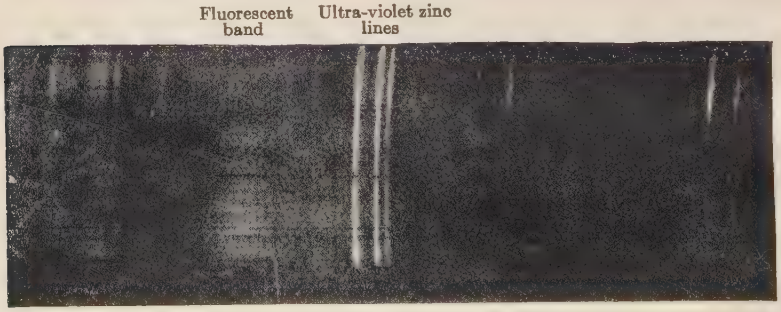


Fig. 353.

zinc arc spectrum is shown above, the fluorescent spectrum below. The banded fluorescent spectrum is strongest at some distance above the zinc lines (*i.e.* on the side of longer wave-lengths). The intervention of a glass plate between the arc and the sodium tube caused the disappearance of the ultra-violet fluorescent bands, showing that they were due to the absorption of ultra-violet and not of visible light.

An attempt was next made to see whether any mechanical connection existed between the mechanisms which produced the *D* lines and the visible channelled spectra (which have been shown

to be connected) and the ultra-violet lines. The zinc ultra-violet triplet was isolated by means of a quartz spectrograph and focussed on the aperture of the sodium chamber. The room was absolutely dark, and all extraneous light carefully screened off. No visible fluorescence was observed. The ultra-violet fluorescence was, however, photographed with a second quartz spectrograph. In this case one is working wholly in the dark, and the difficulties attending the adjustment of the two optical trains may well be imagined.

This experiment indicates absence of connection between the two systems.

The converse experiment was next tried. Powerful stimulation by the entire visible spectrum failed to show any trace of the ultra-violet band in the spectrum of the fluorescent light, even though the visible region was overexposed tenfold. This again indicates absence of connection.

It looks very much as if the different lines of the principal (Balmer) series and their accompanying channelled spectra may be considered as produced by different entities. The enormous increase in vapor density necessary to bring out the higher members of the series may perhaps be ascribed to the possible circumstance that the entities producing them are present in smaller numbers.

It appears to me that there are two hypotheses which we may make: First, that the Balmer lines and their accompanying spectra are caused by atoms which have lost one, two, three, four, etc., electrons. Secondly, that they are produced by aggregates or complexes of one, two, three, or more atoms. In either case it seems probable that the members would be present in continuously decreasing numbers.

These experiments show in a striking manner the complexity of the piece of machinery which we call the sodium molecule. Professor Rowland once said that a molecule is much more complicated than a piano. In most cases all that we have been able to do is to strike the entire keyboard at once, but in the case of sodium it seems possible to strike one key at a time. A study of the fluorescent spectra of other vapors will doubtless do much to clear up the mystery of the mechanism of molecular radiation.

Fluorescence of Mercury Vapor. — The fluorescence of mercury vapor was first observed by Hartley. An extended investigation of it has been made by the author.[1] It is best shown by enclosing a drop of the metal in an exhausted quartz bulb, heating the bulb over a Bunsen burner turned down very low, and passing a powerful spark between zinc or cadmium electrodes placed as close as possible to the bulb. The color of the fluorescent light is a bluish green mixed with a good deal of white, *i.e.* it embraces a region of the spectrum extending from the red-yellow well down into the ultra-violet. The color of the fluorescent light varies somewhat with the nature of the electrodes used. The spectrum appears continuous even with a concave grating of 2 metres radius.

[1] Wood, "The Absorption, Fluorescence, Magnetic Rotation, and Anomalous Dispersion of Mercury Vapor," *Phil. Mag.*, August 1909.

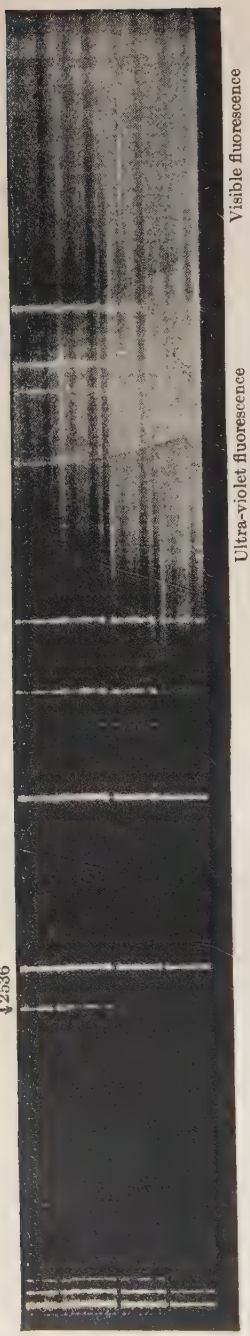


FIG. 354.

Photographed with a small quartz spectrograph, the spectrum is found to be continuous, extending roughly from the yellow down to wave-length 3000, with a very pronounced minimum at wave-length 3600. It was at first suspected that the visible band and the ultra-violet band might be excited by different radiations, and a week or more was spent in photographing the spectrum emitted by the vapor when stimulated by lines isolated from the spark-spectrum by an auxiliary quartz spectrograph. Very little, if any, difference could be detected, however. It was found that if air was in the bulb it was impossible to excite any fluorescence. If, however, we boil the mercury in an open flask of quartz it fluoresces brightly as soon as it is in brisk ebullition, and if the absorption spectrum is photographed at the moment at which fluorescence appears, we find that the absorption-band has contracted on its short wave-length side to its position when the vapor is *in vacuo*, due to the expulsion of the air by the boiling mercury. (See Chapter on Absorption.) In helium gas at 40 cms. we have strong fluorescence, however. The fluorescence spectrum is shown in Fig. 354, excited in this case by the light from the cadmium spark. The cadmium lines appear as well owing to diffused light. As will be seen, there is in addition to the continuous spectrum, a bright line (indicated by an arrow) which is not present in the spark spectrum, and which coincides in position with the sharp absorption-line 2536.7, shown by vapor of small density. It was at first thought that this line was excited by the bright cadmium line which falls within the region of the expanded part of the absorption-band. The zinc spark has a bright line which lies even nearer the mercury line, and it was expected that the emission line would be stronger in the case of zinc excitation. No trace of it was found, however, and its appearance was subsequently proved to result from excitation by the aluminum line at

wave-length 1860, or by a group of cadmium lines of wave-lengths shorter than any usually recorded in the spectrum of the metal.

This bright line appears strongest when the vapor density is such that visible fluorescence does not quite appear. As the temperature is raised, and the visible fluorescence grows stronger, the

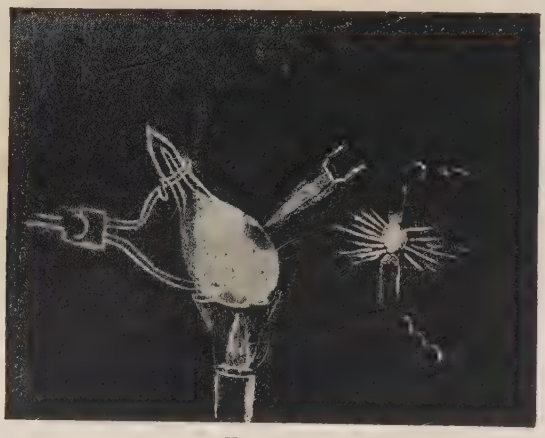


FIG. 355.

2536 line fades away as is shown in the figure, the upper spectrum having been taken at a low temperature, and the succeeding ones with increasing vapor density. The fluorescence of the vapor disappears if it is superheated. Whether this is due to dissociation or not has not as yet been ascertained. The experiment is a very pretty one, and is easily performed with the exhausted bulb previously described.

FIG. 356.

It is heated from below by a small flame from a Bunsen burner until the fluorescence excited by a cadmium spark placed as close as possible is at its brightest. The small pointed flame from a blowpipe is then directed against the side for a few seconds. The fluorescence promptly disappears at the superheated spot, reappearing again as soon as the local heating is stopped. See Fig. 355.

If the ultra-violet light of the spark is focussed at the centre of the bulb by a quartz lens of short focus, a narrow cone of brilliant green light extends from the point at which the light enters the bulb nearly to the opposite wall. The flame of a blast-lamp directed against the wall of the bulb at the point where the fluorescence begins causes the luminous cone to retreat from the wall to a distance of several millimetres, much as the positive column separates from the cathode in a vacuum-tube discharge. Photographs of the fluorescent cone of light in the bulb, with and without this superheating, are shown in Fig. 356. The destruction of the fluorescence by superheating is also shown in Fig. 9 of the frontispiece. For a further discussion of the spectrum of the fluorescent vapor the reader is referred to the original paper.

The Fluorescence of Iodine and Bromine Vapor. — The fluorescence of iodine vapor was first observed by Wiedemann and Schmidt, and subsequently studied by Konen, who found the luminosity too feeble to allow a study, either visual or photographic, to be made of the spectrum. The optical properties of the vapor have been investigated by the author.[1]

The fluorescence manifests itself to the best advantage when the vapor is formed in a high vacuum. A glass bulb from 10 to 30 cms. in diameter, and containing a few flakes of iodine, is highly exhausted on a mercurial pump and sealed. The fluorescence at once appears when a beam of sun- or arc-light is focussed at the centre of the bulb by means of a large condensing-lens, no heating of the bulb being necessary. The cone of fluorescent light is yellowish green and of great intensity, and can be shown to a large audience in a darkened room. Very recently (Dec. 1910) the author has discovered that the vapor gives resonance spectra when stimulated with monochromatic light, analogous in every respect to those found with sodium vapor. The green line of the mercury arc excites a resonance spectrum made up of 9 or 10 lines about 100 Ångström units apart, most of them being in the red and yellow. It will doubtless be possible to effect the analysis of the complicated band spectrum of iodine in the same manner as that of sodium, and study the behavior of resonance spectra in a magnetic field.

The destruction of the fluorescence by air or other gases seemed at first sight to be the result of collisions of the molecules (as first suggested by Lorentz), the phenomena requiring time for their development. The heating of a gas by absorption of radiant energy must in some way result from a transformation of the energy

[1] Wood, "Fluorescence, Magnetic Rotation, and Temperature Emission of Iodine," *Phil. Mag.*, October 1896.

stored in the molecule during absorption (which we may regard as potential) into kinetic energy at the moment of collision. That is, the velocity of the rebound is increased by actions or motions going on within the molecule. If we consider that the molecule is capable of storing energy up to a certain point, after which further absorption results in a reëmission of radiant energy (fluorescence), and if we further assume that at the moment of collision the stored energy is liberated, or spent in increasing the molecular velocity, it is clear, if the time interval between collisions is sufficiently small, that the internal energy of the molecule cannot be increased up to the point at which fluorescence manifests itself.

If we examine bromine, which shows a similar absorption spectrum, in the same way, we find no trace of fluorescence. We can explain this by assuming for the bromine molecule a greater capacity for storing energy. In other words the path cannot be increased sufficiently to allow the saturation point to be reached before a collision occurs.

Some, as yet unpublished, experiments by the author appear to be in accord with this theory. By sufficiently increasing the length of free path, we ought to be able to observe fluorescence, provided that a sufficient number of molecules remain to produce a visible illumination. A small amount of bromine vapor was introduced into a bulb, and condensed upon the wall by the application of solid carbon dioxide and ether. The bulb was then exhausted to the highest possible degree and sealed. On warming it to room temperature, the bromine vaporized, and though it was so highly rarefied that it showed no color, no fluorescence could be detected.

Sunlight was now concentrated at the centre of the bulb by means of a portrait lens having a ratio of focus to aperture of 2.3. Even in a dark room with careful screening off of diffused light no fluorescence could be detected. The outside of the bulb was now touched with a piece of solid carbon dioxide, which gradually condensed the bromine upon the wall. In two or three seconds a faint green fluorescence appeared, which vanished almost immediately, owing to the complete removal of the bromine vapor. There appears then to be one density at which bromine shows a visible fluorescence. At higher densities collisions destroy it, at lower there are too few molecules present. This appears to be in accord with our hypothesis regarding absorption of energy, saturation point, etc., a matter which will be more fully discussed presently. The spectrum of the fluorescent iodine was photographed with a large three-prism spectrograph. It resembles that of sodium vapor and is reproduced in Fig. 357, together with the absorption and magnetic rotation spectra, and the spectrum which the vapor emits when heated to a high temperature, which will be discussed in the Chapter on Radiation.

Destruction of Fluorescence by Other Gases. — A photometric study of the intensity of the fluorescence of iodine as influenced by the presence of various gases mixed with it has been made recently by the author. The results enable us to test any hypothe-

sis made regarding the action of the gas in destroying the fluorescence. The intrinsic intensity of the fluorescence, when the vapor, in a high vacuum, was illuminated by sunlight concentrated by a large F 2.3 Voigtlander portrait objective, was over one quarter that of a Welsbach mantle, the color match being made by a screen of light yellow glass combined with a very dilute solution of prussian blue.

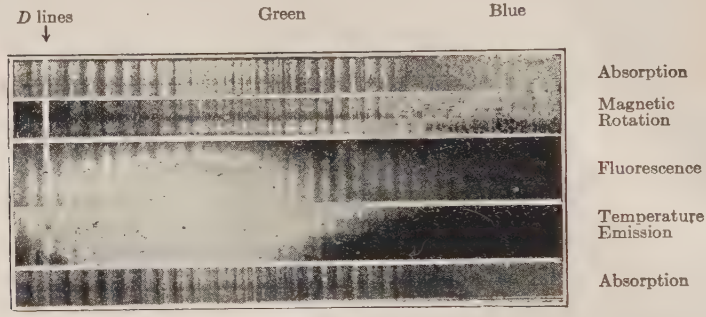


FIG. 357.

The reduction of the intensity caused by the admission of air and hydrogen is shown in the following table:

AIR + IODINE		HYDROGEN + IODINE	
Pressure	Intensity	Pressure	Intensity
0	44.7	0	45
.5 mm.	41.3	1.2	38
1	34.5	2.0	31
2	26.5	3	28
3	22.0	9	18
4	16.5	18	12
5.5	12.8	34	5.8
9	10.6	50	4.3
10	9.5	78	3.0
16	5.8	100	1.5
20	4.3		
40	2.4		
57	1.5		
88	0.7		

If we plot these results we find that the curve for air lies below the one for hydrogen. With CO_2 and ether vapor curves still lower down were obtained, showing that the molecular weight of the gas was operative in destroying the fluorescence as well as the pressure. The subject has been more fully investigated by the author in collaboration with J. Franck (Nov. 1910) who had found that the distances which electrons can travel in a gas are greatly reduced if traces of an electro-negative gas are present. Electro-negative gases, such as chlorine, oxygen, and to a lesser degree hydrogen, exerted very powerful effects in inhibiting the movements of the free electrons, and it appeared possible that they might also be effective in damping the vibrations of bound electrons. The photometric measurements commenced by the author were enlarged upon, the reduction in the intensity of the fluorescence due to helium, argon, oxygen, nitrogen, hydrogen and chlorine being determined. The results are shown graphically in

Fig. 357 *a*. Hydrogen and argon gave superposed curves, notwithstanding the fact that argon is much heavier. Helium, which is also heavier than hydrogen, had a much smaller effect upon the fluorescence. In helium gas, at a pressure of 2 or 3 cms., the fluorescence was distinctly reddish. This was found to be due to the remarkable fact that helium reduced the intensity of the green portion of the fluorescent spectrum to a greater degree than the red. Curves were drawn from observations made through red and green color filters. It is probable that helium reduces the intensity of the fluorescence solely as a result of the damping of the vibrations by collisions. The same may be true of argon. The action of the other gases is to be ascribed to their electro-negative character in part. Chlorine, the strongest electro-negative gas studied, at 2 mms. pressure, reduced the intensity as much as air at 2 cms. There thus appear to be two factors at work. These results explain the effects observed by Elston in the study of the fluorescence of anthracene vapor.[1]

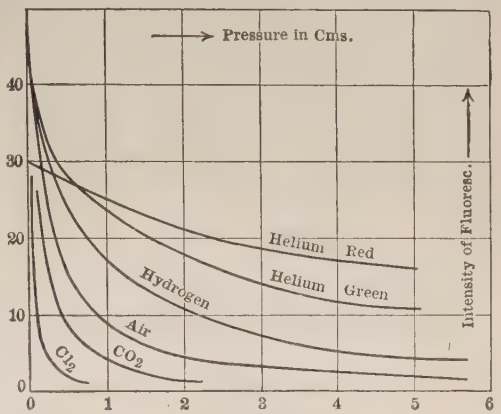


Fig. 357 *a*.

Emission of Polarized Light by Fluorescent Gases. — Until very recently all attempts to detect polarized light in the emission of fluorescent vapors have yielded negative results. Schmidt's observations were made with a Nicol only, and he failed to detect any difference in the brilliancy of the fluorescence as the prism was rotated.

The phenomenon was first detected by the author [2] in the case of potassium vapor by means of a Savart plate which is capable of showing two per cent of polarization. It was immediately picked up in the fluorescence of sodium and iodine, and it is probable that other vapors will be found to show it as well.

The apparatus used in the work consisted of a steel tube with a lateral branch brazed to its centre for the observation of the fluorescent light at an angle of 90° with the exciting beam. This tube was used in the earlier work upon fluorescence, before the expedient of "end-on" examination had been adopted. The metal was contained in a small retort also brazed to the large tube, immediately below the lateral observation-tube (Fig. 351). The light of the arc was focussed by a large lens at the centre of the large tube immediately in front of the lateral branch. Owing to the sensitiveness of the Savart plate it was necessary to eliminate all other

[1] *Astro-Phys. J.*, xxv, 3, p. 155.
[2] "On the Emission of Polarized Light by Fluorescent Gases," *Phil. Mag.*, July, 1908.

possible sources of polarization, such as fog or mist in the tube. Exhaustion with a Gaede mercurial pump while the tube was heated was sufficient to completely banish all trace of fog.

The complete absence of fog can be told by illuminating the vapor with an intense beam of light embracing the spectrum region comprised between wave-lengths 5400 and 5700, obtained by prismatic dispersion. Such a beam is incapable of exciting any fluorescence, and if fog is absent the cone of light is absolutely invisible when viewed through the lateral tube. The Savart fringes were very distinct, and the percentage of polarization was determined by compensating it with a pair of glass plates which could be turned about on a vertical axis furnished with a graduated circle. The first measurements were made with the exciting light polarized (electric vector) vertically.

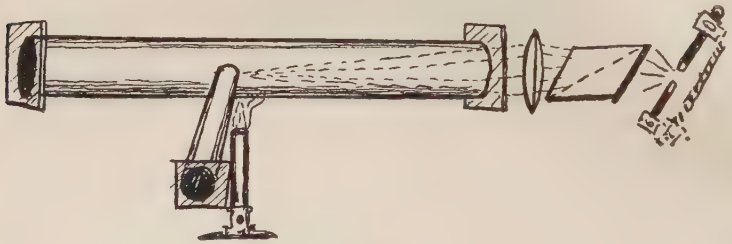


Fig. 358.

It was found that the polarization was strongest when the temperature of the tube was comparatively low, *i.e.* when the fluorescence first appeared. At a higher temperature, with very bright fluorescence, compensation was secured with a single plate at an angle of 59°, or with two plates at an angle of 47°. Taking the refractive index of the glass as 1.52, and making use of the formula which expresses the ratio of the amplitudes of the components of the vibration in and perpendicular to the plane of incidence,

$$\frac{D_s}{D_p} = \frac{E_s}{E_p} \cos(\Phi - \chi),$$

we find the percentage of polarization to be 20.

At the lowest temperature consistent with a fairly bright fluorescence compensation was secured with two plates at an angle of 52°. This gives us 30 per cent of polarized light. The change in the intensity of the fluorescent spot when viewed through a slowly revolving Nicol was easily observed in this case, and a separate determination was made by Cornu's method, which gave 28 per cent of polarized light.

If the exciting beam was polarized with its electric vector horizontal, no trace of the Savart fringes could be detected, which was to be expected, as everything was then symmetrical about the line of sight.

If the exciting light was unpolarized to start with, the fluorescent

light was found to be polarized, but not to the same extent, the percentage in each case being one-half of its former value, which was also to be expected. The question at once arose as to why we have only a partial polarization of the fluorescent light when we start with plane-polarized light.

Excitation of the vapor with monochromatic light causes it to emit this same wave-length and a large number of other wave-lengths, the spectrum exhibiting in the most typical case a number of very sharp lines spaced at nearly equal intervals along a normal spectrum. It was important to see whether the polarization was confined to the line corresponding to the exciting line, for it seemed not impossible that the electron vibrating in synchronism with the exciting light might emit light which was completely plane-polarized, whereas the other disturbed electrons might emit unpolarized light. The vapor was accordingly excited with the light of the cadmium arc, and the spectrum examined with a small spectroscope furnished with a Nicol prism. It was found, however, that the polarization was quite as strong in all parts of the spectrum as it was at the lines coinciding in position with the exciting lines. As a check on this observation the following experiment was then tried. The light from the arc was passed through a filter which removed everything above wave-length 5000, *i.e.* which transmitted a deep blue light. In front of the Savart polariscope a dense screen of aurantia was placed which cut off everything below 5000. This removed all the wave-lengths from the fluorescent spectrum which were to be found in the exciting light, transmitting, however, the upper end of the fluorescent spectrum. This light was found to be strongly polarized, the fringes appearing quite as distinct as in the absence of the screens. It will be observed that this experiment completely eliminates all possible sources of error, such as polarization produced by fog or by reflection from the back wall of the tube.

The *D* lines appear in the fluorescent spectrum stimulated by white light, and the spectroscope showed that they were polarized to about the same degree as the rest of the spectrum. As I have shown, the *D* lines can also be caused to appear by stimulation with blue light, but in this case they are too faint to make a study of their polarization possible.

The foregoing experiments prove that we must seek elsewhere for the cause of the partial polarization.

That there is a depolarizing agent is probable from the most elementary theory. If we assume the electrons to be free to move in any direction, under the influence of the electric forces in the light-waves, it seems probable that with a polarized beam of exciting light, we should have plane-polarized light emitted by the vapor in directions making an angle of 90° with the exciting beam. If the stimulating light were plane-polarized to start with, there would be no fluorescent emission at 90° in directions parallel to the electric vector in the exciting light. In other words, the vapor would behave in much the same way as a fog composed of very fine particles, though the physical processes involved would be quite different in the two cases.

Let us now assume that the electron is able to move along a straight line, the position of which is fixed within the molecule. While it is quite improbable that this condition actually holds, the conception of it will enable us to see how the introduction of constrained motion will decrease the amount of polarized light emitted by the vapor. We can imagine as a rough model of our molecule a hollow sphere with a wire passing through its centre, along which a bead can slide. Consider the polarized exciting waves as travelling along the x axis towards the origin (Fig. 359), with their direction of vibration parallel to the z axis. It is clear that the bead will vibrate with the greatest amplitude when the direction of the wire is also parallel to the z axis. If the wire lies in the xy plane no vibration will occur. In other positions there will be vibrations of greater or less amplitude, according to the component of the force in the light which is directed along the wire. We may represent the fluorescent gas by an enormous number of these vibrators with their wires pointing in all possible directions. The vibrations of the beads along the wires can each be resolved into two components, one parallel to the x axis, the other parallel to the z axis, and the integrated effects of these components, or rather of their squares, will give us the measure of the intensity of the emitted radiation vibrating parallel to the x and z axes. These integrals stand in the ratio of 6 to 2, which means, since the total intensity is 8, and their difference is 4, that we are to expect a polarization of 50 per cent with a vibration direction parallel to the z axis. By experiment we find only 30 per cent.

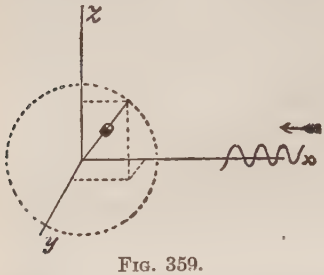


FIG. 359.

As the percentage of polarization appears to decrease as the temperature of the tube is raised, it is possible that at lower temperatures than any which can be employed satisfactorily the percentage may be higher. As to possible depolarizing agencies, rotation of the molecule would doubtless act in this way if the electrons continued to emit light after the cessation of the incident light, or — expressing it in terms of our imaginary model — if the bead on the wire continued to vibrate after the wire turned into the zy plane. It must be remembered, however, that sodium is regarded as monatomic, and the question arises as to whether we can ascribe much of the energy of the gas to rotation of the molecule. Another possible depolarizing factor is what may be termed secondary fluorescence, or fluorescence excited, not by the primary ray, but by the resonance radiation of the gas. Lord Rayleigh has considered the possibility of an analogous action in his treatment of the theory of the color of the sky.

Resonance Spectrum of Iodine Vapor in Helium : Effect of Molecular Collisions. — A very remarkable effect of molecular collisions upon the resonance spectrum of iodine was discovered by the author, working in collaboration with J. Frank (Dec. 1910). The

fluorescence spectrum of the vapor, when stimulated with white light, resembles closely that emitted by sodium vapor under the same conditions. If we stimulate it with the monochromatic green light of the mercury arc, we get a series of 15 bright narrow lines about 70 Ångström units apart, one line for each band of the channelled spectrum. If, however, we have helium gas, which has been shown to affect the fluorescence *not* by electro-negative qualities, present at a pressure of only 3 mms., a most remarkable change in the resonance spectrum is found. The sharp equidistant lines have faded to about $\frac{1}{10}$ of their original intensity and the complete banded spectrum, consisting of innumerable fine lines, appears. The total amount of light emitted is about the same in the two cases. The collisions with the helium molecules appear to couple the electron systems together, so to speak, so that monochromatic stimulation produces a fluorescence spectrum very similar to that excited by white light. Chlorine gas, which affects the fluorescence chiefly by its electro-negative qualities, as has been shown, reduces the intensity of the resonance spectrum *without causing the appearance of the banded spectrum.* These two different effects are very important, and the question should be investigated from a theoretical standpoint.

We have now to consider cases in which stable rearrangements of the molecules or atoms are produced by the action of light, *i.e.* a transformation of light into molecular or atomic energy. Two classes of phenomena come under this head. We may have an element transformed into some allotropic modification, or we may have a chemical compound decomposed into its constituents, or a chemical compound formed from its elements.

Molecular Changes. — We will first consider a few cases in which light produces a specific action upon elementary bodies. Ordinary white phosphorus is transformed into the red modification by the action of light, while sulphur is changed into the insoluble variety.

Ultra-violet light, if sufficiently intense, changes oxygen into ozone. The new mercury vapor lamps, in quartz tubes, emit the short waves in such profusion that the odor of ozone is as noticeable as during the operation of a large static machine.

Amorphous selenium, which is fairly transparent, is a non-conductor of electricity, and under the action of light passes over into what is usually termed the metallic modification, which is opaque and conducts electricity. Still more remarkable is the fact, discovered in 1872 by May, that the metallic form conducts better when illuminated than when in the dark.

Chemical Changes. — Examples of chemical decomposition are very numerous. Peroxide of hydrogen is rapidly decomposed, by exposure to light, into water and oxygen, the recombination of which, or the " burning " of the water if it could be accomplished, would liberate in the form of heat the chemical energy into which the light has been transformed. Chloride of nitrogen decomposes explosively when illuminated.

Many salts of silver, gold, iron, platinum, uranium, etc., are decomposed by the action of light, these actions being at the bottom

of all photographic processes. In certain cases the decomposition may be only started, a molecular instability being imparted by the action of light, the continuation of the process being effected by reducing agents (developers). Another interesting example of photo-chemical decomposition has been pointed out by Tyndall. The vapor of amyl nitrite in a glass flask is colorless and transparent until illuminated by a powerful beam of sun- or arc-light, when a dense white cloud at once forms, the products of decomposition condensing to liquid drops which scatter the light.

A remarkable transformation of luminous into chemical energy on a vast scale is the breaking up of the carbonic acid of the atmosphere, which takes place in the leaves of plants. Some of the oxygen is liberated in the free state, and some enters into the organic compounds which the plant forms, by the subsequent combustion of which we may recover the original energy of the light in the forms of both heat and light.

If ultra-violet rays are greatly in excess other very remarkable transformations are produced. Oxygen is changed into ozone, as we have seen. Permanganate of potash solutions are almost instantly bleached, and many other chemical substances break down, which under ordinary conditions are stable.

Chemical Combination. — Examples of chemical combination resulting from the action of light are not as common. The best-known case is the union of chlorine and hydrogen to form hydrochloric acid. The mixture of the two gases is best obtained by the decomposition of strong hydrochloric acid by electricity, the operation being conducted by feeble lamplight. The gas which comes off first contains an excess of hydrogen, owing to a solution of a part of the chlorine in the acid. Subsequent portions may be collected in small glass bulbs, blown in strings and separated by fine thin-walled tubes, which are broken and closed with warm sealing wax, care being taken not to have the wax on fire. The bulbs should be kept in a dark box until wanted. On exposing one to sunlight or the light of burning magnesium, or the electric arc, a violent explosion is said to occur. Precautions should of course be taken to avoid injury from the flying fragments of glass.

Transformation of Luminous into Electrical Energy. — Cases of this sort are worthy of mention, though in none of them is the transformation direct. In the thermopile, when illuminated by light, we have a liberation of electrical energy, which is, however, due only to the heating action of the radiation. In the photo-electric cell, which consists of two silver plates coated with silver chloride and immersed in dilute acid, a feeble current flows through a wire joining them, when one of them is illuminated by light. In this case the action of the light is primarily chemical, the current being the result of the chemical decomposition. The case is somewhat analogous to phosphorescence, the difference being that here the reversion of the process set up by the light liberates electricity instead of light.

CHAPTER XXI

LAWS OF RADIATION

In the present chapter we shall discuss the laws which govern the emission of light by bodies in virtue of their temperature. Of the physical processes which are at work we know but little. In the heated body we believe that the molecules are in a rapid vibratory motion, which increases in violence as the temperature is raised, but the precise mechanism by which this energy is transformed into radiant energy is but imperfectly understood. The simplest assumption appears to be that the molecular collisions in some way throw the electrons into vibration, or increase the amplitude of their vibrations, and that these radiate energy into space. The same thing does not, however, occur in the case of a gas, for the same molecules can be heated to a much higher temperature — several thousand degrees even — without emitting light. Mercury may be heated " white-hot " in a sealed quartz tube, but mercury vapor can be heated to the highest temperatures at our command without emitting any visible light. It would be extremely interesting to study the emission of light by some fluid which had a critical temperature of about 700°, observing the luminosity as the liquid passed over into the gaseous state. The phenomenon might be studied in a quartz tube (which emits but little light, for reasons which will appear presently) if a suitable substance could be found. Strutt has experimented with mercury with a view of measuring the electrical conductivity of the vapor at the critical temperature, but even thick-walled capillaries of fused quartz exploded, or yielded to the pressure, before any evidence of the approach of the critical state appeared.

Nothing is to be gained by increasing the thickness of the walls beyond a certain point, since the inner layers give way or " tear " before the strain reaches the outer layer. Possibly the thing could be accomplished by enclosing the tube in an outer tube containing air at a pressure of several hundred atmospheres. This principle is the one used in the wire-wound guns.

The electrons of the gas molecules can be made to emit light by the stimulus of electrical discharges, or chemical changes, and in one or two cases (iodine and sodium) by virtue of temperature alone.

All substances in the *solid or liquid* state emit light as soon as their temperature is raised above 500° C. The intensity of the light varies, however, with the nature of the substance, being greatest for substances which absorb light strongly. A *perfectly* transparent solid or liquid would not emit light even at the highest temperature. No such substance is known, however, though a bead of microcosmic

salt, heated in a loop of platinum wire by means of a blast lamp, comes pretty near to fulfilling the required conditions. We shall presently investigate the laws which govern the emission of light by various substances, and in particular the emission by a substance which is perfectly black, *i.e.* perfectly absorbing. No substance has this property, though by an experimental artifice we can produce a radiator which will give out radiation identical in every respect with the radiation which would be given out by a perfectly black body at the same temperature.

The Relation between Emission and Absorption. — We will now investigate the very intimate relation which exists between the emission of heat or light waves by a substance, and its power of absorbing the same waves.

That some relation existed between the emissive and absorbing power in the case of radiant heat was indicated by the experiments of Leslie, Melloni, Provostaye and Desains, and others. Their measurements were, however, made for the most part with apparatus of insufficient sensitiveness, the spectrum regions being only roughly determined by means of absorbing screens. It was determined, however, without question, that bodies which possessed a strong emissivity acted also as powerful absorbers of the radiant heat which they emitted, and the approximate equality of the emitting and absorbing powers was recognized.

Ritchie's Experiment. — This relation was shown by a very simple and ingenious experiment devised by Ritchie (*Pogg. Ann.*, 28, p. 378, 1833). Two air-tight metal chambers were connected by a glass tube containing a drop of fluid, the whole forming an air thermometer. Between them a third metal chamber of the same size was mounted, which could be heated by filling it with boiling water. One surface of this heat radiator was covered with lampblack, the other with the substance under investigation, for example powdered cinnabar. (See Fig. 360.) The surface of the air thermometer which faced the radiating lampblack surface was coated with cinnabar, while the surface which faced the cinnabar radiator was coated with lampblack. With the apparatus arranged in this way no movement of the fluid drop occurred when boiling water was poured into the radiator, which established the fact that the emitting and absorbing powers were equal. Let us assume that the lampblack radiates powerfully, the cinnabar feebly. The powerful radiations coming from the former are but slightly absorbed by the latter, while the feeble radiations from the latter are strongly absorbed by the former, the heating of the two chambers of the thermometer being equal. Calling E the amount of heat emitted by the cinnabar and A its absorbing power, e and 1 the corresponding expressions for the lampblack, the lampblack surface emits an amount of radiant heat e, of which the cinnabar surface absorbs the amount eA. The cinnabar surface emits an

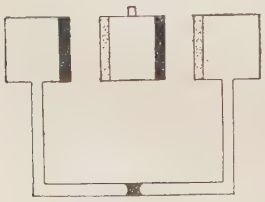


Fig. 360.

amount E, which the lampblack completely absorbs (since its absorbing power = 1). The equality of temperature indicated by the thermometer shows us that $eA = E$ or $\frac{E}{e} = A$. Now $\frac{E}{e}$ is the ratio of the amount of heat emitted by cinnabar to the amount emitted by lampblack at the same temperature. This ratio we will call the emissivity of the cinnabar, and our equation shows us that it is equal to the absorbing power.

Kirchhoff's Law. — This relation was reduced to a more definite form in 1859 by Kirchhoff, and independently by Balfour Stewart, who showed that it must be true for each wave-length in the emitted spectrum, and formulated the law which has since gone by his name. "*At a given temperature the ratio between the emissive and absorptive power for a given wave-length is the same for all bodies.*" The theoretical considerations from which Kirchhoff's law is deduced will be discussed later on in the chapter. It may be remarked that we frequently meet with the statement that the absorption of light by flames which contain the vapors of metals is a necessary consequence of this law. This is by no means the case, for Kirchhoff's law is only to be applied to radiation which results from temperature. In the case of the emission by flames, the phenomena are probably connected with chemical changes which are taking place.

Cotton has called attention to the fact that there are two distinct relations which are almost invariably confused: a qualitative rule, which connects the absorption and emission for a given substance, and a quantitative rule, which establishes relations between different bodies. From the former we can only draw the conclusion that if a body emits certain radiations it absorbs them when they come from without. It may, however, absorb other radiations, as is illustrated by the selective absorption of colored substances. For these radiations $\frac{E}{A} = 0$, since at the temperature in question there is no emission of visible radiations.

Absorption by Flames. — The absorption by flames and the reversal of spectral lines are special cases of the qualitative rule. These cases have been studied by Gouy, who sought to determine by experiment whether flames were transparent to the radiations which they emitted. His method consisted in comparing photometrically the light of different thicknesses of radiating gas. If no absorption occurred, doubling the thickness should double the intensity of the illumination. He found, in the case of every line examined, that after the line had attained a certain brightness absorption manifested itself. If E and A be the emissive and absorptive powers of unit thickness of the flame and we increase the thickness, we shall find that the emissive power (*i.e.* the intensity sent out by the thick layer) approaches $\frac{E}{A}$ as a limit.

This can be readily seen by calculating the amount of light sent out in the direction B by a flame of thickness 6 (Fig. 361).

Assume unit thickness to emit light of intensity 100 ($E = 100$) and to have the power of absorbing one-half, of the light of the same wave-length, which traverses it. The intensity of the light from the element 1 is reduced to 50 by the absorption of element 2, and on emergence from element 6 has an intensity of only 3.37. The total amount of light emerging from element 6 is obviously the sum of the numbers in the different elements, which in this case is 197.62. If the number of elements were increased indefinitely this sum would be 200, the limiting intensity which is equal to $\frac{E}{A}$, as defined above.

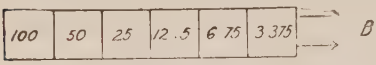


Fig. 361.

It should be possible therefore to reverse all lines by a sufficient increase in the thickness of the flame, for the line will appear darker than the background of the continuous spectrum when the intensity of the light of the given wave-length emitted by the flame is less than the intensity of the continuous spectrum at the point in question. By increasing the thickness we progressively increase the absorption, while the intensity of the flame remains practically constant after a certain thickness is reached.

It by no means follows, however, that all vapors which emit radiations of definite wave-length will show the corresponding wave-lengths reversed, even when a very thick layer is used. The above argument is based on the fact that there is a certain absorbing power. If $A = 0$ reversal can never occur. Now it is a noteworthy fact that reversals are only obtained in the case of the bright lines in metallic spectra. The lines in the spectra of the metalloids have never been reversed. This shows that Kirchhoff's law does not even apply qualitatively in these cases; in other words, there is powerful selective emission without any corresponding selective absorption. In such cases $\frac{A}{E}$ is infinite. Summing up, we find that for colored substances, absorbing without a corresponding emission, $\frac{E}{A} = 0$, for flames which show reversal of spectrum lines $\frac{E}{A}$ is finite, for luminous gases which show no absorption $\frac{E}{A} = \infty$.

Temperature Radiation of Gases. — Kirchhoff's law states that in the case of radiation which results solely from temperature, $\frac{E}{A} = \text{const.}$ for all bodies at the same temperature. The value of the constant is a function both of the temperature and the wave-length, and is equal to the emissive power of a perfectly absorbing body, that is, a body which at the temperature in question completely absorbs, without reflection, all radiation falling up, no matter what its wave-length. This amounts to saying that at a given temperature no substance can emit more light of a given wave-length than a perfectly black body. Paschen has compared the intensity of

the light of the two D lines in the sodium flame with the total intensity of a region, completely enclosing the D lines, in the continuous spectrum of a black substance heated in the same flame. The total intensity of the D radiation was more than twice as great as that of the region of the continuous spectrum which enclosed them, from which the inference can be drawn that something other than temperature is concerned with the emission of light by the sodium flame. The same thing was found by Kayser and Paschen in the case of the ultra-violet bands of the arc, which were much brighter than a corresponding region of the spectrum of the positive crater, notwithstanding the fact that the temperature of the latter is higher than that of the arc proper.

The experiments of Pflüger and of Ladenburg on the emission and absorption of ionized hydrogen in vacuum tubes described in the chapter on absorption have been considered by Ladenburg to prove that the quotient E/A is not a constant, and that the emission is therefore not a true temperature effect. Pflüger, however (*Verh. der d. phys. Ges.*, 12, 208, 1910), and Konen (*Phys. Zeit.*, August 1910) claim that this is by no means the case. The source of light in these experiments did not emit a continuous spectrum, for there were strong maxima at the positions occupied by the hydrogen lines. Konen shows that all of the double reversal effects, and the behavior of the absorption lines, when the intensity of the source is diminished by rotating one of the Nicols, can be explained, if we have emission and absorption curves of certain types. His treatment is too long to be included here, and the reader is referred to the original paper. Ladenburg has apparently defended his position successfully, however, and in the opinion of the author it seems very improbable that the emission of hydrogen is a pure temperature effect. The question cannot, however, be regarded as quite settled at the present time.

There are certain cases in which we have a true temperature emission of a gas. Carbon dioxide, when heated, emits an infra-red radiation, the spectrum showing a very sharp band at $\lambda = 4.3\,\mu$. Paschen (*Wied. Ann.*, 51, p. 1, 1894) found that a layer of the gas 7 cms. thick emitted and absorbed as strongly as a layer 33 cms. thick.[1] This indicated that the radiation from a 7 cm. layer could be regarded as the equivalent of the radiation from a layer of infinite thickness, or in other words, the radiation of a black substance at the same temperature. By heating the gas in a tube, and measuring the intensity of the emitted radiation with a spectro-bolometer, Paschen found that for all temperatures between 150° and 500° the intensity of the radiation of wave-length $5.12\,\mu$ was only a little below that of a black body at the same temperature. As a black body he used a smoked strip of platinum heated by an electric current. The proof of the law follows from the fact that for a layer

[1] This holds, however, only for the absorption of a narrow range of wave-lengths. Rubens and E. Ladenburg (*Verh. der Deutsch. Phys. Ges.*, 1905) have made a study of the absorption of CO_2 for the long waves isolated by a prism of sylvite, and also for the total radiation. In the latter case the absorption increased with the logarithm of the thickness of the gas layer. In the case of homogeneous radiation the logarithm of the absorption is proportional to the thickness.

from which the radiation is the equivalent of that from an infinitely thick one, we have the relation $\frac{E}{A} = e$, the emissivity of a black body, as we can at once see by comparing the equation $eA = E$, given at the beginning of the chapter with the relation previously deduced, namely, that as the thickness of an emitting absorbing layer increases, the intensity of the radiation leaving it approaches the value $\frac{E}{A}$ as a limit. An interesting conclusion has been drawn by Kayser from Paschen's experiment. The thickness of the layer necessary in order that the radiation may equal that of a black body will vary for the different lines in the spectrum, the greatest thickness being necessary for the wave-lengths for which the absorption is a minimum. If we employed a layer so thick that the conditions were realized for all of the lines, and plotted the intensities of the lines as ordinates, and the wave-lengths as abscissae, the curve joining the points should be the emission curve of a black body at the same temperature as shown in Fig. 362.

Fig. 362.

We may perhaps apply the same reasoning to the case of the bright lines of flames. The sodium flame, for example, exhibits in addition to the D lines a very faint line in the green. By increasing the thickness of the flame the ratio of the intensity of the D lines to that of the green line should become less. Multiplying the flame by repeated reflections between two parallel mirrors amounts to the same thing as increasing its thickness, and Wanner (*Wied. Ann.*, 68, p. 143) found that a sodium flame placed between two concave silvered mirrors showed the green line with great distinctness. The D lines were not increased in brilliancy in anything like the same proportion, and appeared much broadened, a faint continuous spectrum appearing in addition. Attempts made by the author to repeat this experiment both by the use of mirrors and a sodium flame, over a metre in length, gave negative results.

The most complete investigation of the subject of the relation between the emission and the absorption of gases has been made by H. Schmidt (*Ann. der Phys.*, 29, 1909). He studied the absorption and emission of the flame of a Bunsen burner, measured the temperature of the flame most carefully with a thermo-element, and found that the laws of black-body radiation could be applied to it, at least within the region of the strong absorption and emission bands at 2.8 and 4.3 μ. By comparing the radiation of the flame with that emitted by a black body at a temperature of 980°, Schmidt was able to calculate the temperature of the flame; the value found was 1670°, in close agreement with the observed value of 1640°.

Very few cases are known in which visible radiations can be obtained by merely heating a gas or vapor. An immense amount of work has been done by Pringsheim and others in endeavors to obtain a luminous emission from gases as a result of high temperature alone. Efforts in this direction have been, almost without exception, in vain, and Pringsheim came to the conclusion that, at least for temperatures which could be commanded in the laboratory, gases remained dark.

There are, however, one or two exceptions to this rule, which were studied by Salet and Evershed. Iodine vapor when heated to a temperature of only six or seven hundred degrees gives off a reddish orange light. The experiment is easily performed by arranging a small spiral of platinum wire, which can be heated by a current, in a test-tube in which a little iodine is vaporized by means of a Bunsen burner. An orange-colored flame is seen to rise from the hot wire. A still better method is to enclose a few crystals of iodine in an exhausted bulb of fused quartz. The body of the bulb is heated to a high temperature by the flame of a blast lamp, and the iodine in the narrow neck then vaporized by the application of a small flame as shown in Fig. 363. The quartz bulb emits little or no light until the vapor enters it, when it immediately glows with an orange-red light. A similar phenomenon has been observed in the case of sodium vapor, but is not as easily reproduced as the iodine emission. The best arrangement is a long steel tube containing metallic sodium, and highly exhausted, heated by a row of burners, or better in one of the electrical ovens made by Heraeus of Hanau. Evershed was of the opinion that the spectrum was continuous, but by employing a vapor of small density Konen (*Wied. Ann.*, 65, p. 256) succeeded in resolving it into bands, which corresponded to the bands seen in the absorption spectrum. It is not difficult to see how a continuous spectrum might easily result when a thicker layer or a denser vapor was used, for we have only to apply the principles involved in the case suggested by Kayser to a spectrum consisting of bands, the intensity having a finite, though different value, for each wave-length.

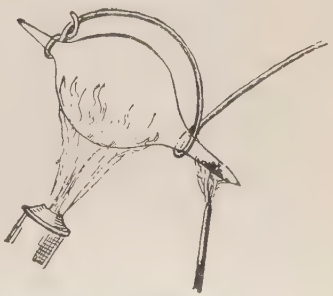


Fig. 363.

Temperature Radiation of Solids and Liquids. — The radiation of solids and liquids is especially adapted to the proof of Kirchhoff's law, since in these cases we can be sure that it is the result of temperature alone.

That the emission of light by heated substances is proportional to the absorption can be easily shown by heating a fragment of a piece of decorated china in a blast-lamp. The design emits much more light than the white background, owing to its stronger absorbing power.

The more powerfully a body absorbs, the more powerfully will

it emit when heated, this relation holding for every individual wave-length. Black bodies then give out the most light when heated. The fact that a white block of lime is far more luminous than a carbon rod when heated in the oxy-hydrogen flame is not usually cited in support of this law, while the fact that the most luminous body of all, the Welsbach mantle, is also quite white is equally unsatisfactory as an illustration, for white bodies are in reality transparent, that is, they are made up of masses of small transparent particles, and transparent bodies ought not to emit at all. It is of course necessary to define just what we mean by transparency in this case, and it may be well to consider first a somewhat analogous case. The absorption which is accompanied by high emissivity is true absorption, and not selective reflection — which is sometimes confused with absorption. A highly reflecting polished metal surface is a poor radiator, but by properly constructing its surface we may give it the power to absorb and emit. A bundle of polished steel needles with their points all turned towards the source of light reflects scarcely any light at all, the rays undergoing multiple reflections between the conical ends of the needles. Such a bundle of needles should emit much more powerfully than a polished steel surface, and it is easy to see just why it should do so. Each needle, seen end on, sends not only emitted light to the eye, but reflects rays coming from its neighbors. The surface formed by the points of the needles can be regarded as an absorbing surface, which absorbs in virtue of its structure: it is analogous to the hollow "black bodies" with which we are now familiar.

The point to be emphasized is that such a surface, which absorbs but little in virtue of its molecular nature, may be also a powerful radiator, the mechanism by which its radiating power has been increased being as indicated above. Suppose now we take a perfectly transparent body, which like a perfect reflector has no emitting power. A bead of microcosmic salt (sodium pyro-phosphate) heated in a blast lamp is a good example. Though the platinum wire which supports it glows with vivid incandescence, the bead remains perfectly dark. A glass bead, however, emits a good deal of light, doubtless from the fact that its transparency is much less at high temperatures, a very common behavior of transparent substances. The microcosmic salt on cooling becomes traversed by hundreds of cleavage planes, which give it a milky appearance. On reheating it it emits light strongly until it finally fuses into a transparent drop, when it instantly becomes dark again. The reason for this behavior is not quite as apparent as in the case of the needles. In fact I am not quite sure that I understand it at all. Quartz behaves in the same way. A drop of clear fused quartz, heated in the blast, emits little or no light, but if it contains spots made up of an emulsion of quartz and air, these spots emit strongly. In other words an opacity resulting from a pulverization of the transparent medium seems to be accompanied with a strong emitting power. Apparently we cannot apply the same reasoning as in the case of the needles, and it looks rather as if the radiation was largely a surface effect.

We have perhaps a better illustration, and one which is easier to handle, in the case of mica.

A thin film of mica, when heated, becomes silvery white and almost opaque. The change appears to be due to the fact that the mica splits into a multitude of parallel laminae, in other words its opacity is due to the presence of a large number of reflecting planes. If we hold it at the polarizing angle it becomes fairly transparent again. The mica in this state glows brilliantly when heated in a flame, while ordinary mica shows little or no luminosity. Calcined mica films are useful for demonstrating polarization by a pile of plates. It is best to place the film in a piece of thin platinum foil, folded once, and heated in a Bunsen burner. At normal incidence, a film of suitable thickness is opaque as a result of the reflecting planes. Turned to the polarizing angle, it becomes partially transparent, as the planes refuse to reflect the perpendicular components of the vibrations at this angle. The mica shows the same peculiarity as the microcosmic bead: an emission of light resulting from cleavage planes.

There is, of course, the possibility that the roasting of the mica has increased its power of absorption, but of this there appears to be no evidence, for the reflected + the transmitted light appears to equal the incident light in intensity. A flake placed between two equally illuminated white planes disappears, *i.e.* has the same intensity as the background. Such a flake if placed within a hollow body heated to a uniform temperature should disappear for the same reason. It is difficult to see how the reflecting planes can increase the emitting power, for if we consider that the elements within the mass radiate with equal intensity, the introduction of reflecting laminae cannot increase the total output, unless we assume that the *surface* has a higher radiating power than the interior. Professor Jeans suggested to me that this might well be the case, since the electrons will have their largest accelerations at the free surface. If this is the case, any circumstance which increases the amount of surface should increase the emitting power. Returning now to the mica with its numerous parallel cleavage planes and the enclosed air films,—assume it introduced into the interior of a hollow white-hot body. It should disappear when viewed through a small aperture, since the reflected + the transmitted light equals the incident light in intensity, even before its temperature is raised in the slightest degree. Now let its temperature go up until it reaches that of the hollow body. We have now, in addition to the reflected and transmitted light, the emitted light, and the mica should appear brighter than the background. This would seem to be a test of a surface emission. The experiment was tried and the mica remained invisible; that is, it behaved like any other substance placed within a uniformly heated enclosure. The microcosmic salt bead behaved in the same way. The action of the cleavage planes in the microcosmic salt remains therefore unexplained, and requires further investigation.

Returning now to the subject of the powerful emission of light by the lime block and the feeble glow of the carbon, we may find

the following analogy helpful in explaining some of the apparent anomalies.

Hydraulic Analogy of a Radiating Body. — The radiator is represented by a tall hollow cylinder, open at the top and closed at the bottom, provided with a number of outflow pipes of different sizes. Water flows into the cylinder at a certain definite rate from a horizontal pipe or flume, the height of which (T_2) above the base of the cylinder represents the temperature of the flame. Obviously the level of the water in the cylinder will rise until the rate at which the water flows out exactly equals the rate at which it flows in. This height (T_1) is the temperature which the radiator acquires in the flame. The jets of water which issue from the tubes represent radiation of different wave-lengths, the small jets representing the short waves. Their velocity corresponds to *intensity* of radiation. We will first suppose our hydraulic radiator to represent a black body, say a lump of carbon. In this case all of the pipes at the bottom are wide open and we have the maximum outflow of all wave-lengths for any given temperature, *i.e.* for any given height of the fluid within the cylinder. (If we take the cylinder empty, and plunge it into water, jets will squirt into it through the pipes; that is, it is a perfect absorber for all wave-lengths.) With all of the pipes open, however, the level of the water within the cylinder will not rise to any great height, owing to the limited rate at which water flows in from the horizontal pipe. This means that the lump of carbon in the flame does not rise to a very high temperature because it radiates energy at a high rate. At the low temperature there is comparatively little visible light in the radiation, for the shorter waves only appear in quantity at high temperatures. We can imitate this condition in our hydraulic model if we choose by putting valves on the inside of the tubes, those on the small tubes opening only at high pressures.

To make our model imitate the bead of microcosmic salt we plug up all of the pipes. The cylinder now represents a transparent body. If immersed in water it absorbs nothing through the pipes, and no matter how high the level of the water rises in it when water is poured in there is no emission of fluid, in other words no radiation. The body rises in temperature until the temperature is equal to that of the flame, but there is no radiation. Take next the case of the lime in the oxy-hydrogen flame. It is a partially transparent substance, and we can imitate it by plugging the tubes with glass beads or cotton. Owing to the lesser rate at which the water now flows out through the tubes, the level rises much higher than when the tubes are all open, and owing to the greater pressure (temperature) we have liquid jets through the small tubes (short wave-length radiation). The inferiority in the emissivity is more than made up for by the higher temperature which the body can acquire. We are now ready for the Welsbach mantle. It has been conclusively shown by Rubens that the peculiar brilliancy of the thorium mantles, caused by a small trace of cerium, is due to the fact that the cerium makes the thorium selectively absorbing for the short waves at high temperatures. If we wave a Bunsen flame over a

mantle in a brilliantly lighted room, it will be seen to turn yellow at a temperature a little below a red heat. In other words it becomes a strong absorber for the short waves. It is, however, transparent for the long waves, consequently it does not emit energy at anything like the rate at which a black body does, and in consequence can rise to a high temperature in the flame, exactly as a pure thorium mantle does. Its band of absorption in the blue region enables it to pour out visible radiations nearly as powerfully as those which a black body at the same temperature would emit, hence its enormous brilliancy. Our hydraulic model with all of its tubes plugged with cotton represents the mantle of pure thoria, while to transform it into the Welsbach mantle we have only to pull out the porous plugs from some of the smaller tubes. In this condition, owing to the impeded flow in the large tubes, the water will rise in the cylinder to a great height, and we get very powerful jets from the small tubes which we have opened, much more powerful than in either of the previous cases considered. Of course with all of the tubes open we could get equally intense small jets if we poured the water in at the top at a sufficient rate. There is a limit to this rate, however, for it is obvious that the rate at which the water is poured in at the top corresponds to the rate at which the flame can pour energy into the radiating body, a circumstance which depends on the conductivity of the body for heat and other circumstances.

Emission of Substances having Absorption Bands. — It is frequently stated that ruby glass when heated emits an excess of green light, but the phenomenon is not very striking, if it exists at all. Cobalt glass was investigated by Rizzo (*Atti acc. Torino*, 29, 424, 1894), who was unable to establish any relation between its emitting and absorbing power. His apparatus was not very sensitive, however, and it is difficult to draw very definite conclusions from his observations.

The emission and absorption of rock-salt has been studied by Abramczyk (*Wied. Ann.*, 64, p. 625, 1898). Unfortunately he made use of absorbing screens, instead of spectroscopic dispersion, and his results cannot on this account be regarded as wholly trustworthy. He found, however, that the heat emission consisted of two parts, one of which was stopped by a salt plate, while the other was freely transmitted. He found that 40 per cent of the radiation from the salt was reflected by a polished plate of the same material, from which he inferred that there was a selective emission at a region in the spectrum corresponding to the position of the band of metallic reflection observed by Rubens and Nichols. Rubens and Aschkinass have pointed out, however, that a strong emission is not necessarily to be expected at a band of metallic reflection. Though this region is one of relatively strong absorption, the absorbed portion may be only a small fraction of the amount reflected, and a high value of the emissivity is not to be expected, as it is the absorption proper, and not reflection, that is related to the emission. In fact high reflecting power is usually associated with low emissivity, as is shown by the small ratio of the emissivity of a polished

metal surface to a surface of the same metal brought into a spongy condition, *i.e.* into the state of platinum black. A hint has been given in the Chapter on Absorption as to the physical explanation of the increased absorption in this case, and it is instructive to reverse the reasoning and apply it to emission. As we shall see later the case is not unlike that of a hollow body, the emission from the interior of which we shall show is equal in intensity to that of the radiation of a perfectly black body at the same temperature.

In the case of substances which are not transparent, as we have seen, a portion of the incident energy is reflected and a portion absorbed. If the intensity of the light is 1, the absorbed fractional part A, and the reflected part R, we have $1 = R + A$ or $A = 1 - R$. Substituting this value in our formula $\frac{E}{A} = e$ gives us an expression for Kirchhoff's law, in which the relation between emissivity and reflecting power is established:

$$\frac{E}{1-R} = e, \text{ the emissivity of a black substance.}$$

This formula has been verified by Rosenthal (*Wied. Ann.*, 68, p. 783), who investigated the emission and reflection of quartz, mica, and glass, with a spectrometer and thermo-element, and compared the results with the values calculated from the above formula. The low emissivity at regions of the spectrum corresponding to those of the maxima of the reflection curve is clearly shown in Fig. 364, and the close agreement between the observed

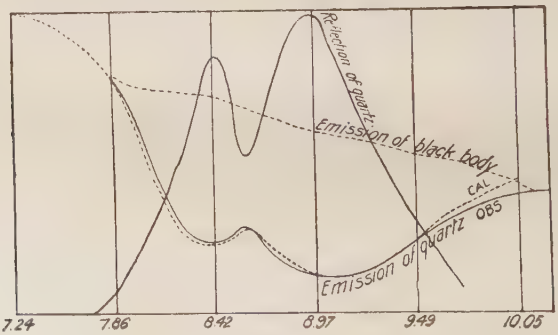


FIG. 364.

values and those calculated from the above formula may be regarded as a most excellent proof of the law. As will be seen, the law has been proven quantitatively for but few substances. That there is a relation between absorption and emission in a large number of cases, where Kirchhoff's law cannot be expected to hold, is evident. Some of these cases we have already considered. A noteworthy example often quoted as a proof of the law is the remarkable relation between the emission of heated oxides of erbium and didymium,

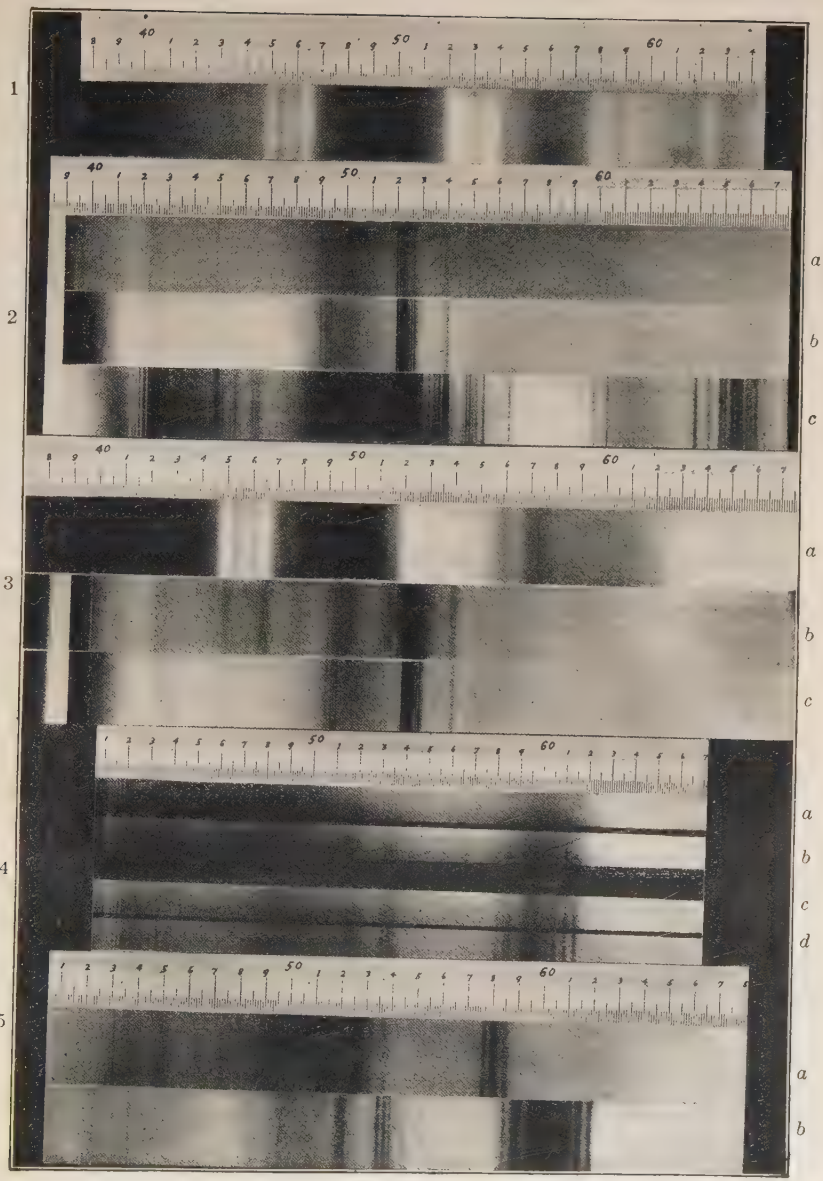


PLATE 9.

and the absorption spectra of the same oxides and solutions of the salts of the metals. The oxides, unlike all other solid substances, when heated to incandescence, show in addition to a continuous spectrum a number of bright bands, which correspond in position to the absorption bands, at least approximately. This can be easily shown by dipping a platinum wire in a concentrated solution of erbium chloride and heating it in a Bunsen burner, the continuous spectrum of the white-hot oxide being crossed by a number of bright bands.

A study of the absorption and emission spectra of the oxides of some of the rare earths has been made by Anderson (*Astrophys. J.*, XXVI, No. 2, 1907), who found that the absorption spectrum, obtained by the " body-color " method, that is by illuminating the powdered or fused substance with white light, was very different at different temperatures. The emission spectrum of heated neodymium oxide showed a very broad and hazy band at $\lambda = 58$–60, the spectrum being almost continuous. If the oxide existed as an impurity in erbium oxide the bands were much narrower, as shown on Plate 9, Fig. 1. When mixed with calcium oxide, however, the bands were similar to those obtained with the pure oxide. The absorption spectrum obtained by illuminating the substance with white light varied with the nature of the surface, as is shown on Plate 9, Fig. 2, in which *a* is the spectrum obtained with the powder; *b*, with a rod of the substance heated for 100 hours in a Bunsen flame; and *c*, with a rod the surface of which was fused with the oxy-hydrogen flame. The bands are much blacker and more numerous in the case of the fused oxide, a circumstance which may be ascribed to the penetration of a greater thickness of the substance by the light in this case. In Fig. 3 we have spectrum *a*, the emission of incandescent erbium oxide, *b* its absorption at a high temperature, and *c* its absorption at room temperature.

As will be seen the absorption spectrum at a high temperature is complementary to the emission spectrum. Fig. 4 shows the absorption of the gray oxide of neodymium at the following temperatures, *a* 600°, *b* 400°, *c* 200°, *d* below 100°. The absorption spectrum depends upon the treatment of the oxide, which when first formed by careful ignition of the oxalate is pink, becoming bluish gray by prolonged heating. In Fig. 5 *a* and *b* we have the absorption spectra of the pink and gray forms respectively. It is not known positively whether they have the same chemical composition.

Selective Emission and Absorption of Incandescent Salts

Some extremely interesting experiments have been recently made by Lenard (*Annalen der Physik*, 17, p. 197, 1905) with beads of the fused salts of the alkali metals supported upon platinum wires in the Bunsen flame. The emitted light was found to be strongly colored, the color depending upon the metal. The sulphates of K, Rb, and Cs emitted green light, while sodium sulphate shone with a bluish tinge. An examination of the absorption

spectra of the fused salts showed that the color of the transmitted light was complementary to that of the emitted light, as should follow from Kirchhoff's law. The salts were colorless when cold, however, showing that some sort of dissociation resulted from the high temperature, metallic ions being set free which had the property of absorbing and emitting radiations of the same frequency. The failure of the ions to show color when the salts are dissolved in water is ascribed by Lenard to a loading of the ions with water. In the case of most of the salts examined the color was found to depend upon the metal, *i.e.* upon the cathions: the borates and phosphates were marked exceptions, however, the color being chiefly due to the anions, the nature of the metal being immaterial. In the same paper will be found a number of very interesting conclusions regarding the emission of colored light by flames and the probable centres from which radiate the different lines of the spectra.

Emission of Polarized Light. — Certain crystals, tourmaline for example, have an absorbing power which differs according to the plane in which the vibrations are taking place. Suppose the crystal to be so oriented that its absorptive power is greatest for horizontal vibrations. We might expect, on heating the crystal, to find a preponderance of horizontal vibrations in the emitted light. This was found to be the case by Kirchhoff, who heated a crystal in a Bunsen flame and found that, on viewing it through a double-image polarizing prism, one of the images was distinctly brighter than the other.

A quantitative proof of Kirchhoff's law in the case of glowing tourmaline has been made by Pflüger (*Annalen der Physik*, 7, p. 806, 1902), who measured with a spectrophotometer the absorption and emission of the crystal at the same temperature and for the same wave-length. If J is the intensity of the incident light, JD that of the transmitted light, and JR that of the reflected, the intensity of the absorbed light $A = J(1-R-D)$. R was calculated from the reflection formula $R = \left(\frac{n-1}{n+1}\right)^2$, while D and E were observed with the spectrophotometer. Designating by E_o and E_ϵ the emissivity for vibrations parallel respectively to the vibrations of the ordinary and extraordinary rays, and by A_o and A_ϵ the corresponding absorptive powers, we have, if Kirchhoff's law can be extended so as to include the direction of the vibration,

$$\frac{E_\epsilon}{A_\epsilon} = \frac{E_o}{A_o} \text{ or } \frac{E_\epsilon}{E_o} = \frac{A_\epsilon}{A_o}.$$

After eliminating all sources of error, Pflüger obtained as final values for the two ratios,

$$\frac{A_\epsilon}{A_o} = .650, \quad \frac{E_\epsilon}{E_o} = .641,$$

a very beautiful verification of the law as applied to anisotropic media.

Deduction of Kirchhoff's Law. — The law of Kirchhoff can be deduced from purely theoretical considerations. Kirchhoff's method is free from serious objections, but assumptions are made which cannot be regarded as truths without further treatment. Moreover, his method involves the consideration of bodies which really have no existence, such as perfect reflectors and perfectly transparent substances.

The most logical and concise treatment is due to Pringsheim (*Verh. d. deutsch. physik. Ges.*, 3, pp. 81–84, 1901). Consider a ball κ composed of any material enclosed in a hollow vessel, opaque to radiation of all wave-lengths, and uniformly heated to any given temperature. The ball emits in unit time the total radiation E, while there falls upon it from the walls in the same time the amount e, of which the fraction Ae is absorbed. Since by Carnot's principle the temperature cannot change, the amount of radiation emitted by the ball must equal the amount absorbed, so that $E = Ae$. If the ball is made of a conglomerate of different substances, some parts of its surface may absorb more strongly than others. Suppose we rotate the ball: the amount of energy e falling upon it will only be changed by an infinitely small amount, since only the part of the radiation which came originally from the body, and is reflected back from the walls, can be responsible for the change: the amount of this which falls across the body is of course very small. If, however, the radiation from the walls is not uniform, *i.e.* if it has especial states of polarization, or is more intense in certain directions than in others, the amount of heat absorbed by the body would be changed by its rotation. If, for example, the rotation brought a strongly absorbing surface element into the path of an especially intense ray coming from the wall, the absorption of heat would be increased.

In this case A is variable, and the heat absorbed is $\int Ade$ taken over the body κ. Since this is equal to E, it follows that $\int Ade$ is constant for all positions of the ball. If A is variable this can only be the case if e is independent of direction.

The total radiation e is made up of waves of all possible lengths between 0 and ∞. Call e_λ the radiant energy comprised between the limits λ and $\lambda + d\lambda$, then

$$e = \int_0^\infty e_\lambda d\lambda.$$

Further, let A_λ be the absorption coefficient of κ for waves of length λ: the total absorbed energy is then

$$Ae = \int_0^\infty A_\lambda e_\lambda d\lambda,$$

or

$$E = \int_0^\infty A_\lambda e_\lambda d\lambda.$$

Now let κ be brought into another hollow vessel of different

material but at the same temperature. The emission of κ remains the same, also its absorption coefficient A_λ, for waves of the designated length.

If in the present case the radiation $e_{1\lambda}$ which falls upon κ is different from e_λ (in the previous case), we should have

$$\int_0^\infty A_\lambda e_\lambda d\lambda = \int_0^\infty A_\lambda e_{1\lambda} d\lambda .$$

Since, however, A_λ is quite independent of e_λ, the above equation can only hold if

$$e_\lambda = e_{1\lambda}.$$

By comparing this with $E = Ae$ we see at once that the radiation e is equal to that which κ would emit if it were perfectly absorbing, *i.e.* if $A = 1$. This shows us that the radiation within a hollow vessel heated to a uniform temperature is independent of the material and shape of the vessel, and is identical in every respect with the radiation emitted by a perfectly absorbing body at the same temperature.

Consider now the radiant energy which a surface element ds_1 of the body κ sends to the distant surface element ds_2 of the vessel. We will define the emission coefficient E_λ of κ as the single radiation $E_\lambda d\lambda$, which in unit time reaches ds_2 from ds_1. This radiation has a wave-length λ and any state of polarization. In a hollow vessel of uniform temperature, ds_1 gives out a radiation similar to that of a black body at the same temperature. The *total* energy of wave-length λ and of a given state of polarization which reaches ds_2 from ds_1 is therefore $e_\lambda d\lambda$, if e_λ is the emission coefficient of a black body under similar conditions.

We thus have $e_\lambda = E_\lambda + G_\lambda$ if we define $G_\lambda d\lambda$ as the energy of wave-length λ, and of a definite state of polarization, which, coming originally from the walls of the vessel, and transmitted by, or reflected from, the body κ, reaches ds_2 from ds_1.

We must now determine the value of G_λ and substitute it in the above equation. The radiation from the inner surface of the vessel which reaches ds_2 by reflection or refraction from ds_1 is equal to the amount which, leaving ds_2, reaches the inner surface by way of ds_1. Of all the waves of length λ which leave ds_2 the amount reaching ds_1 in unit time is $e_\lambda d\lambda$, of which an amount equal to $A_\lambda e_\lambda d\lambda$ is absorbed, while the remainder $(1 - A_\lambda)e_\lambda d\lambda$ are in part reflected and in part transmitted, and pass off to the inner wall again. This remainder is equal to $G_\lambda d\lambda$, and substituting the value $G_\lambda = (1 - A_\lambda)e_\lambda$ in the equation

$$e_\lambda = E_\lambda + G_\lambda,$$

gives us $$e_\lambda = E_\lambda + (1 - A_\lambda)e_\lambda,$$

or $$E_\lambda = A_\lambda e_\lambda,$$

an equation which expresses Kirchhoff's law, showing that the emission coefficient of any substance for any given wave-length, divided by its absorption coefficient for the same value of λ, is equal to

the emission coefficient of a perfectly black body, for, as we have seen above, e_λ represents the latter quantity.

That the intensity of the radiation from the inner surface of a hollow vessel is independent of the nature of the material can be shown by placing a fragment of decorated china in a porcelain crucible heated over a Bunsen burner. If the cover of the crucible is put on, a small opening being left through which the interior can be viewed, and the flame of a second burner be directed upon it so as to bring the whole to a nearly uniform temperature, the decorations on the china will be quite invisible, the radiations from them being equal to the radiation from the rest of the surface. As we have seen, if the china is heated in the open air the dark portions radiate more strongly, the design appearing brighter than the background. The cause of the equality in the case of an enclosed radiator can be very simply stated. The radiation is made up of two parts, the emitted and the reflected, the latter coming from the heated walls. Dark portions of the material emit more powerfully than white portions, since their power of absorption is greater; on the other hand, they reflect scarcely any of the radiation from the walls. The white portions, which emit feebly, reflect powerfully, and, owing to the proportionality between emission and absorption, a perfect balance is secured.

This principle is now made use of in experiments pertaining to radiation. In studying the nature of the radiation of perfectly absorbing bodies as a function of temperature, it was formerly the custom to make use of an electrically heated strip of platinum with a smoked surface. Such a radiator cannot, however, be brought to a high temperature, owing to the oxidation of the carbon. For high temperature work it was customary to coat the strip with platinum-black, or copper oxide.

Such radiators cannot be regarded as perfect, and at the present time the heated hollow chamber is almost exclusively used, the radiation to be examined escaping through a small hole.

The Perfect Black Radiator. — While the principle that the radiation within a closed space at a uniform temperature is identical with the radiation of a perfectly black body had been recognized for many years, Wien and Lummer (*Wied. Ann.*, 56, p. 451, 1895) were the first to actually prepare radiators acting on this principle, and make use of them in experimental work.

For studying the intensity of the radiation at low temperatures and the distribution of energy in the spectrum of the radiation, a hollow cylinder of brass, blackened on the inside, can be used. The cylinder is provided with a small aperture, and is surrounded by a steam jacket, or embedded in a mixture of sodium and potassium nitrate, and the whole packed in felt. The smaller the size of the hole in comparison to the internal capacity of the cylinder, the more nearly does the emerging radiation compare with that of an ideal black body. For high temperature work a cylinder of platinum or porcelain, electrically heated, can be employed, or even a hollow iron ball heated in a gas furnace. Kayser has proposed a very simple device, which, though superior to an electrically heated strip

of blackened platinum, is not as good as a hollow vessel. Two strips of platinum, one provided with a narrow slit, are mounted opposite to one another and heated to the same temperature by a current. The principle is of course the same as that of the device just considered.

Paschen (*Wied. Ann.*, 60, p. 719, 1897) has proposed still another device. A glowing carbon filament is mounted at the centre of a hollow silvered sphere. Assuming the silver to reflect all of the energy, it can be regarded as a hollow vessel having the same temperature as the carbon filament. The radiation escapes as before through a small hole. This same device has been applied to the bolometer, the absorbing strip being mounted at the centre of a hollow spherical chamber silvered on the inside. All radiation not absorbed at once by the bolometer is returned to it by the reflecting surface. In this way it is possible to prepare a perfectly black bolometer.

Equilibrium between Radiation and Material Bodies. — In the deduction of the remaining laws of radiation we shall employ largely a conception due to Bartolli, which, though it cannot be carried out experimentally, leads to important laws which can be verified in other ways. The idea in brief is to apply the principles of thermodynamics to radiation, performing a cyclical process similar to Carnot's cycle, employing vibrating ether instead of a gas as the working substance.

The radiation within a hollow vessel can be in equilibrium with the walls or with bodies in the interior, only when it is of the same nature as the radiation emitted by the walls or the bodies contained within the vessel.

To get an idea of exactly what we mean by equilibrium between radiation and a material body we will consider the following case:

Suppose we have a hollow vessel the walls of which are perfect reflectors, which contains only ether. If we fill this cavity with monochromatic radiation, say that of the sodium flame, by opening a door in the wall and allowing the light to enter, which, of course, can be done perfectly well in theory, the radiation will, if we close the door, be reflected back and forth within the vessel forever. It will neither change in intensity nor alter its wave-length; in other words, it is in equilibrium with the reflecting walls. We shall now prove that a perfect reflector is the only body with which this radiation can be in equilibrium, with the exception of the flame which originally emitted the light. Suppose we introduce a small fragment of absorbing matter within the cavity of the reflecting vessel. It will immediately absorb the monochromatic sodium radiation as fast as this radiation falls upon it, and in a very short space of time the monochromatic waves will have vanished completely. The temperature of the absorbing body will be slightly elevated, and it will emit long heat-waves, the energy being distributed over a wide range of wave-lengths, the range and distribution depending on the temperature of the body. This radiation will now fill the cavity in place of the sodium radiation, and it will be in equilibrium with the absorbing body, *i.e.* a permanent state is speedily reached, after which there is no further change.

Pressure of Radiation. — The radiation within the vessel exerts a pressure upon the walls and upon the surface of the absorbing body. As we are to make use of this pressure in the derivation of laws it will be well to investigate it somewhat in detail.

Maxwell, in his electro-magnetic theory, showed that radiation must exert a pressure when it falls upon a reflecting or absorbing surface. As this pressure is the foundation upon which the laws of radiation have been built, we will briefly consider the phenomenon.

Maxwell showed that when plane electro-magnetic waves fall in a normal direction upon a perfectly absorbing surface, the pressure exerted on unit area is equal to the energy contained in unit volume of the vibrating medium.

That a pressure is exerted by heat (or light) waves may be proven by making use of the idea of Bartolli. Consider a cylinder, composed of some material which reflects perfectly, closed at the ends by black plates at temperatures $T_1 > T_2$ (Fig. 365). Introduce a screen S, made also of a reflecting material, which divides the cylinder into two compartments. The body at temperature T_1 will fill the upper compartment with radiation of energy corresponding to its temperature. T_2 (at a lower temperature) will fill the lower compartment with radiant energy of less density. Let B represent a movable reflecting diaphragm, provided with a sliding door, which, when open, allows the energy from T_2 to fill the middle compartment. Now close the door and raise the diaphragm or piston. The volume of the middle compartment is decreased, and the density of the radiant energy "trapped" within it is increased. On removing the screen S laterally, which we can do without performing work, the diaphragm will drive the radiant energy above it into the body T_1. We have thus taken energy (or heat) from a body at low temperature and carried it to one of high temperature, which by the second principle of thermo-dynamics is impossible, unless mechanical work is done in the operation. This work can have resulted only from the overcoming of a pressure exerted upon the diaphragm, the vibrating medium resisting compression in the same way that a gas does. This pressure becomes greater as the volume is diminished owing to the increase in the energy density. In the case of the compression of a gas, the molecules rebound from the moving piston with increased velocity, consequently the force of each blow, and the number of blows per second, are increased.

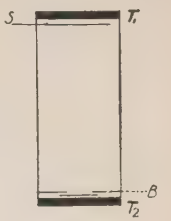


FIG. 365.

In the case of compressed radiation the mechanism is not so easy to follow; as we shall see presently, reflection from a moving diaphragm decreases the wave-length by an amount proportional to the distance through which the diaphragm moves (provided the rest of the vessel is reflecting). This means that the number of waves which strike it per second will be increased. The amplitude, as we shall see presently, remains the same, and the increase of energy density is due solely to the fact that more waves are present

in unit length of the train after the compression than existed before the motion of the diaphragm. Such a process as that described above cannot be even approximately realized experimentally. It is no less valuable, however, as our inability to carry it out is due solely to mechanical difficulties and our inability to obtain a substance which reflects perfectly.

An admirable treatment of the mechanical pressure of radiation has been given by Larmor (*Encycl. Brit.*, vol. 32, "Radiation"). Consider a wave train travelling along the x axis incident upon a perfect reflector, which is travelling in the opposite direction with a velocity v. The displacement in the incident wave train is

$$\xi = a \cos m(x+ct),$$

and in the reflected train

$$\xi' = a' \cos m'(x-ct).$$

The position of the reflector at time t is given by

$$x = vt.$$

The disturbance does not travel into the reflector, and must therefore be annulled at its surface. Thus when $x=vt$ we must have $\xi+\xi'=0$. This gives us $a=-a'$ and $m'(c-v)=m(c+v)$.

The amplitude of the reflected disturbance is therefore equal to that of the incident one, while the wave-length is altered in the ratio $\frac{c-v}{c+v}$ or $1-\frac{2v}{c}$, approximately, when $\frac{v}{c}$ is small.

The energy of the wave-train is half potential and half kinetic, and is given by the integration of $\rho\left(\frac{\partial\xi}{\partial t}\right)^2$ along the train, in which ρ = density.

In the reflected train it is therefore augmented, when equal lengths are compared, in the ratio $\left(\frac{c+v}{c-v}\right)^2$, but the length of the train is diminished by the reflection in the ratio $\frac{c+v}{c-v}$. This increase in energy per unit time can arise only from work done by the advancing reflector against pressure due to the radiation. The pressure per unit surface must therefore be equal to the fraction $\frac{2}{c-v}$ of the energy in the length $c+v$ of the incident wave-train; thus it is the fraction $\frac{c^2-v^2}{c^2+v^2}$ of the total density of energy in front of the reflector belonging to both the incident and reflected trains.

When v is small compared with c this makes the pressure equal to the density of the vibrational energy, in accordance with Maxwell's electro-dynamic formula.

The pressure due to light was, for a long time, sought for in vain. The disturbing effects of "radiometric action," or the reaction pressure of gas molecules rebounding from the surface heated by the

radiation, completely masked the very small effect which was looked for. As early as 1754 an attempt was made by DeMairan and Du Fay to detect the pressure of light. This was of course in the days of the corpuscular theory, and the looked-for pressure was that due to the arrest of the flying corpuscles. Fresnel, Zöllner, Bartolli, and Crookes also searched in vain for evidences of the pressure, the experiments of the latter, however, resulting in the discovery of the radiometer. The pressure was first observed by Lebedew in 1900 (*Rapp. près au Congres de Phys.*, 2, 133, Paris, 1900) and by Nichols and Hull independently at about the same time. Though the latter investigators were anticipated by Lebedew by some months, their investigation was conducted with greater care, and the errors due to gas action were more carefully eliminated.

Experiments of Nichols and Hull.[1] — The inability of previous observers to measure the pressure due to radiation was due to the apparent impossibility of separating the effect from the so-called "radiometer" action. Thin vanes were employed to detect the pressure, and the radiation warmed the side on which it fell. When such a condition exists the gas exerts a greater pressure on the warm than on the cold side, and in general this pressure is vastly greater than the true radiation pressure.

Nichols and Hull finally succeeded in eliminating the gas action by employing a suspended vane made of two circular disks of thin glass silvered on one side. By employing a reflecting surface the pressure is double that exerted upon a black surface, and the heating is reduced to a minimum. By measuring the deflections when the glass and silver sides were illuminated in succession the gas action could be calculated, for the silver surface is the one heated in both instances. This is due to the fact that the radiation, before its impact upon the vane, has passed through a number of lenses and plates of glass, and is consequently robbed by absorption of all rays capable of heating a glass surface. It is at once apparent that when the radiation falls upon the glass surface the gas pressure and the light pressure are opposed, while when the silver surface is illuminated they act together, *i.e.* in the same direction. Larger deflections are of course observed in the latter case than in the former. To still further eliminate gas action, the ballistic method was adopted; it had been observed that some seconds' or even minutes' exposure to the radiation were required before the gas pressure reached its maximum, while the radiation pressure is of course instantaneous. Very short exposures were consequently given, and the ballistic deflection of the vane was observed by means of a mirror and scale.

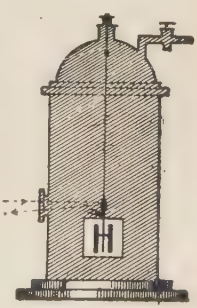


Fig. 366.

By an elaborate series of experiments the investigators determined the most suitable pressure for the air in the chamber in which the vane was suspended, the pressure, in other words, at which the gas action was at a

[1] *Proc. Am. Acad.*, xxxviii., April 1903; *Phys. Rev.*, xiii. 293, [1901].

minimum. This pressure proved to be about 16 mms. of mercury. After measuring the value of the radiation pressure, the energy of the radiation was determined, by allowing it to fall upon a blackened silver disk. The rise of temperature of the disk was determined by means of iron-constantan thermo-junctions imbedded in the disk. From these energy measurements the pressure to be expected was calculated.

The radiation employed was that of an arc-lamp, either with or without absorbing screens. After correcting for all possible sources of error, the following values were obtained:

Radiation	Pressure in 10^{-5} dynes observed	Pressure calculated from energy measurements
Through air only	$7.01 \pm .02$	$7.05 \pm .03$
Through red glass	$6.94 \pm .02$	$6.86 \pm .03$
Through water cell	$6.52 \pm .03$	$6.48 \pm .04$

These experiments can be regarded as establishing in a quantitative manner the existence of the Maxwell-Bartolli pressure, which measured in dynes per sq. cm. is equal to the energy contained in unit volume of the radiation. A reflecting surface doubles the energy density in the medium in front of it by superposing the reflected beam upon the incident.

The gas action was subsequently eliminated by Hull (*Phys. Rev.*, May 1905) by enclosing the reflecting and absorbing surfaces in thin glass cells, as proposed in the earlier paper. The silvered side of a thin cover glass was placed in contact with the blackened side of a similar glass, and the whole enclosed by means of two other thin glasses as shown in Fig. 367. Two cells of this description were mounted upon opposite ends of a torsion arm suspended in a receiver from which the air could be removed. When the light falls upon the blackened surface and is absorbed, the temperature of the two outer glass surfaces of the cell are the same, since they are separated from the heated surface by equal thicknesses of glass and air. The gas action should therefore be equal on the two surfaces. Any gas action occurring within the cell will produce no effect, owing to the equality of action and reaction. Hull found that the ratio of the deflections obtained when the silvered and blackened surfaces were illuminated in succession, agreed with the calculated ratio to within 2 per cent, showing that the "radiometer" action had been practically eliminated.

Fig. 367.

Tangential Component of Radiation Pressure. — An interesting experiment was described by Poynting at the Cambridge Meeting of the British Association in 1904, in which gas action is completely eliminated.

When radiation is incident upon an absorbing surface in an oblique direction, the pressure has a component parallel to the surface. In

the case of a reflecting surface this tangential force cannot be detected, since the incident and reflected beam give rise to equal and opposite forces parallel to the surface. The magnitude of the force, when E is the energy density, μ the fraction reflected, and α the angle of incidence, is given by

$$F = \frac{E}{2}(1-\mu)\sin 2\alpha.$$

The existence and magnitude of the force was observed with the apparatus shown in Fig. 368. Two thin glass disks were mounted on the ends of a fine glass rod, the system being suspended by a quartz fibre in a brass box provided with glass windows. One of the disks was silvered, the other blackened, and the pressure within the case was reduced to 1 cm. Sunlight, or the beam from an arc lamp, was directed against the black disk at an angle of 45°. Gas action due to heating will give rise to a pressure normal to the surface, but there will be no tendency to rotate the suspended system. The tangential component of the radiation pressure, on the other hand, will produce a deflection, the magnitude of which can be read with a mirror and scale. E was calculated from the observed deflection, and was found to be $5.8 \cdot 10^{-6}$ dynes, while a direct measurement of E, by the heating of a silver plate, gave the value $6.5 \cdot 10^{-6}$.

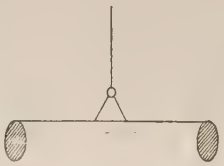


Fig. 368.

Pressure of Radiation on an Absorbing Gas. — The repulsion of the tails of comets by the sun has been explained as the result of the pressure exerted by the solar radiation. If we reduce the size of an obstacle its mass becomes less in proportion to its surface, and the pressure of the radiation may eventually become greater than the attraction of gravitation. It has been shown by Schwartzchild, however, that there is a critical size at which the ratio of pressure to gravitational attraction has its greatest value. In other words, if we make the particles too small the radiation no longer exerts any pressure on them, for they no longer act as obstacles, or diffract light. Now the spectroscope shows us that the tail of a comet is gaseous, and the gas molecule is very much smaller than the smallest obstacle capable of feeling the pressure of radiation. There is this difference, however. The gas molecule may be capable of stopping the radiation by resonance, *i.e.* the gas may absorb, and it seems quite probable that radiation may exert a measurable pressure on the molecules of a gas, in spite of the fact that a cloud of material particles, each one of which is vastly larger than the molecule, experiences no pressure at all. This question has been very successfully attacked by Lebedew,[1] who has measured the pressure which the radiation from a Nernst lamp exerts upon various absorbing gases, such as CO_2, methane, butane, propane, etc. The gas was enclosed in a cell provided with fluorite windows, and was

[1] *Annalen der Physik*, April 1910.

set in motion by the pressure of the radiation. This motion was communicated to a very light torsion balance made of magnesium, by which it was rendered visible.

The Stefan-Boltzmann Law. — An empirical law was deduced by Stefan from observations made by other observers on the intensity of the total radiation from bodies at different temperatures. The law states that the complete emission S of a black body is proportional to the fourth power of the absolute temperature T, or

$$S = aT^4,$$

in which a is a constant.

This same law was subsequently deduced from theoretical considerations by Boltzmann (*Wied. Ann.*, 22, p. 291, 1884), who availed himself of the ingenious conception by which Bartolli proved that radiation must exert a pressure. Consider a hollow cylinder of unit cross-section, the walls of which are black, and of infinitely small heat capacity. The ends of the cylinder are also black, but of infinitely large heat capacity. Within the cylinder is a frictionless black piston, in contact with the left-hand end plate of the cylinder, which has an absolute temperature T_0. The opposite end plate has a lower temperature T (Fig. 369). The radiant energy exerts a pressure on the piston, which in the case of plane-waves parallel to the surface is equal to the radiant energy in unit volume of the ether. Let $\Psi(T)$ be the energy in unit volume. Since the energy is travelling in all possible directions, the pressure on unit surface will not be $\Psi(T)$ but $\frac{1}{3}\Psi(T)$. (Compare with the calculation of the pressure due to molecules moving in all directions, in the Kinetic Theory of Gases.) We have then $\frac{1}{3}\Psi(T) = f(T)$, the pressure at temperature T.

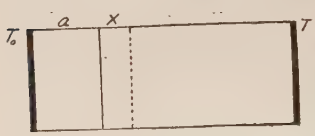


FIG. 369.

Now let the piston move forward a distance a, under the influence of the pressure of the radiant energy coming from the high temperature plate T_0. The heat energy leaving T_0 is partly spent in doing the work $af(T_0)$ on the piston, and partly in filling up the volume "a" with radiant energy. This process obviously corresponds to the isothermic expansion of the Carnot cycle, the filling of the space a with energy corresponding to the heating of the gas. The amount of heat which leaves the plate T_0 is $a[\Psi(T_0) + f(T_0)]$.

We will now introduce a screen impervious to heat immediately in front of T_0, which prevents further radiation into the space to the left of the piston. This corresponds to placing the cylinder in Carnot's cycle upon an insulating stand. The pressure to the left of the piston is greater than that to the right, owing to the higher temperature of the plate which filled this portion of the cylinder with radiation. The piston will therefore move forward until the energy per unit volume is the same on both sides. Allowing this adiabatic expansion to take place we have $d[(a+x)\Psi(T)] = -f(T)dx$, the characteristic equation for an adiabatic process (see any

Thermodynamics). In this expression T is of course variable. During this process the volume to the right of the piston has been still further diminished, and an amount of heat energy represented by $(a+x)[\Psi(T)+f(T)]$ due to diminishing the volume and work done enters the plate at temperature T.

Since the process is reversible we have, by the second law of thermodynamics,

$$\frac{(a+x)[\Psi(T)+f(T)]}{T} = \frac{a[\Psi(T_0)+f(T_0)]}{T_0} = C,$$

in which x and T are variables.

Writing $$(a+x)[\Psi(T)+f(T)]=CT,$$

or, for simplicity, $$(a+x)(\Psi+f)=CT,$$

$$(\Psi+f)d(a+x)d(\Psi+f)=\frac{(a+x)(\Psi+f)}{T}dT,$$

and subtracting, $\Psi d(a+x)+(a+x)d\Psi=-fdx$ (adiabatic equation),

gives us $$(a+x)df=\frac{(a+x)(\Psi+f)}{T}dT,$$

$$(\Psi+f)dT=Tdf,$$

or, inserting the (T) which we omitted above,

$$\Psi(T)dT+f(T)dT=Tdf(T).$$

Substituting for $f(T)$ its equivalent value $\frac{1}{3}\Psi(T)$

gives $$\tfrac{4}{3}\Psi(T)dT=\tfrac{1}{3}Td\Psi(T),$$

$$\frac{d\Psi(T)}{\Psi T}=4\frac{dT}{T},$$

$$\Psi=aT^4.$$

Proof of Stefan's Law. — The law was first deduced empirically from observations made on the rate of cooling of a blackened thermometer bulb.

Lummer and Pringsheim (*Wied. Ann.*, 63, p. 395, 1897) proved the law over a range of temperatures included between 100° and 1300° C. by measuring the intensity of the radiation from a hollow chamber (black body) by means of the bolometer.

The constant a has been determined in absolute measure by Kurlbaum (*Wied. Ann.*, 65, p. 746, 1898), who heated the bolometer strip (screened from the radiation) by means of an electric current of known strength to the same temperature to which it was raised by the radiation. The radiation was thus determined in absolute units by calculating the Joule heat developed by the current. The value found was

$$a=1.71\cdot 10^{-5}\frac{\text{erg}}{\text{sec.}}=0.408\cdot 10^{-12}\frac{\text{gr. cal.}}{\text{sec.}}.$$

Optical Pyrometers. — Various types of pyrometers have been designed for measuring high temperatures by optical methods. Féry's instrument is based upon the law of total radiation. It consists of a telescope with a fluorite objective, in the focus of which is mounted a sensitive thermo-couple as shown in Fig. 370. To use the instrument one has only to point it at the object, the temperature of which is to be measured, *e.g.* the interior of a blast-furnace, and focus the image upon the thermo-junction by means of the eye-piece, which is moved with the latter by means of the rack and pinion wheel at *B*. The temperatures are read with a galvanometer.

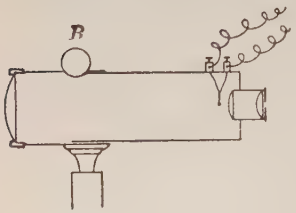


Fig. 370.

Other optical photometers have been devised, based upon the laws which we are about to study.

Temperature of the Sun. — The sun's temperature has been computed by measuring the total radiation. Assuming the solar disk to be a black body, and taking for the value of the solar constant 3 gr. cal. per minute, the computed temperature comes out a trifle over 6000°.

Change in the Spectrum of a Black Body with the Temperature. Wien's Laws. — Making use of a conception similar to the one by means of which Boltzmann deduced Stefan's law, but extending it by the introduction of the consideration of the change in wave-length which occurs when radiation is reflected from a moving mirror, Wien (*Wied. Ann.*, 46, p. 633; 52, p. 132) arrived at a formula which expressed the change in the spectrum of a heated black body with its absolute temperature. As is well known, when a solid or liquid is heated the longer heat-waves appear first, then red light, and finally at still higher temperatures the violet and ultra-violet. If we measure the energy at different points in the spectrum with the bolometer and plot these values as ordinates, with the wave-lengths as abscissae, we obtain the energy curve for the emission at the temperature in question. The maximum of this curve moves towards the region of the shorter waves as the temperature is increased, but there is an increase in the height of every ordinate; in other words, the curve does not move bodily down the spectrum. It seems very remarkable that the form and position of this curve can be determined by considering merely the motion of reflecting pistons moving in a closed cylinder, the ends of which radiate at different temperatures. This, however, is precisely what was done by Wien, whose treatment we will now consider. Consider a cylinder of unit cross-section, the walls of which reflect diffusively all of the incident energy, while the ends are composed of black material, of infinite heat capacity, at temperatures

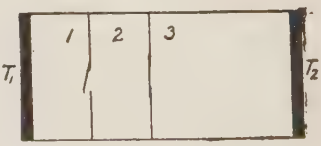


Fig. 371.

T_2 and T_1 ($T_2 > T_1$). The cylinder is divided into three compartments by means of movable pistons composed of perfectly reflecting material and furnished with openings which can be closed by means of trap-doors. At the beginning we have things arranged as shown in Fig. 371 and the perfect radiator T_1 fills up compartments 1 and 2 with radiant energy, of density $\Psi(T_1)$. The density in compartment 3 is greater, namely $\Psi(T_2)$.

The trap-door is now closed and the partition moved towards 3, a distance dx, such that the radiation in 2 has the same density as that in 3. The distribution of energy in the spectrum is now the same in compartments 2 and 3, since, if this were not the case, there would of necessity be rays of a certain wave-length λ in compartment 3, having a greater energy density than the waves of corresponding λ in compartment 2. This being the case, we could cover the opening in the moving plate with a screen composed of some material transparent to waves of this particular length, but reflecting all others. On opening the trap-door more energy would pass from 3 to 2 than passed back in the opposite direction, and the density in 2 would become greater. On closing the door and removing the screen, the plate would be moved to the right until the pressure became equal on both sides, furnishing an amount of work A. A small amount of heat leaves the black body at temperature T_2 in the form of radiation of wave-length λ, to restore the original condition. This heat is the equivalent of the work A. The trap-door is now opened and the plate brought back to its original position, no work being required.

The door is now closed, and the plate which separates 1 and 2 is driven back to its original position (distance dx), by which the work is gained which was originally spent in moving the plate through the distance dx. If now we open the door in the plate, we have the original state of things, the body at temperature T_1 neither having given up nor received heat, while the other body at temperature T_2 has given up heat corresponding to the amount which passed through the selectively transparent screen, and furnished the work A. By the second law of thermodynamics work cannot be derived by a cyclic process in the case in which a single reservoir gives up heat in such a manner that all of it is transformed into work.

We conclude therefore that when the energy density is the same in compartments 2 and 3, the distribution of energy in the spectrum is also the same.

Going back now to the main part of the problem. The motion of the piston which condensed the energy in 2 until it had the same density as that in 3 is accompanied by a shortening of the wave-lengths of the reflected energy, for a moving mirror will, by Doppler's principle, alter the lengths of all waves incident upon it. Though the shortening depends on the velocity of the mirror, in the present case it depends only upon the total distance through which the mirror moves. This is due to the fact that the rays are repeatedly reflected from the mirror, and if the mirror moves with slow velocity more reflections will occur from it during its movement, the increased number of reflections compensating for the

slower velocity. For normal incidence, if the velocity of the mirror is v, the wave-length after one reflection is

$$\lambda' = \frac{c-2v}{c}\lambda.$$

As we have seen, we can regard $\frac{1}{3}$ of the total radiation as incident normally, and we will assume that a single reflection shortens the waves by an amount h.

Plotting the original density distribution for wave-lengths λ and $\lambda + h$, we will determine the effect of a single reflection upon this portion of the energy curve. In Fig. 372 the curve AB represents the energy distribution before reflection, the ordinates representing total densities of energy for different values of λ. Since we can regard $\frac{1}{3}$ of the energy as incident normally, if we disregard the rest we shall have $\frac{2}{3}$ of the energy at λ unaffected, represented by the ordinate a. Of the energy at wave-length $\lambda + h$, $\frac{2}{3}$ is unaffected (ordinate b), while $\frac{1}{3}$ is shortened in wave-length to the value λ; consequently this portion must be added above a as indicated. If we do this for all values of λ we shall obtain the dotted curve, which represents the energy distribution after one reflection. If $f_1(\lambda)$ represents this distribution, and $\Phi(\lambda)$ the original distribution, we can express the above change as follows:

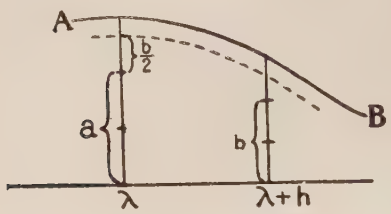


Fig. 372.

$$f_1(\lambda) = \tfrac{2}{3}\Phi(\lambda) + \tfrac{1}{3}\Phi(\lambda + h) = \Phi\left(\lambda + \frac{h}{3}\right).$$

If the radiation is reflected n times, we have

$$f_n(\lambda) = \phi\left(\lambda + \frac{nh}{3}\right).$$

The change in the distribution of energy can thus be represented by considering $\frac{1}{3}$ of the rays as shortened by an amount nh.

If now $(a - x)$ is the distance between the pistons, we have for n, while the piston moves a distance dx, $n = \dfrac{dx}{2(a-x)}\dfrac{c}{v}$, in which c is the velocity of the radiation and v the piston's velocity. After n-fold reflection, we have

$$\lambda_n = \left(\frac{c-2v}{c}\right)^n \lambda = \left(\frac{c-2v}{c}\right)^{\frac{dx}{2(a-x)}\frac{c}{v}} \lambda = \left[\left(1 - \frac{2v}{c}\right)\right]^{\frac{dx}{(a-x)}\frac{c}{v}} \lambda.$$

For the limit $c = \infty$,

$$\lambda_n = e^{-\frac{dx}{a-x}}\lambda, \text{ since limit of } \left(1 + \frac{\theta}{n}\right)^n = e_\theta, \text{ when } n = \infty,$$

or writing $\lambda_n = \lambda + nh$, in which nh is infinitely small, of the order dx,

$$nh = -\frac{dx}{a-x}\lambda.$$

Now $f(\lambda)$, the distribution of energy after n-fold reflection, is given by

$$f(\lambda) = \Phi\left(\lambda + \frac{nh}{3}\right) = \Phi(\lambda + d\lambda);$$

$$\therefore d\lambda = -\frac{dx}{3(a-x)}\lambda,$$

which gives us the change of wave-length due to the motion of the piston through a distance dx.

Integrating the above, $\lambda = \sqrt[3]{\frac{a-x}{a}}\lambda_0$, in which λ_0 is the value for $x=0$, *i.e.* before the motion of the piston commenced.

Let E be the total energy in compartment 2, when $x=0$: its density is then

$$\Psi = \frac{E}{a-x}.$$

If x increases by dx, the energy intensity is increased by diminution of volume, and work done against the radiation pressure, by an amount

$$\frac{d\Psi}{dx}dx = \left\{\frac{dE}{dx}\frac{1}{a-x} + \frac{E}{(a-x)^2}\right\}dx = \left\{\frac{dE}{dx} + \Psi\right\}\frac{dx}{a-x}.$$

The pressure on the piston is $\frac{1}{3}\Psi$, therefore the work is

$$\frac{dE}{dx}dx = \tfrac{1}{3}\Psi dx, \text{ and } d\Psi = \frac{4}{3}\frac{\Psi}{a-x}dx,$$

$$\Psi = \left(\frac{a}{a-x}\right)^{\frac{4}{3}}\Psi_0,$$

in which Ψ_0 is written in place of the original $\Psi = \frac{E}{a-x}$, for $x=0$,

$$\frac{\Psi}{\Psi_0} = \left(\frac{a}{a-x}\right)^{\frac{4}{3}}, \text{ and as we have seen above, } \frac{\lambda}{\lambda_0} = \left(\frac{a-x}{a}\right)^{\frac{1}{3}};$$

$$\therefore \frac{\Psi}{\Psi_0} = \frac{\lambda_0^4}{\lambda^4}.$$

We can now write the expression $\frac{\Psi}{\Psi_0}$ in terms of the absolute temperatures, by applying the Stefan-Boltzmann law. If Ψ_0 is the energy at T_0 and Ψ the energy at T,

$$\frac{\Psi}{\Psi_0} = \frac{T^4}{T_0^4},$$

and we get at once $T\lambda = T_0\lambda_0$.

This expression may be interpreted in the following way: On raising the temperature of a black body from T_1 to T_2 the ordinates of our energy curve move towards the short wave-lengths by an amount such that the product of the corresponding abscissa and the temperature remains constant for each ordinate. The maximum ordinate at say wave-length λ_m for temperature T_1 will therefore at temperature T_2 occupy the position λ_m' such that $\lambda_m T_1 = \lambda_m' T_2$: in other words, as the temperature rises, the summit of the energy curve drifts towards the region of shorter wave-lengths.

We can find, by the aid of this displacement law, the distribution of energy for any temperature, if the distribution for some given temperature is known.

Plot as before a $\Phi(\lambda)$ and λ energy curve; the area of the curve is equal to the total energy Ψ.

To pass to a curve for another temperature, we take a vertical strip at λ_0 of width $d\lambda_0$, the area of which is $\Phi_0 d\lambda_0$; this strip is displaced by the temperature change to, say, λ.

The quantity of energy in the strip $\Phi_0 d\lambda_0$ must remain constant.

$$\therefore \Phi d\lambda = \Phi_0 d\lambda_0,$$

$$\Phi = \Phi_0 \frac{d\lambda_0}{d\lambda} = \Phi_0 \frac{T}{T_0}.$$

It will be observed that thus far we have neglected the circumstance that the total energy increases with the temperature, as represented by Stefan's law.

Taking this into account by itself, we have

$$\Phi = \Phi_0 \frac{T^4}{T_0^4}, \text{ and by combining this with } \Phi = \Phi_0 \frac{T}{T_0},$$

we obtain as the complete expression

$$\Phi = \Phi_0 \frac{T^5}{T_0^5},$$

and our new ordinate at λ must be equal to ordinate at λ_0, multiplied by the ratio $\frac{T^5}{T_0^5}$.

Wien's two laws as applied to the wave-length at which we have the maximum energy may be written as follows:

$$\lambda_m T = A \text{ (const.)},$$

$$E_m T^{-5} = B \text{ (const.)},$$

in which E_m is the energy at the maximum.

Lummer and Pringsheim have tested these two laws by measuring the energy curves of a heated black body over a range 621–1650 Absolute. Their results are given in the following table:

T	λ_m	E_m	$A = \lambda_m T$	$B = E_m T^{-5}$
1650	1.78	270	2928	$2246 \cdot 10^{-17}$
1260	2.35	69	2959	2176
1094	2.71	34	2956	2166
908	3.28	13.6	2980	2208
723	4.08	4.3	2950	2166
621	4.53	2.03	2814	2190

Neither of the two equations, however, give us any information regarding the actual distribution of energy in the spectrum of a black body.

To express this we require an expression which represents E as a function of λ and T.

A number of formulae have been developed which we will briefly discuss.

Complete Radiation Formulae. — Wien (*Wied. Ann.*, 58, p. 662), by the consideration of a peculiar type of radiator, deduced a formula connecting E with λ and T. The radiator is considered as a hollow vessel filled with a gas mixture capable of emitting waves of all lengths. He assumes that every molecule emits only a single wave-length, which depends on its velocity, the intensity of which wave is a function of this velocity. Further, the intensity $\Phi(\lambda)$ of the radiation between the limits λ and $\lambda + d(\lambda)$ is proportional to the number of molecules vibrating with periods corresponding to wave-lengths within this range. From these assumptions he derived the formula

$$E = \frac{c}{\lambda^5} e^{-\frac{\beta}{\lambda T}}.$$

This formula represents the energy distribution very well if it is not applied to too long waves. Lord Rayleigh has pointed out that the energy at a definite wave-length, as represented by the formula, does not increase indefinitely with the temperature, but approaches a limit. For visible waves this limit would only be reached at temperatures far beyond our command, but for $\lambda = 60\,\mu$ (Rest-strahlen from sylvite) the limiting value of the intensity would occur at about 1000°.

Planck has deduced a radiation formula of different form from electro-magnetic considerations:

$$E = \frac{c\lambda^{-5}}{e^{\frac{c^2}{\lambda T}} - 1}.$$

This formula has been confirmed in a remarkable manner by the work of Rubens and Kurlbaum (*Ann. der Physik*, 4, p. 649, 1901), who measured the intensity of the radiation of $\lambda = 51\,\mu$ emitted by a black body over a temperature range comprised between 85 and 1773 Absolute. Their results are given in the following table,

together with values calculated from the formulae of Wien and Planck:

T	E obs.	E cal. (Wien)	E cal. (Planck)
85	−20.6	−107	−21.9
193	−11.8	− 48	−12
293	0	0	0
523	+31	+ 63	+30.4
773	64.5	96	63.8
1023	98.1	118	97.2
1273	132	132	132
1523	164	141	166
1773	196	147	200
∞	—	194	∞

This table shows not only the close agreement between the observed values and those calculated from Planck's formula, but also that at a temperature of 1773 a value of E was obtained, larger than the limiting value 194 for infinite temperature, calculated from Wien's formula.

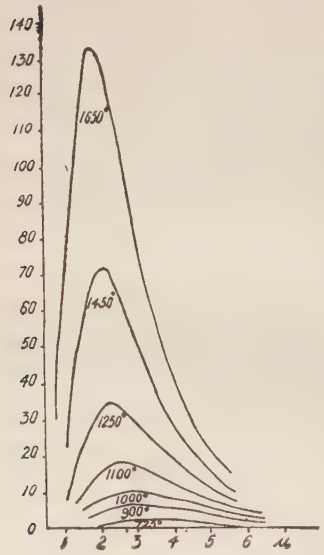


Fig. 373.

For short waves Wien's formula is perfectly satisfactory. It is questionable, however, whether it is anything more than an empirical formula, for many objections have been raised against the methods employed in its deduction. The energy curves for a black body are represented in Fig. 373 for temperatures between 750 and 1650. The shift of the maximum towards the region of shorter wavelengths with increasing temperature is clearly brought out. These curves were made from observations by Lummer and Pringsheim.

In the case of the sun's spectrum the point of maximum energy is shifted much nearer to the visible region. The solar intensity curve, as actually measured, is of course greatly modified by atmospheric absorption. Fig. 374 is from a record obtained by Langley, and shows the relative energy distribution in the solar spectrum and in the spectrum of a black body at various temperatures. The spectra were obtained by means of a prism, which by crowding the energy, so to speak, in the infra-red portion, on account of the small dispersion in this region, does not give us a true value for the position of the maximum. The deep valleys in the solar curve represent atmospheric absorption.

Energy Distribution in the Spark Spectrum. — As we have seen, there is a very rapid drop in the energy as we pass from the red to the violet of the spectrum of a white-hot body. Pflüger,[1] in a series of remarkably interesting experiments, has found that in general the reverse is true in the case of the bright-line spark spectra. He investigated the energy distribution in the spark spectra of a large number of metals with a bolometer, and found that the largest deflections were obtained *in the remote ultra-violet:* in other words, the ultra-violet lines were "hotter" than the red or infra-red. We have

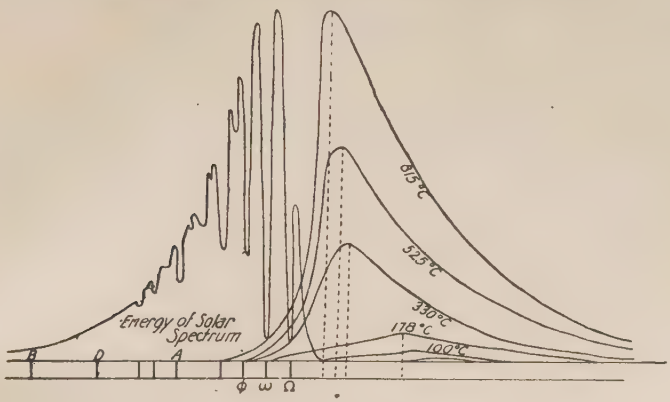


FIG. 374.

as yet no law governing the energy distribution in discontinuous spectra, but Pflüger's experiments probably may be taken as marking the beginning of a subject to which a chapter may be devoted in text-books twenty years hence. His results for the zinc spark are given in the following table, from which an energy curve can be constructed. It is instructive to compare the curve thus obtained with the curves in the last section.

WAVE-LENGTHS	GAL. DEFL.	WAVE-LENGTHS	GAL. DEFL.	WAVE-LENGTHS	GAL. DEFL.
		Deflections under 10		Deflections under 10	
199	50	244	35	515	80
200	195	252	85	570	45
203	225	261	20	650	50
204	205	275	25	800	80
206	280	335	60	900	85
206+	160	360	15	1.μ	80
208	220	395	45	1.2	75
211	60	465	40	1.4	60
				From here on, decreasing	

[1] *Annalen der Physik*, 13, p. 890, 1904.

CHAPTER XXII

SCATTERING OF LIGHT BY SMALL OBSTACLES. OPTICAL RESONANCE

Scattering of Light by Small Particles. — If a beam of light is passed through a transparent medium containing in suspension small particles, the refractive index of which differs from that of the surrounding medium, light will be given off by the particles in all directions. If the particles are very small, the color of the scattered light is blue, and it is more or less completely plane-polarized, the direction of vibration being perpendicular to the direction of the incident beam. If the incident beam is plane-polarized to start with, no light is scattered by the cloud of particles in directions parallel to that of the incident vibration. In the case of particles of the order of magnitude of the light-waves, the amount of light scattered increases as the wave-length is decreased, which explains the preponderance of blue always observed in these cases. The subject was investigated experimentally by Tyndall with clouds precipitated by the chemical action of light upon the vapor of iodide of allyl. Whenever the particles obtained were sufficiently small, the laterally emitted light was blue in color and polarized in a direction perpendicular to the incident beam. Tyndall was unable to explain the polarization, and imagined that it contradicted Brewster's law, there being no angle of maximum polarization, as in the case of reflection from flat surfaces of isotropic media. That there is really no contradiction here is at once apparent when we consider that the nature of the phenomenon is radically different in the two cases. In ordinary reflection we are dealing with surfaces large in comparison with the wave-length, and the amount of reflected light is independent, or nearly so, of the wave-length. While, in the case which we are considering, regular reflection in the ordinary sense does not occur, and the intensity of the light is a function of the wave-length. By making the particles sufficiently small we may obtain a violet of great intensity and purity. The fog formed by the condensation of sodium vapor has been observed by the author to give a deeper color than any of the other media heretofore employed. The experiment, however, is rather difficult to perform, and a description of it would be out of place here. The blue color is easily seen in tobacco smoke rising from the end of a lighted cigar. On standing, the smoke particles appear to collect into larger aggregates and the blue color disappears. This is usually the case with smoke exhaled from the mouth. The blue color of the

sky has its origin in a similar action exerted either by small dust particles or even by the molecules of air themselves. The absence of long waves in the light coming from the blue sky on a clear day is well shown by Plate 10, which is reproduced from a photograph made by the author with a screen or ray filter made by combining a sheet of very dense cobalt glass with a deep orange aniline dye. This screen absorbed all rays below wave-length 6900. The spectrum of the sun or arc photographed through this screen on a Cramer or Wratten and Wainwright spectrum plate is reduced to a band extending from $\lambda = 6900$ to $\lambda = 7400$. The rays within this region are visible to the eye, if all other rays are excluded, but they play little or no part in ordinary vision, on account of their very feeble action upon the retina of the eye. Landscapes photographed through the screen present a remarkable appearance. The grass and trees in full sunlight appear as if snow-white, while the sky is as black as midnight. This is due to the fact that the chlorophyl of the vegetation reflects the infra-red light very powerfully, while the light of the blue sky is nearly or wholly wanting in it. The shadows in all of the infra-red photographs are very black, since practically no light comes from the sky, and it is the skylight which is chiefly responsible for the illumination of the shadows.

The scattering of light has been exhaustively studied by Lord Rayleigh, who explains the phenomenon in the following way: "Conceive a beam of plane-polarized light to move among a number of particles all small compared with any of the wave-lengths. The foreign matter may be supposed to load the ether so as to increase its inertia without altering its resistance to distortion. If the particles were away the waves would pass on unbroken, and no light would be emitted laterally. Even with the particles retarding the motion of the ether the same will be true if, to counterbalance the increased inertia, suitable forces are caused to act on the ether at all points where the inertia is altered. These forces have the same period and direction as the undisturbed luminous vibrations themselves. The light actually emitted laterally is thus the same as would be caused by forces exactly the opposite of those acting on the medium otherwise free from disturbance, and it remains only to see what the effect of such forces would be. In the first place there is necessarily a complete symmetry around the direction of the force; the disturbance consisting of transverse vibrations is propagated outwards in all directions from the centre; and in consequence of the symmetry the direction of vibration in any ray lies in the plane containing the ray and the axis of symmetry; that is to say, the direction of vibration in the scattered or refracted ray makes with the direction in the incident or primary ray the least possible angle. The symmetry also requires that the intensity of the scattered light should vanish for the ray which would be propagated along the axis. For there is nothing to distinguish one direction transverse to the ray from another. Suppose for distinctness of statement that the primary ray is vertical, and that the plane of vibration is that of the meridian. The intensity of the light scattered by a small particle is constant, and a maximum for rays which lie in the ver-

tical plane running east and west, while there is no scattered ray along the north and south line. If the primary ray is unpolarized, the light scattered north and south is entirely due to that component which vibrates east and west, and is therefore perfectly polarized, the direction of its vibration being also east and west. Similarly any other ray scattered horizontally is perfectly polarized, and the vibration is performed in the horizontal plane. In other directions the polarization becomes less and less complete as we approach the vertical."

What actually occurs as a physical process, that is, the exact manner in which the particles load the ether, is not definitely stated. We shall have no difficulty in remembering that the direction of vibration of the light scattered in a direction perpendicular to the incident beam, is the direction in which no light is scattered when the incident beam is plane-polarized, if we imagine the obstacle actually set in vibration. Transverse waves would then not be given off in the direction in which the vibration takes place, *i.e.* in the direction parallel to the vibrations of the incident light. But it is inconceivable that particles as small even as the molecules are actually thrown into vibration as rapid as those of light. We may imagine if we like that the contained electrons are set into vibration as a whole, *i.e.* that the centre of gravity of the system is periodically displaced by the electric forces in the light-waves. This would be equivalent to "loading the ether," and though it may not be a correct conception, serves perhaps as something tangible to fix the mind on. This conception neglects everything analogous to dispersion, the free periods of the electrons not being considered, and forces of restitution being neglected. We are merely concerned with the inertia of the system as a whole, which we conceive as having no free period of vibration.

The intensity of the scattered light as a function of the wave-length, for obstacles of fixed size, small in comparison to λ, was calculated by Lord Rayleigh (*Phil. Mag.*, xli., pp. 107–120, 274–279, 1871).

Let i be the ratio of the amplitude of the incident to that of the scattered light, and T the volume of the disturbing particle. If r is the distance from the particle of a given point, the value of i at this point is shown to be proportional to $\frac{T}{\lambda^2 r}$, *i.e.* the amplitude varies inversely as the square, and the intensity inversely as the fourth power of the wave-length.

Observations were made of the distribution of energy in the spectrum of the light of the blue sky by comparing its spectrum with the spectrum of direct sunlight, diffused by white paper. These values were compared with values calculated on the assumption that the intensity of the scattered light (*i.e.* blue-sky light) varied as $\frac{1}{\lambda^4}$. The two sets of values are given in the following table for four of the Fraunhofer lines:

PLATE 10. — Foliage and blue sky photographed by infra-red rays.

C	D	b_3	F	
25	40	63	80	calculated.
25	41	71	91	observed.

Lord Rayleigh's formula for the intensity of the scattered light in a direction making an angle β with the incident ray is, if the incident light is unpolarized,

$$A^2\frac{(D'-D)^2}{D^2}(1+\cos^2\beta)\frac{m\pi T^2}{\lambda^4 r^2},$$

in which A^2 is the intensity of the incident light, D' and D the densities of the particles and the medium in which they are immersed, m the number of particles, and λ the wave-length.

The formula shows that the intensity is twice as great in the direction from which the light comes originally as in a direction perpendicular to it.

The Residual Blue. — Tyndall found that as the particles in his precipitated clouds increased in size, the blue color disappeared, the scattered light appearing white. If, however, it was received through a Nicol prism held in the position in which it would ordinarily extinguish the scattered light, the blue color appeared again in increased splendor. This blue color he named the "residual blue."

Lord Rayleigh considers this phenomenon in a subsequent paper (*Phil. Mag.*, xii., p. 81), and shows that if the incident light is polarized with its vibrations parallel to the z axis, the intensity of the light scattered along the z axis varies as the inverse 8th power of the wave-length, so that the residual blue is purer than the blue seen under ordinary conditions. With smaller particles no light at all would be seen in this direction under the conditions specified. It remains to be seen whether in this case there is any direction in which the scattered light vanishes. Lord Rayleigh derived an equation which showed that zero illumination was to be expected in a direction inclined backwards, *i.e.* towards the source of light, and this was found to be the case. The experiments were made with a precipitate of sulphur, obtained by adding a small quantity of dilute sulphuric acid to a weak solution of hyposulphite of soda. The more dilute the solution, the slower is the process of precipitation, and the slower the change in the size of the particles. Solutions of such strength that no precipitate appears for four or five minutes will be found to give the best results, and the process can be arrested at any stage by the addition of a few drops of ammonia. The experiment should be performed in a dark room, a beam of sunlight rendered convergent by means of a long focus lens being passed through a glass tank containing the solution. The scattered light should be examined by a Nicol prism. Quoting from the paper above referred to, "In the early stages of the precipitation polarization is complete in a perpendicular direction and incomplete in other directions. After an interval the polarization begins to be incomplete in a perpendicular direction, the light which reaches the eye when the Nicol is in the position of minimum trans-

mission being of a beautiful blue, much richer than anything that can be seen in the earlier stages. This is the moment to examine whether there is a more complete polarization in a direction somewhat more oblique, and it is found that with θ positive (*i.e.* towards the source) there is in fact an oblique direction of more complete polarization, while with θ negative the polarization is less perfect than in the perpendicular direction itself."

The mathematical treatment of the subject is too long to be given in detail, and as an abbreviated treatment is unsatisfactory it is omitted entirely. Reference should be made to the original papers. In one of his more recent papers Lord Rayleigh has shown that the blue light of the sky can be regarded as caused by the scattering power of the air molecules themselves, in the absence of any suspended particles.

Abnormal Polarization and Color of Light scattered by Small Absorbing Particles. — The light scattered by clouds of smoke or very finely divided precipitates in transparent liquids is usually plane polarized to a greater or less degree, as we have just seen, the plane of the vibration being *vertical*, *i.e.* perpendicular to the direction of the incident ray.

In the course of some experiments by the author on the fluorescence of iodine vapor, the precipitation of what appeared to be an iodine fog in one of the glass bulbs was observed. This fog scattered powerfully light of a deep red color, and on examining it with a Nicol prism it was found to be plane-polarized in a direction at right angles to that which is usually observed in the case of light scattered by small particles. When a powerful beam of light was sent through the bulb in a horizontal direction, the scattered light came off at right angles, with its direction of vibration (electric vector) horizontal instead of vertical. If the light was polarized before it entered the bulb, the light was scattered laterally in the directions of the vibration in the incident light. The phenomenon was first observed in a bulb containing a small flake of iodine, and air at atmospheric pressure, cooled by immersion in a beaker of water to which ice had been added. The best method of producing the colored fog is to precipitate the iodine upon a smoke-cloud already existing in the bulb. If iodine is vaporized in a bulb and then cooled, it crystallizes either upon the walls, or in the form of floating spiculæ, too large to show the scattering effects. If, however, the bulb is filled with a light smoke and the iodine vaporized, on cooling each smoke particle forms a nucleus upon which the iodine crystallizes. The smoke which appeared best was that from hot sealing-wax. Tobacco-smoke did not work well, and it seems probable that the smoke particles must be transparent. The best plan appears to be to introduce a bit of sealing-wax the size of a pin-head, and a similar speck of iodine, into a small glass bulb, and then heat them both with a very small flame. It is a good plan to support the bulb in a clamp-stand in a dark room and focus sun- or arc-light at its centre. The red fog persists for some time, once it is formed. When at its best, its scattering power is so great that a reddish color is seen within the bulb at a distance of a metre from the arc without any

concentration. Examined by transmitted light in a well-lighted room, no trace of color is to be seen, which proves that the red light is selectively scattered, and not produced by the absorption of iodine vapor. With the concentrated beam from the arc the scattered light is blood-red and of great intensity. A Nicol placed with its long diagonal horizontal, nearly, but not quite, extinguishes it.

It seemed probable that the polarization might be produced by refraction of the light within the small particles. The light which is given off at right angles to the direction of the incident beam is in part reflected, of course. This portion is uncolored, and polarized with its vibration perpendicular to the direction of the incident rays. Another portion of light may reach the eye after two refractions and one internal reflection, and this portion will be colored, and oppositely polarized. We are of course considering the particles as partially transparent, and of such size as to permit them to act as the water-drops in the theory of the rainbow. A small glass bulb filled with a red liquid can be used to show the phenomenon on a large scale; if it is placed in a beam of sunlight and viewed at an angle slightly greater than 90° with the incident rays, two spots of light will be seen — one uncolored, due to reflection, the other colored, due to refraction; and examination with a Nicol shows that they are more or less completely polarized in opposite directions. Attempts were made to obtain fogs of other strongly absorbing media. Precipitation of the aniline dyes was tried, without much success, however. Nitroso-dimethyl aniline, a substance which has proved of interest in connection with many optical problems, was found to give excellent results. A few grains are placed in the bottom of a large test-tube, and heated with a small flame. A yellow fog at once forms in the tube, which exhibits the same phenomena shown by the iodine fog. Examined with a Nicol with its short diagonal horizontal, the scattered light is yellowish green and very bright; if the Nicol is turned through a right angle the scattered light is white and much fainter. The nitroso absorbs the blue end of the spectrum. The fog is best examined through the open mouth of the tube, the arc-light being focussed at the centre. Under certain conditions, these nitroso fogs showed brilliant colors which were not due to absorption. If the test-tube is fastened at an angle of 45° in a clamp-stand, and the light focussed about an inch below the open mouth, the fog will be found to whirl about in a vortex; and if it is examined with a Nicol, the stream lines will often be found to show brilliant colors. One stream will be bright red and the adjacent stream deep green, and the colors appear to change to their complementary tints when the Nicol is rotated through 90°; though it is difficult to be sure of this effect, owing to the rapid changes which take place in the distribution of the colors.

These changes reminded me of the curious effects observed with thin films of collodion deposited on silver surfaces described previously. When a powerful beam of unpolarized white light is concentrated normally upon the film, and the light scattered at grazing emission is examined with a Nicol, it is found that one-half

of the visible spectrum is polarized vertically, *i.e.* parallel to the surface, the other half horizontally.

The diameter of the particles in the nitroso fogs averaged about .003 mm., as was determined by inverting the tube over a glass plate, allowing the particles to settle, and examining the deposit with a microscope. The particles are much larger than the ones treated by Lord Rayleigh, which are too small to regularly refract or reflect the incident light, and which operate in a wholly different manner.

Somewhat similar clouds can be formed by shaking up some of the finely powdered aniline dyes in a test-tube, the finest particles remaining suspended in the air. Soluble Prussian blue makes a good cloud, the scattered light being deep blue when examined with a Nicol with its short diagonal parallel to the incident rays, and white under the reversed conditions. The polarization effects are not as marked, however.

Scattering of Light by Metal Spheres. — The case where light is scattered by metal spheres, small in comparison to the wave-length, has been treated theoretically by J. J. Thomson (*Recent Researches*, p. 437). He finds that the scattered light produced by the incidence of plane-polarized light vanishes in the plane through the centre, at right angles to the magnetic induction in the incident wave along a line making an angle of 120° with the radius to the point at which the wave first strikes the sphere, and it does not vanish in any direction other than this. The direction in which the scattered light is plane-polarized will thus be inclined at an angle of 120° to the direction of the incident light. It is thus seen that the law is quite different from that which holds in the case of non-conducting particles, when the scattered light vanishes at all points in a plane normal to the magnetic induction, where the radius vector makes an angle of 90° with the direction of the incident light. When the light is scattered by a conducting sphere, the points of complete polarization are on the surface of the cone, whose axis coincides with the direction of propagation of the incident light, and whose semi-vertical angle is 120°.

Optical Resonance. Electrical Resonance. — In the chapter on Dispersion we have seen that the velocity of waves in dispersing media is modified by the presence of charged electrons, which have definite periods of vibrations, and which are thrown into oscillation by the periodic electric forces of the light-waves.

We have an analogy in the case of electro-magnetic waves, which differ from light-waves only in the comparative slowness of the period. Oscillatory electrical disturbances are set up in strips of metal of suitable dimensions by the passage of Hertzian waves, and we shall see presently that the velocity of the waves will be changed if the metal strips or resonators are sufficiently numerous. The maximum resonance effect will occur when the natural period of the resonator agrees with that of the wave. Garbasso (*Atti. Acc. di Torino*, 28, 1893) found that a system of tin-foil strips was much more opaque to the radiation of a Hertz oscillator when the period of the latter agreed with the natural period of the electrical vibration

in the tin-foil strips. The greater part of the incident energy was reflected, the phenomenon being the electrical analogue of surface color.

Garbasso and Aschkinass (*Wied. Ann.*, 53, p. 534, 1894) subsequently found that if the tin-foil strips were distributed in space in the form of a prism, the phenomena of dispersion were exhibited, the electro-magnetic waves being deviated in different degrees, according to their lengths.

If the resonators were immersed in fluid dielectrics, such as benzol, ether, or acetone, the free periods were altered and the position of maximum absorption in the electrical spectrum was shifted towards the region of longer waves (Aschkinass and Schaefer, *Ann. de Physik.*, 5, p. 489, 1901). This is analogous to the shift in the position of the absorption band, which occurs when a dye is dissolved in different solvents.

In the above case we are clearly dealing with electrical vibrations which occur in the metal strips taken as a whole, and not with vibrations going on within the molecule, as is the condition assumed in the electron theory of dispersion.

The question naturally arises as to whether it is possible to reduce the dimensions of the metal strips to such a point that their free periods are of the order of magnitude of the periods of light- or heat-waves.

Resonance for Heat-waves. — The first attempts in this direction were made by Rubens and Nichols (*Wied. Ann.*, 60, p. 456, 1897), who ruled a thin silver film, deposited on glass, into narrow strips, and then cut the strips into suitable lengths by cross ruling. The reflecting power of the plates, for the very long heat-waves (Reststrahlen) obtained by multiple reflections from fluorite surfaces, was studied, with the direction of the strips first parallel and then perpendicular to the direction of the electric vector in the heat-waves. The width of the strips averaged about $5\,\mu$, and the lengths were made equal to 6, 12, 18, and $24\,\mu$, the cross ruling being omitted in one case, which gave a plate with resonators infinitely long in comparison with the waves.

In all cases the plates showed a higher reflecting power when the direction of vibration in the heat-waves (polarized by reflection) coincided with the long axis of the resonators. The highest reflecting power was exhibited by the plate on which the cross ruling had been omitted, which suggested the polarizing power of wire gratings observed by Rubens and Du Bois (*Wied. Ann.*, 49, p. 593, 1893). The authors were of the opinion that unquestionable evidence of electrical resonance was shown, which seems highly probable. For many reasons more conclusive data could be obtained with resonators of smaller dimensions, which would respond to waves corresponding to regions of the heat spectrum where there was a greater abundance of energy, and accordingly a series of experiments were commenced by Nichols and Hull in collaboration with the author, which are still in progress. By depositing films of gold on glass by means of the cathode discharge and ruling under oil (which prevented the tearing of the film by the diamond), it was

found possible to rule resonators measuring only 1.6 by 3 μ. These plates were ruled with one of Rowland's machines, and were far superior in appearance to the earlier plates used by Rubens and Nichols, which were ruled by hand with a small dividing engine. Under the microscope the plates appeared almost flawless, the rectangles being sharply cut even at the corners. Unfortunately, however, no trace of resonance could be found with these plates, which appears to be due to the excessive thinness of the films, which were not wholly opaque to light. Rubens has, moreover, pointed out that, in order to secure sharp resonance, it is necessary that the clear spaces between the strips should be much wider than the strips, the reverse being true in the present case. In the course of some experiments with heat waves of very great wave-length (110μ) described in the chapter on dispersion theory, the author observed a curious phenomenon which appears to have a direct bearing upon electrical resonance. Polished marble, which reflects over 40 per cent of the radiation, when reduced to an impalpable powder and pressed into a flat cake with a smooth surface was found to reflect practically nothing, though the irregularities of the surface were much too small to account for the absence of specular reflection. The particles were apparently so small that the resonance necessary for selective reflection could no longer take place. Similar results had previously been obtained with films of very finely divided aniline dyes which showed colors quite different from those exhibited by continuous films. Metal powders were found to behave in the same way. A film was obtained by shaking reduced copper in a jar and then allowing the finely divided dust to settle from the air upon a quartz plate. This film was absolutely opaque for visible light, yet it *transmitted 90 per cent of the long-wave radiation*, and reflected practically nothing. A *continuous* film of copper of much less thickness would reflect nearly 100 per cent and be absolutely opaque to the radiation. Further experiments showed that with thinner deposits of the copper powder, with clear spaces between the grains, resonance occurred somewhere between 1μ and 10μ. The subject has not however been very thoroughly investigated at the present time, but the field appears promising.

Polarizing Action of Gratings. — Closely related to the phenomena which we have just discussed is the polarizing action which gratings exert on light. Fizeau (*Ann. de Chim. et Phys.*, (3), 63, 385, 1861) noticed as early as 1861 that light is partially polarized by passage through a narrow slit. The same phenomenon was found by H. Du-Bois (*Wied. Ann.*, 46, 548, 1892) in the case of gratings made of fine silver wire. In all of these cases the direction of polarization (*i.e.* electric vector) of the transmitted light was parallel to the slit or wires. Fizeau, however, recorded that light which had passed through an extremely narrow slit less than .1 μ in width was polarized perpendicular to the direction of the slit. On widening the slit to the dimension of the light-wave .5 μ, the direction of polarization of the transmitted light turned through a right angle.

Du-Bois and Rubens have (*Ber. der Deut. Phys. Gesell.*, 2, 2, 1904,

p. 77) investigated the polarizing action of wire gratings for the long heat-waves reflected from fluorite and rock-salt. They found that as the wave-length was increased the polarization of the transmitted light, which was parallel to the wires, increased to a maximum, then diminished, becoming zero for a certain wave-length, after which further increase in the wave-length was accompanied by a polarization of the transmitted light perpendicular to the wires, which increased in amount with increasing wave-length. This is in agreement with the investigation of Hertz, who found that a wire grating was more transparent for electric waves when the direction of the wires was perpendicular to the direction of the electric vector than when it was parallel to it. The investigations were confined to the energy directly transmitted by the grating, *i.e.* observations were made of the central image, and *not* in the diffraction spectra. The results found with a platinum grating are given in the following table:

Wave-length λ	Q_m	Q_p	Q_s	$u^2=\frac{Q_p}{Q_s}$
5	.223	.198	.248	.80
25.5	.297	.230	.265	.63
51.2	.535	.332	.738	.45

In this table Q_m represents the percentage transmitted when the incident light is unpolarized, Q_s and Q_p the transmission of light, the vibrations of which are perpendicular and parallel respectively to the direction of the wires.

We see in the first place that the transparency for all three types of radiation increases with increasing wave-length, but that the increase is greater for the perpendicularly polarized light. The ratio $u^2 = \frac{Q_p}{Q_s}$ therefore decreases with increasing wave-length. The increase in the transparency for unpolarized radiations with increasing wave-length is due to the fact that the diffraction spectra disappear, with the exception of those of the first order, with the result that the intensity in the central image is increased. For the longest waves, 51 μ, reflected from rock-salt, which are larger than the grating-constant, diffraction in the ordinary sense no longer occurs, and the grating transmits practically one-half of the incident light.

It will be observed that in these experiments the state of polarization of the transmitted light is similar to that found by Fizeau with a slit smaller than the wave-length of light, *i.e.* the polarization direction of the transmitted heat-waves is at right angles to polarization direction of transmitted light.

As we shall see presently, the condition which prevails in the present case can be observed with visible light, if the elements of the grating are made small enough.

The Color of Light diffracted by Screens, showing Selective Absorption.—Some remarkable effects were observed by Gouy (*Comptes Rendus*, xcvii., p. 1573, 1884), and more carefully studied by Wien (Inaug. Diss., Berlin, 1886), of the colored light diffracted into the region of the shadow by thin sheets of copper, gold, silver, etc.

Wien focussed sunlight upon the highly polished edges of thin plates of various metals, and observed that light was diffracted far into the region of the shadow, the edge of the plate appearing luminous. The color of the light varied in a remarkable manner with the nature of the metal, appearing red with copper and gold screens, orange with silver, yellow and yellow-green with platinum and tin-foil. The color only appeared when the edge was clean and quite free from dust; it was complementary to the color most strongly absorbed by the metal, and polarized with the vibration perpendicular to the diffracting edge. If the incident light was polarized to start with, the color was only seen when the vibration was perpendicular to the edge. The phenomenon is evidently related in some way to resonance, vibrations being set up in the metal along the edge, which emit energy into the region behind the screen. In addition to the colored light, Wien found that white light was also present, and that it could also be cut off by a Nicol prism, though its plane of polarization appeared to depend upon the azimuth in which the incident light was polarized, and also upon the angle of diffraction.

Still more recent observations have been made by F. Braun (*Ann. de Phys.*, 16, p. 1, 1905), who worked with extremely fine metal gratings obtained by passing the discharge of a large battery of Leyden jars through platinum or silver wires held in contact with a glass plate. The metal vaporized by the discharge deposited itself in the form of fine metal striae perpendicular to the wire. An examination of these plates between crossed Nicols showed that certain spots were transparent for light polarized with the electric vector perpendicular to the direction of the metal strips.

The best results were obtained with platinum wires .044 of a millimetre in diameter, through which the discharge of a battery of twenty jars (capacity 40,000 cm.) was passed. This phenomenon had previously been observed by Kundt in metal films obtained by the cathode discharge from a fine wire, which stood perpendicular to a glass plate. When observed between crossed Nicols the film appeared as a brightly illuminated disk, with a dark cross, the arms of which were parallel to the planes of polarization, the centre of the cross being always at the point of the conical metal film, *i.e.* the point above which the cathode stood. Kundt regarded the phenomenon as a consequence of the orientation of the metal particles, and designated it as double refraction, although he clearly recognized the difficulty of the conception of double refraction in an isotropic metal. Although his films appeared homogeneous under the microscope, it appears probable, in view of the more recent investigations, that they were in reality metal gratings, the elements of which were arranged radially around a central point.

Another method of obtaining metal gratings of extreme fineness was devised by Braun, based upon an observation made by Ambrown, that thin sections of pine wood stained with chloride of gold exhibited dichroism. The color of the transmitted light varied with the direction of the polarization of the incident light. Experiments were made with sections of various kinds of wood, stained by immersion in a 2 per cent solution of chloride of gold, followed by exposure to light, which reduced the metal in the microscopical cells of the material. The results of the experiments indicated that the metal was deposited in the form of microscopic strips, the whole constituting a metal grating of extremely fine structure. It was also shown by Braun that the absorption of the light by the grating was greater when the direction of polarization was parallel to the direction of the metal strips than when perpendicular to them. This was accomplished by mounting the grating in the bulb of a small air thermometer, and observing the motion of the fluid index when the direction of polarization was changed.

Optical Resonance. — The first experiments on optical resonance were made by Wood (*Phil. Mag.*, April, 1902, October, 1902, August, 1903). It was observed that films of the alkaline metals, deposited on the inner walls of exhausted glass bulbs, exhibited brilliant colors which could not be explained by any of the known laws of interference or diffraction. Examination with the highest powers of the microscope revealed the fact that the deposits were granular, the size of the metal grains being of the order of magnitude of the light-waves. It seemed highly probable that the phenomenon was the optical analogue of the experiment performed by Garbasso with the tin-foil strips, the metal granules having free periods of electrical vibration of the order of the periods occurring in the case of visible radiation. The films are very easily prepared in the following manner:

A number of small bulbs are blown of the form shown in the figure, and a piece of sodium or potassium is cut up under ligroin into blocks about 3 mm. on each edge. These are introduced into the bulbs as quickly as possible after wiping off the fluid, and the stems of the bulbs drawn down to a small bore for subsequent sealing.

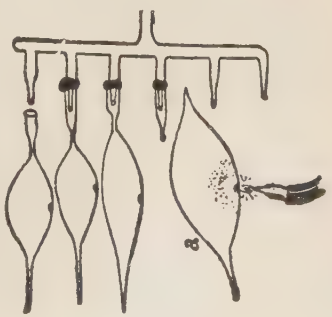


Fig. 375.

They are quickly fastened to the branched tube and exhausted. It is a good plan to heat the metal until it fuses, while the bulb is still on the pump; the whole bulb may be warmed by a Bunsen flame to drive off absorbed air. If the exhaustion is carried down to the point where the mercury begins to hammer, it is generally sufficient. The bulbs are now sealed off from the pump, and may be put away for future use, or experimented with at once. A burner should be made by drawing out a glass tube, which will give a pointed flame about half a centimetre high, and the tip of the flame

allowed to play against the spot on the bulb where the metal lies (Fig. 375, *a*). Sometimes the whole bulb will suddenly flash a deep violet or blue, and sometimes the film will develop more slowly. A chain of three or four bulbs may be made, the sodium heated in one, and the clean molten metal shaken into the others, drops of various sizes sticking to the bulbs. Colored films can then be formed by heating these clean drops in the bulbs. This shows that the oxide present in the first lump has nothing to do with the production of the color.

If the metal is heated at one end of a rather long tapering bulb, the color is most intense near the metal and gradually fades away to nothing at the other end of the bulb. If the bulb is placed in strong sunlight with a black background, it is seen that in some places where the deposit is too slight to show much color by transmission, the light is scattered or diffused, and this diffused light is colored. The claret-colored or purple film, where the deposit is slight, scatters a green light, the surface appearing as if fluorescent. Now the spectrum of the transmitted light in these purple films has a heavy absorption-band in the yellow-green, consequently the scattered light is the complementary color of the transmitted. If the film is greenish blue, the scattered light is reddish. The microscope shows that in these deposits, which have the power of scattering light, the individual particles are rather widely separated, that is, the distance between them is large in comparison to their diameters. The appearance of a bulb in strong light is very much as if certain portions of its interior surface had been painted over with a solution of fluorescein. No trace of regular reflection is shown by these films, except of course the reflection due to the glass. The particles are so far apart that they apparently act as independent sources, the interference necessary for rectilinear propagation not being present. If the incident light is polarized, the fluorescent light (as we may call it for convenience) is also polarized, which is not the case for ordinary diffuse reflection. On the resonance theory, we may regard this fluorescent light as the energy radiated from the resonators, as a result of their forced vibrations. Passing now to a part of the film where the color of the transmitted light is deeper, we find that there is no longer any trace of this fluorescent light. The color absent in the transmitted light is now regularly reflected, the particles being so close together that interference, as imagined by Huygens, takes place.

It appears as if the case was very similar to the hypothetical one considered by Planck in his paper on absorption, which has been already alluded to. It will be remembered that Planck considers the energy stopped by his resonators as reëmitted by them, either as diffused light or regularly reflected light, the diffusion and reflection being of course selective. This seems to be precisely what occurs in the present instance, the particles diffusing or reflecting regularly according to their proximity.

On portions of the bulb close to the heated spot, the metal is deposited in granules too large to show resonance colors, a silky lustre being exhibited by reflected light. This is obviously the ordinary diffusion or diffraction produced by small particles.

Effect of Changing the Surrounding Medium. — It has been shown by Aschkinass and Schaefer that the length of electromagnetic waves to which a system of resonators respond is increased by immersing the resonator system in a medium of high dielectric constant. The same phenomenon occurs in the case of the sodium and potassium films. The bulbs usually contain traces of hydrocarbon vapor, which can be condensed upon the inner wall by touching a spot on the outside with a piece of ice. It is a good plan to moisten the bit of metal with a little ligroin before its introduction into the bulb. The color changes are most remarkable. Pink and purple films become blue, while pale apple-green films change to a deep blue-violet, as deep as dense cobalt glass. Blue films often became perfectly transparent, the absorption-band, originally in the red, moving out of the visible spectrum entirely. Spectroscopic examination showed that the immersion of the resonators in the liquid dielectric caused the absorption-band to move towards the region of longer wave-lengths, as it should do according to theory.

Changes produced by Oxidation. — If the tip of one of the bulbs is cut off, the entrance of the air causes the colored film to vanish like a flash. In some instances a momentary change of color was noticed before the film disappeared. To lengthen the process the expedient was adopted of drawing the end of the bulb out into a long fine capillary, with a bore less than one one-hundredth of a millimetre. In this bulb a film of a deep pink color was formed, and on cutting off the tip of the capillary the color changed to blue, and the film vanished almost immediately. The small amount of air necessary to efface the films is indicated by the fact that at the end of an hour there was still a fairly good vacuum in the bulb, notwithstanding the fact that the end of the capillary had been open all the while. If the process of oxidation be made very slow, by employing a very long capillary, and the process watched under the microscope, the black particles slowly become dim, and finally fade away. The microscope merely shows us the diffraction-disk due to the opaque particle. This becomes dimmer as the size of the particle is reduced, without any apparent change of size. Examination with the spectroscope shows us that, during the process of oxidation, the absorption-band sometimes moves out of the spectrum through the red end, and sometimes merely fades away without any motion.

Colors of Granular Films of Gold and Silver. — To ascertain whether the color effects were common to all metals in a state of fine subdivision, experiments were made with gold and silver, the former obtained by electrical discharges in high vacua from a gold cathode, the latter by employing Carey Lea's solution of allotropic silver.

The color of the gold deposit varies with the conditions under which the deposition takes place. Gold cathodes of two forms were employed, a flat plate about 3 cms. square, and a thick wire, screening off the radiation from all but the tip with a mica screen. The most interesting deposits were obtained from the small source.

In one instance the film showed a brilliant green surface color, resembling fuchsine, the transmitted light having a purple tint. Owing to the transparency of the film a good deal of white light is mixed with the selectively reflected light; this can be cut off with a Nicol, if the reflection takes place at the polarizing angle for glass, and the colored light from the film, which is unpolarized, then appears in great purity. One plate showed patches of brilliant carmine red, deep blue, and green, of a surprising intensity and saturation. The color of the selectively reflected light depended somewhat on the angle of incidence, a phenomenon observed also in the case of the sodium and potassium films. Increasing the angle of incidence changed the color from green to blue; the period of vibration of the resonator system appears therefore to be less when the angle of incidence is large.

If the glass plate is placed near the tip of the gold wire, the green deposit, similar to gold leaf in its optical properties, is deposited at the centre. The color of the green film is probably due to the same causes which operate in the case of gold leaf, *i.e.* to molecular resonance. These films are not granular, the metal vapor not condensing into drops before reaching the glass. Surrounding this is a film appearing light yellow by transmitted light, and bluish by reflected light. This seems to be what we should expect, for the smallest particles, which will resonate for blue light, will be deposited when the distance from the cathode is a little greater than that at which the molecular deposit occurs. Increasing the distance, we get larger particles, and the point of maximum resonance moves up into the green, giving us a purple film with green surface color. At a still greater distance we get particles large enough to resonate for red, and the film appears deep blue by transmitted light. All of these variously colored films can be changed into the green structureless film by heating. We may regard the change as due to the fusing together of the resonators. Silver films showing brilliant colors can be prepared by employing a solution of so-called allotropic silver described by Lea. Three solutions are prepared: a 30-per-cent one of ferrous sulphate, a 40-per-cent one of sodium citrate, and a 10-per-cent one of silver nitrate. Fourteen c.c. of the citrate solution are mixed with 10 c.c. of the ferrous sulphate solution, to which are then added 10 c.c. of the silver nitrate solution. A dense black precipitate at once forms, and the whole is at once poured into a filter. As soon as the liquid has entirely run through, the precipitate is washed with 10 c.c. (*not more*) of distilled water. This removes the salts which make the precipitate insoluble. After the water has entirely passed through the filter, about 25 c.c. of distilled water are poured into the filter, and the blood-red solution which runs through collected. As it does not keep very well, it is best to prepare it on the day on which it is to be used.

A sheet of glass is washed clean, rinsed with fresh water, and the wet surface rubbed over with some shreds of gelatine. It is then drained for a few seconds and dried on a hot plate. A little of the silver solution is flowed over it, the surplus being drained off. If too much gelatine has been used, precipitation is apt to take place,

the deposit taking the form of floating shreds of a reddish membrane. If no considerable precipitation occurs, the plate, which should have been quite warm when flowed, is placed once more on the hot plate until dry. The films formed in this way are usually deep red in color, though sometimes patches of deep violet form, with sharply defined edges. Violet patches may be easily formed in the following way: When the plate is about half dry, with a steaming film and a few small pools of the hot solution, it is removed from the hot plate, held at an angle, and treated with a few drops of alcohol, which are allowed to run down across the still damp portion of the plate. These portions speedily dry into a most gorgeous mosaic of red, purple, and violet patches, the experiment being especially striking in the lantern, as it occupies but a few seconds, and the color-display spreads over the plate like the blaze of a sunset.

Any desired depth of color can be obtained with these films by merely allowing more or less of the solution to remain on the plate. Some are of such a deep red that they are almost opaque, a gas-flame being barely visible through them. The light which does get through is regularly transmitted, that is, the films are not turbid media. The spectroscope shows that the absorption band is wider and less sharply defined than is the case with some of the purple potassium films, which have a rather narrow and very black band in the yellowish green. This can be explained by assuming that there is not a great regularity in the size of the particles, and consequently less sharp selective resonance. These films are transparent to the entire ultra-violet region, even down to the last cadmium lines, of wave-length 22 or thereabouts.

Anomalous Dispersion of the Films. — Prisms were made by the evaporation of the silver solution between a plate of glass and a piece of glass tubing, the method being similar to the one employed by Pflüger in the preparation of cyanine and fuchsine prisms. If the colors are due to resonance, anomalous dispersion should be observed in crossing the absorption-band, which in the case of the silver films occurs in the green.

This was found to be the case, the prisms deviating red light in a measurable degree, though transmitting blue light without sensible deviation. It is highly probable that a similar phenomenon would be shown by the sodium and potassium films, though these have not been investigated up to the present time owing to the experimental difficulties involved.

The Scattering of Light by Sodium and Potassium Fogs. — Allusion has been made to the deep violet light scattered by a condensing cloud of sodium vapor. The author has frequently observed that the color of the light transmitted through the sodium tubes, in the experiments upon the optical properties of the vapor, was colored a deep yellow instead of blue, as is usually the case. It was difficult to understand this at first, since the vapor is perfectly transparent to blue light, and somewhat less so to yellow-green light. The cause was finally found to be a scattering of the violet and blue rays by the fog of condensing vapor, which was so powerful that none of these rays was transmitted. The phenomenon was

investigated further with the large tube used in the experiments on fluorescence, some potassium being introduced into the retort. The light from the arc was focussed about 20 cms. in front of the retort and the tube powerfully and rapidly heated by a blast lamp. Under this condition the vapor is puffed out in clouds from the aperture of the retort, and the condensing clouds are most wonderfully colored, red and orange predominating. It was found that if the heating was carefully regulated, a steady state could be maintained in which the fog scattered red light at the outer boundary, where the cone of rays entered it, yellow a little farther in, and green at the point where it was first forming and where the cone of rays passed out into the vapor in which it was of course invisible. The cone of rays seen from the side, which can be accomplished by looking in at the edge of the glass window, resembled a spectrum, the blue end of which was wanting. A potassium fog thus scatters longer waves than a sodium fog, and the scattering power is confined to a comparatively narrow region of the spectrum, which apparently varies with the size of the particles in the fog. It is probable that we are dealing with something analogous to the phenomenon exhibited by the granular deposits of these metals already described. Further investigations along these lines should be made, especially with polarized light. A large brass tube without a retort, or even a glass tube, could be used.

Colors of Lippmann Photographic Plates. — An exhaustive study has been made by Kirchner (*Ann. der Physik*, 13, p. 239, 1904) of the colors exhibited in transmitted light by Lippmann plates. These plates, after development, appear reddish brown, green, or blue, according to the developer used. The color phenomena have nothing to do with the presence of silver laminae, which are chiefly responsible for the colors seen in reflected light in Lippmann's color photographs, for they are equally pronounced in the case of plates immersed in benzol during their exposure to light. Standing waves and the formation of the laminae are of course out of the question in such a case.

Kirchner came to the conclusion that the color was due to the resonance of the minute silver aggregates reduced in the film by the action of the developer. In ordinary photographic plates these are too large to serve as resonators for visible radiations, and the plates appear black in consequence, the light being stopped by the opaque masses.

The position of the absorption-band was found by Kirchner to shift when the dry films were moistened, the band moving towards the blue. This is in accordance with theory, for the refractive index of wet gelatine is much less than that of dry. The dispersion of the colored films was also investigated and found to be anomalous. The refractive indices for the various wave-lengths are given in the table on opposite page.

The absorption-band begins in the red and extends to the blue, increasing in intensity. There is a decrease in the values of n, as the observations are pushed into the band, which is in accordance with theory, and the maximum value of n is found on the red side of the

DISPERSION OF REDDISH BROWN SILVER FILM

λ	500	525	550	560	570	575	589	600
n	1.542	1.557	1.567	1.585	1.595	1.601	1.555	1.535

band, though perhaps farther within the band than is usually the case with absorbing media.

These results are interesting, as confirming those obtained by Wood with granular deposits of the alkali metals and allotropic silver. Kirchner's work was practically completed when the paper above referred to was published, his results having been obtained independently.

The subject of optical resonance has also been studied extensively by Kossonogoff, whose papers appeared in the *Physikalische Zeitschrift* for 1903. Resonating films were prepared by blowing a spray from an atomizer charged with a solution of the metal salt upon a strongly heated glass plate. He also obtained evidences of resonance in granular deposits of non-metallic substances, and made a careful study of the colors of the wings of butterflies, which he regards as due to similar causes. Bock, in the same journal, has published results which are claimed to show that minute water-drops also exhibit the phenomenon.

Colors in Metal Glasses and Metallic Films. — An extremely interesting treatment of the colors exhibited by glasses which are stained with metallic oxides, and by the films of the alkali metals deposited in vacuo, has been given by Garnett (*Phil. Trans. of Roy. Soc. Lond.*, Series A, vol. 203, p. 385). Colored glasses are supposed to owe their color to the presence of minute globules of the metal. Colloidal solutions of the metals act in a similar manner, and recently the presence of the particles has been detected in both cases by Siedentopf and Szigmondy, by means of their so-called ultra-microscopic method, which is merely oblique illumination pushed to the limit. A powerful beam of light issuing from a horizontal slit is brought to a focus by means of a microscope objective, within the glass or liquid under investigation. The small particles scatter some of this light and appear as minute diffraction disks of light, when a microscope is focussed on the illuminated plane (Fig. 376). The phenomenon is analogous probably to the scattering of light by the air molecules or small particles suspended in the atmosphere, the action of which has been exhaustively treated by Lord Rayleigh. A simpler device, due to Cotton, is shown in the lower part of the figure, in which the illuminating beam is prevented from entering the microscope by total reflection.

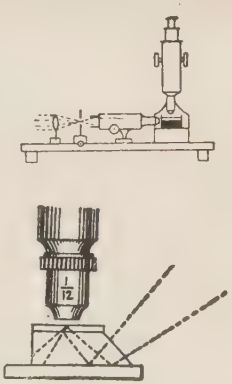


FIG. 376.

Garnett shows that the colors of the glasses can be accounted for by the presence of the small metal spheres, and explains a number of very curious effects observed by Siedentopf and Szigmondy, some of which we will now consider.

Polarization Effects of Ultra-Microscopic Particles. — Some very curious and interesting effects were observed by Siedentopf and Szigmondy in the case of gold particles when the illuminating beam was plane-polarized. If the plane of the vibration was perpendicular to the plane containing the illuminating ray and the microscope, the little diffraction disks appeared everywhere in the field, and were of uniform illumination. The scattered light was polarized in the same plane as the incident. This condition is shown at *a* (Fig. 377). The field of the microscope is represented by the large circle, and the direction of vibration of the scattered light by the arrows. The appearance of the diffraction disks is shown below. If, however, the incident light vibrated parallel to the above defined plane, the particles scattered no light in the vertical direction, and the diffraction disks were formed by the oblique rays gathered in by the objective. In this case each diffraction disk appeared with a black spot at its centre as shown. Moreover, since the light scattered in any given direction by the particles comes to a focus at a given point in the second focal plane of the microscope, a black spot will be found at the centre of this plane. This condition is shown at *b*. If the incident vibration is inclined at 45°, the spot appears on the sides of the diffraction disks, and there is a dark region on one side of the second focal plane as in *c*. The direction of the vibration at the second focal plane is indicated in each case by the double-headed arrows in the large circles.

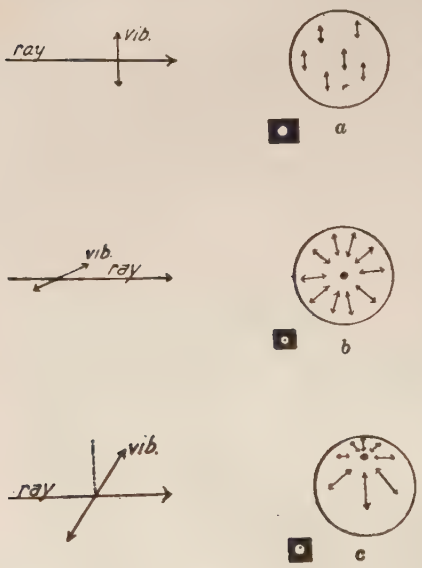


Fig. 377.

The explanation of these appearances was not given by Siedentopf and Szigmondy, but Garnett discusses them very fully in the paper referred to. Another very remarkable effect observed was the splitting of each diffraction disk into a pair, when the gold particles were not smaller than 1 μ. One of the disks was reddish, the other yellow-green, and they were all oriented in the same way, showing that the effect was one of diffraction, and not due to any actual dual structure of the minute particles. No explanation of this curious phenomenon has been given, but it is probably related in some way

to the diffraction of colored light (red) by thin plates of gold observed by Wien.

The effects described indicate that the gold particles must be spherical, for oblong particles would not completely polarize the scattered light (as Lord Rayleigh has shown in his papers dealing with the polarization of the light of the sky), and we should not observe the dark spot in the diffraction disks, or the dark region in the second focal plane of the microscope.

We will now consider the effects of these small gold spheres upon the color of the transmitted light. The subject has been very fully discussed by Garnett, who has investigated it from the standpoint of the electro-magnetic theory. His treatment is much too long to give in full, but we can examine to advantage the general method of attack and some of the conclusions.

Let light of wave-length λ fall on a metal sphere of radius a, refractive index n, and absorption coefficient κ. Further, let

$$N \equiv n(1-i\kappa) = \sqrt{\epsilon'},$$

ϵ' being the complex dielectric constant.

This case has been considered by Lord Rayleigh (*Phil. Mag.*, xliv., p. 28, 1897), who showed that the sphere excited by a periodic electric force E_0 emits the waves which would be emitted by a Hertzian doublet, which at time t was of moment equal to

$$\frac{N^2-1}{N^2+2}a^3E_0.$$

If there are a large number of spheres in close proximity, the electric force exciting each one will be E', *i.e.* the force E_0, together with forces due to the neighboring doublets. This force E' causes the polarization

$$f(t) = a^3\frac{N^2-1}{N^2+2}E'.$$

If the average moment of a doublet be $f(t)$, and there are n doublets per unit volume, the polarization of the medium will be $f'(t) = nf(t)$.

By means of analyses by Lorentz and by Larmor it can be proved that

$$E' = E_0 + \frac{4\pi f'}{3} = E_0 + \frac{4\pi}{3}na^3\frac{N^2-1}{N^2+2}E',$$

provided the doublets are distributed through a space large in comparison to the wave-length.

This gives us

$$E' = \frac{E_0}{1-\frac{4\pi}{3}na^3\frac{N^2-1}{N^2+2}}, \text{ so that } f = \frac{E_0\frac{N^2-1}{N^2+2}a^3}{1-\frac{4\pi}{3}na^3\frac{N^2-1}{N^2+2}}.$$

By substitution of these units in Maxwell's equation, the complex dielectric constant of the medium containing the spheres is found to be

$$\epsilon' = 1 + \frac{3D\dfrac{N^2-1}{N^2+2}}{1-D\dfrac{N^2-1}{N^2+2}}.$$

in which D is written for $\frac{4\pi}{3}na^3$, denoting the volume of the metal per unit volume of the medium. This is for spheres in vacuo: in glass of refractive index μ the equation becomes

$$n'(1-i\kappa')^2 = \epsilon' = \mu^2 + \frac{3\mu^2 D\dfrac{N^2-\mu^2}{N^2+2\mu^2}}{1-D\dfrac{N^2-\mu^2}{N^2+2\mu^2}}.$$

The optical constants of the medium n' and κ' thus depend only on D, the relative volume of the metal, and not on the size of the spheres, restricting them, however, to sizes small in comparison to λ. By reducing the above equation and substituting in it the values for N and μ, the absorption coefficient can be found for any given value of D.

Now D varies with the nature of the glass. The gold glass as first prepared is colorless, becoming red on reheating, the process causing the metal spheres to form within the body of the glass. "Excretion of the metal" Garnett calls it. Colorless gold glass turned red on exposure to the emanation of radium, and it is probable that the blue color of X-ray tubes, and tubes which have contained radium, is due to the excretion of metallic potassium or sodium by the radiation. Sir William Ramsay exposed glass containing silver to radium rays and found that it turned yellow. Quartz glass is not colored, as no metal is present.

Elster and Geitel (*Wied. Ann.*, 59, p. 487, 1896) found that salts of the alkaline metals, colored by the action of cathode rays, exhibited photo-electric properties, which suggested the presence of free metal; this supports the view held regarding the coloration of glasses by X-rays and radium rays.

Garnett took the values of N calculated from Drude's tables of the optical constants of the metals, and the values of D calculated from the total gold content of the glass, and the observations of Siedentopf and Szigmondy, and showed that the medium should be much more transparent for red than for yellow light.

Values of n and κ for green and blue light not being available, the best that could be done was to infer that, since yellow is less freely transmitted than red, the medium is still more opaque to green and blue.

Garnett next develops an expression for the intensity of the

scattered light, and finds that the amplitude at any point of the light emitted from a sphere is proportional to

$$\left|\frac{N^2-\mu^2}{N^2+2\mu^2}\right|\frac{a^3}{\lambda^2}.$$

The λ^2 in the denominator indicates that the scattered intensity increases with the inverse 4th power of the wave-length, but that it is also dependent on N, *i.e.* on the optical constants of the metal.

Calculations showed that yellow light would be scattered more powerfully than red, from which it was inferred that green would be still more powerfully scattered, which is in agreement with the observations of Siedentopf and Szigmondy.

Certain types of gold glass scattered a muddy red light, however. In this case the particles are probably so large that they reflect light in the ordinary sense, and, as we know, gold reflects red light in greater excess than any of the other colors.

In an appendix to the paper, the transmission of gold and silver glass has been calculated for red, yellow, green, and blue light, from values of n and κ given by Rubens. The colors, in the order of the degree in which they were transmitted, were found to be: for gold glass, — red, yellow, blue, green; for silver glass, — yellow, red, green, blue. Certain gold glasses appear blue by transmitted light, and it appears probable that large particles (diameter $>.0001$), by reflecting out the red and orange, give the glass a blue color.

Colors of Granular Metallic Films. — In the second part of the paper above referred to, Garnett examines the conditions which hold in cases where the metallic granules are deposited in thin films. The equations already given were developed on the assumption of a uniform polarization in the medium, which is only the case when the spheres are distributed in three dimensions. For a two-dimensional distribution, in the xy plane, it is shown that the complex dielectric constant in the direction of the xy axes is the same as for the medium in bulk, whereas the constant in the direction of the z axis may be quite different. If this were the case the film would behave like a doubly refracting substance, the " optic axis " being perpendicular to the film.

It is found that for films of thickness greater than $\frac{2}{3}$ of λ the absorption is governed by $n\kappa$, while in the case of films less than $\frac{1}{2\,5}\lambda$ it is governed by $n^2\kappa$. Curves are given showing how the absorption depends on D, the volume of metal per unit volume of the medium. The values of $n\kappa$, or $n^2\kappa$, are plotted as ordinates, and the values of D as abscissae. In the case of a non-granular film of solid metal it is evident that $D=1$.

Garnett was able to explain all of the effects observed by Wood in the case of the sodium and potassium films deposited in exhausted bulbs, at least all of the effects which fell within the scope of his equations. The curves for a potassium sodium amalgam are given in Figs. 378, 379, and show how the absorption depends on the value of D. The upper curve represents the condition for a thick film. For $D=1$, *i.e.* solid metal, the absorption is strongest for red and

weakest for blue. For $D = .5$, that is, for a film with equal volumes of metal and empty spaces, the absorption is strongest for yellow,

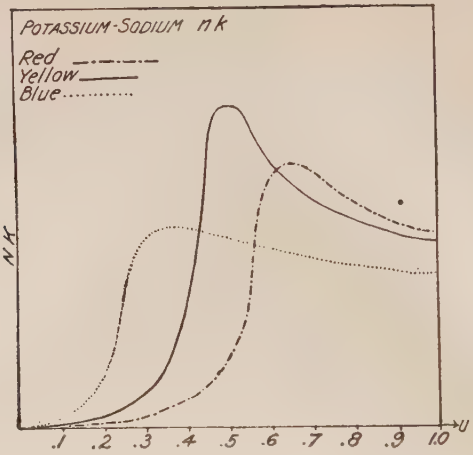


FIG. 378.

while for $D = .3$ the blue is practically the only color absorbed. For thin films we find that for $D = .5$ the yellow is very powerfully

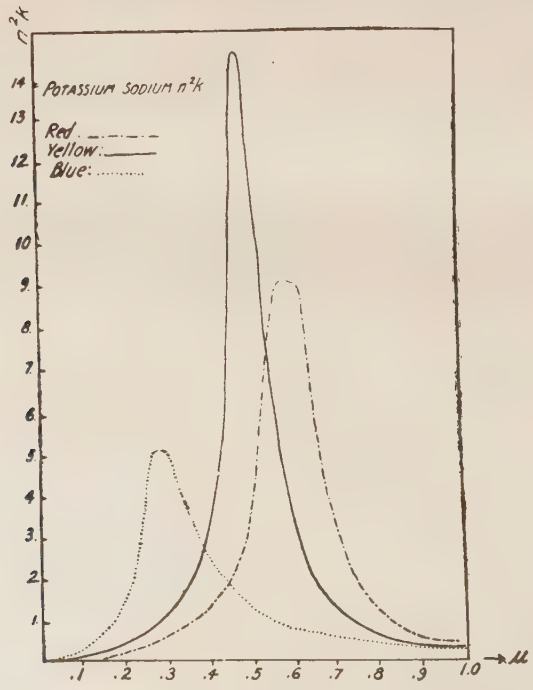


FIG. 379.

absorbed, which agrees with Wood's observation that when the conditions were such as to cause an absorption-band in the yellow, the band was much blacker and narrower than when it occurred in some other part of the spectrum.

The graphs for gold indicated that for $D=1$ the color in the case of very thin films of gold leaf should be blue. This was the color observed by Mr. Beilby in the case of the thinnest leaf which could be procured. For thick films the graphs showed that the color of the transmitted light should be green. Many very curious and interesting changes, observed by Mr. Beilby (*Proc. Roy. Soc.*, 72, p. 226), in the colors of thin gold films, produced by heating and annealing, are discussed and explained by Garnett, whose paper is by far the best on the optical properties of metallic films which has appeared up to the present time.

It appears that the theory of optical resonance can be regarded as in a measure confirmed by these investigations, though the phenomenon is very much more complicated than in the case of large resonators and electro-magnetic waves. The optical constants of the metal enter as a factor, and for very small particles at least the absorption depends not on their size, but on the total bulk of metal in unit volume.

It is probable that very definite and more easily interpreted results can be obtained by experimenting with very long heat-waves, either with cross-ruled films of metal or fine metal powders, along the lines indicated on page 632.

CHAPTER XXIII

THE NATURE OF WHITE LIGHT

Previous to Newton's classic experiments on the decomposition of white light by a prism, it had been supposed that every refraction of light actually produced color, that is, the color was produced within the dispersing piece. Newton's experiments apparently showed, however, that the colors were actually present in the original light, the function of the prism being merely to separate them. At the present time, however, the view most generally held resembles in some respects the idea which prevailed previous to the time of Newton. Colored light implies a greater or less amount of regularity in the sequence of the waves. If a grating or prism yields us light which the eye is unable to distinguish from the light of the sodium flame, it is pretty certain that the luminous impulses are following one another at approximately the same intervals, as in the case of the radiation from the sodium molecule. The question now is, " Were these regular wave-trains present originally in the white light, or have they been manufactured by the grating? " If the former is true, how are we to regard a vibration which is made up of an infinite number of regular trains of waves, each, however, of different wave-length? Obviously the motion of the ether must be regarded as the resultant of all the component trains, and the regularity of sequence, as we usually understand it, would disappear wholly in wave-motion of this nature.

If, however, the regular wave-trains, or colored rays, are manufactured by the prism, we very naturally ask whether Newton's discovery was really a discovery after all.

The problem can be attacked both from the experimental and theoretical standpoint. Let us see first what evidence experiment furnishes.

Interference fringes were obtained by Fizeau and Foucault with white light and with a path-difference of some 50,000 wave-lengths. This experiment has been held, by many eminent authorities, to indicate that the colored components of the white light consist of regular wave-trains, the vibrations of which take place without sensible change of phase for at least 50,000 complete periods.

The first to question this conclusion was Gouy (*Journ. de Phys.*, 5, p. 354, 1886), who showed that the experiment of Fizeau and Foucault showed nothing whatever about the regularity of the vibrations in white light, the number of fringes observable, or the allowable path-difference, depending solely upon the resolving power of the spectroscope used for their detection.

Gouy made use of Fourier's theorem, by which any function $f(x)$, between limits $x=-c$ and $x=+c$, may be represented by the sum of a series involving the sines and cosines of $\frac{\pi x}{c}$.

$$f(x)=a_0+a_1\cos\frac{\pi x}{c}+a_2\cos 2\frac{\pi x}{c}+\cdots$$
$$+b_1\sin\frac{\pi x}{c}+b_2\sin 2\frac{\pi x}{c}+\cdots.$$

If the medium is free from dispersion we can determine the shape of the resultant disturbance at a point beyond $+c$, if the component waves are travelling in the positive direction, by substituting $x-vt$ for x. If we now add the series we find it equal to $f(x-vt)$, which shows us that waves of any shape are transmitted in a non-dispersive medium without change of form. If, however, the medium is dispersive, the component waves, each one of which is represented by a term in the series, travel with different velocities, and the form of the resultant disturbance changes with the time, *i.e.* it is not propagated with its type unchanged.

Gouy showed that the fringes observed by Fizeau and Foucault could be explained as well on the hypothesis that the original disturbance was a single pulse, or a series of waves of irregular form, *i.e.* not sine waves. A pulse or single irregular wave can be represented by Fourier's theorem, as the resultant of a large number of sine waves which extend to infinity on either side of the pulse. The spectroscope will spread this disturbance out into a spectrum, and at every point of the spectrum we shall have a periodic disturbance. In other words, the spectroscope will sort out the Fourier components into periodic trains of waves, just as if these wave-trains were really present in the incident light. We can perhaps get a clearer idea of the two theories of white light in the following way: Suppose our source of light to consist of a number of electrons vibrating in simple harmonic motion, but with different periods. The resultant disturbance at any point will have an irregular form, which we can represent by a curve something like that shown in Fig. 380. The regular periodicities due to the electrons are present in the disturbance, and a prism or grating will sort them out into a bright line spectrum of simple harmonic trains, precisely as Fourier's analysis does.

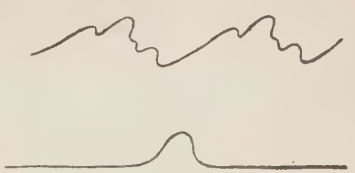


Fig. 380.

We can, however, consider the same type of disturbance as given out by a single electron, if it is constrained to vibrate in an irregular manner, and the spectroscope must necessarily resolve this disturbance in precisely the same manner as before, since the disturbances are identical.

Suppose now that our disturbance consists of a single pulse, such as is shown in the lower part of Fig. 380. By Fourier's analysis it

can be represented as the resultant of a large number of simple harmonic waves, differing in wave-length by infinitesimal steps, and with amplitudes suitably distributed. We can think of this disturbance as originated in two different ways: from a group of continuously vibrating electrons, or from a single electron executing a half vibration with a motion obeying such a law as to give a wave-form of the requisite shape. In the first case the component wave-trains can be regarded as having an actual existence, the periodicities being actually present both in the source and in the disturbance. In the second, there is no periodicity at the source, and consequently none in the disturbance. Though this way of looking at the matter is not quite rigorous, it may help us in getting a clearer idea of the distinction between the old and new idea regarding white light. In the last two cases considered, the spectroscope will give us a continuous spectrum, in which the intensity at any point is proportional to the square of the amplitude of the corresponding wave-length. In the first case this wave was present in the train, and had a definite origin; in the second, it can only be considered present in a mathematical sense.

Gouy's conception of white light was criticised by Poincaré (*Compt. Rendus*, 120, p. 1895), both from a mathematical and experimental standpoint. Since the Fourier components representing the disturbance must extend to infinity in both directions, they ought to appear in the spectroscope not only after the source of light has been extinguished, but even before the source is lighted, an obvious absurdity. Poincaré was of the opinion that the experiment of Fizeau and Foucault indicated the presence of a high degree of regularity in the disturbances constituting white light.

These objections were met by Gouy, and also by Schuster (*Compt. Rend.*, 120, pp. 915, 987), who had arrived at conclusions similar to those held by Gouy. We find the same idea expressed by Lord Rayleigh in his article on "Wave Theory," written in 1888. "The light," he says, "may be highly composite and homogeneity brought about with the aid of a spectroscope. The analogy is closest if we use a spectroscope to give us a line of homogeneous light in simple substitution for the sodium flame (as a source of light producing interference fringes). Or following Foucault and Fizeau, we may allow the white light to pass (*i.e.* enter the interference apparatus), and subsequently analyze the mixture transmitted by a narrow slit in the screen upon which the interference bands are thrown. *In either case the number of bands observable is limited solely by the resolving power of the spectroscope, and proves nothing with respect to the regularity or otherwise of the vibrations of the original light.*" He shows further that when achromatic bands are formed by using a diffraction spectrum as a source, and duplicating it with Lloyd's mirror, the number of bands possible is still limited by the resolving power of the instrument used to form the spectrum.

If we go back to the source of white light and try to picture the nature of the disturbances there which would be necessary to give us the two types of radiation which we have discussed, we shall incline towards the ideas of Gouy and Rayleigh. If the light con-

tains periodic trains, which are regular over a length of some thousands of wave-lengths, there must be present in the source electrons or vibrators of some sort vibrating with all possible periods, otherwise there would be dark lines in the spectrum corresponding to the absent frequencies. The smallest particle of solid matter which we can command, when raised to incandescence, gives us a continuous spectrum. The visible region of the spectrum, say 7000 to 4000, comprises 3000 Ångström units. With the best spectroscopes we can easily resolve to $\frac{1}{10}$ of a unit, consequently the continuity of the spectrum implies that there must be at least 30,000 electrons, no two of which vibrate in the same period. If the molecules are moving to and fro, the Doppler effect will of course alter the wave-length, consequently we can diminish the necessary number somewhat. From what we know about electrons, however, it is difficult to see how we can have a very large number vibrating regularly in totally different periods, and we naturally incline towards the idea that the disturbances in a source of white light are irregular in character.

Theory of Damped Vibrations. — Another idea was put forward by Garbasso (*Arch. de Genéve*, vol. 4, p. 105, 1897), who considered white light to be the result of a heavily damped vibration, represented by

$$f(t) = e^{-kt} \sin ht.$$

We have seen that the damping due to radiation is very small, but collision between the molecules, if frequent enough, may accomplish the desired result. In gases the time elapsing between molecular impacts is very large in comparison to the period of the vibration, and we have long trains of approximately homogeneous waves thrown off between collisions. In the case of solids and liquids the conditions are quite different, the molecular excursions being extremely short.

One way of testing the hypothesis is to develop the expression representing the damped vibration by Fourier's theorem, and see whether the intensity distribution in the resulting mathematical spectrum corresponds with the distribution of energy observed in the case of incandescent solids. This was done by Carvallo (*Compt. Rend.*, 130, p. 79, 1900).

$$f(t) = e^{-kt} \sin ht \text{ (for } t > 0 \text{ but } f(t) = 0, \text{ for } t < 0).$$

Developed by Fourier's theorem.

$$f(t) = \frac{1}{\pi} \int_0^\infty \frac{h dq}{\sqrt{(q^2 - h^2 - k^2)^2 + 4k^2q^2}} \cos\left[qt - \arctan \frac{2kq}{q^2 - h^2 - k^2}\right].$$

The intensity of a vibration of period $\frac{2\pi}{q}$ is, according to the formula,

$$y = \frac{h^2q^2}{(q^2 - h^2 - k^2)^2 + 4k^2q^2}.$$

If we put $k^2 = a^2h^2$ and $q^2 = (1+a^2)h^2u^2$, the above becomes

$$y = \frac{1}{(1+a^2)\left(u - \frac{1}{u}\right)^2 + 4a^2}.$$

In this form the properties of the function appear. For $u=0$ and $u=\infty$, $y=0$. It attains its maximum $\frac{1}{4a^2}$, for $u=1$.

Finally, two values of u which are equal but of opposite sign give us the same intensity. The same is true for equal but opposite values of the logarithm of the wave-length.

Carvallo then constructed a curve with the intensities as ordinates and the values of log λ as abscissae. The lack of agreement between this curve and the curve plotted from the observations of the energy distribution in the spectrum, made by Mouton and by Langley, indicated that the hypothesis of a damped vibration was inadmissible. Another objection was raised by Carvallo, who showed that a grating would yield a band of white light instead of a spectrum if the incident light consisted merely of damped vibrations.

Gouy (*Compt. Rend.*, 130, p. 241) comes to a different conclusion, objecting to the treatment of Carvallo, in that he extended his analytical treatment from $-\infty$ to $+\infty$, a condition which could not be realized in experiment. Limiting the number of disturbances falling on the grating to a small number n, which must be the case when the vibrations are heavily damped, he shows that the disturbance at any point will have the periodicity calculated from the ordinary laws of the grating, and not, as imagined by Carvallo, the same periodicity as the original damped vibration.

Carvallo in his reply (*C. R.*, 130, p. 401) proposes an interesting acoustical experiment to settle the question. Let the source of light be represented by a large tuning-fork, driven electrically. The waves from this are to be received by a large concave grating made of broad slats with open spaces between. As long as the vibrations are maintained by the electrical mechanism we should find points of maximum intensity with silent spaces between, at the focus of the grating, corresponding to the spectra produced by a grating when illuminated with monochromatic light. If, however, the current is suddenly turned off, the vibration is damped, and we should, if Gouy's hypothesis is correct, find a faint sound of varying pitch all along the region between the points previously occupied by the maxima: in other words, a sound spectrum. This effect might be detected by a Helmholtz resonator placed at the proper point in the spectrum. If, however, Carvallo's notion is the correct one, the region between the maxima would still be a region of silence, or at least only yield a faint sound corresponding in pitch to the pitch of the fork.

The experiment would be a difficult one to perform on account of the enormous dimensions of the apparatus and the difficulty of protecting the ear from the direct sound of the fork.

Type of the Impulse constituting White Light. — If we regard

white light as a series of impulses, without regularity, the impulses cannot be regarded as arbitrary, *i.e.* of any form, for, as Lord Rayleigh has pointed out, there would in this case be no way of distinguishing the radiations corresponding to different temperatures. He considers (*Phil. Mag.*, xxvii., p. 460, 1889) the simplest type of impulse that could meet all the requirements of the case to be the one with which we are familiar in the theory of errors, viz. (Fig. 381):

$$y = e^{-c^2x^2}.$$

Such an impulse, he remarks, can be considered as the resultant of a very large number of localized simultaneous impulses, all *aimed* at a single point ($x = 0$), but liable to deviate from it, owing to accidental causes. This disturbance he resolves into its elements by means of Fourier's theorem, and then finds the energy carried by each component. By assuming an infinite number of these impulses, of the same form but unequal magnitude, he obtains probable values of the partition of the energy among the various wave-lengths, which agreed fairly well with Weber's law, which at the time best expressed the energy distribution in the spectrum. Wien's law could be satisfied by an impulse of some other definite form.

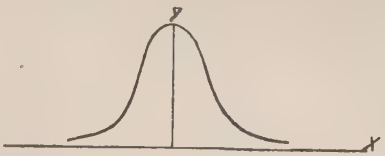


FIG. 381.

The character of the disturbance is thus fixed by this distribution of energy in the spectrum, and Wien's law marks the limit of our knowledge regarding the nature of white light. Planck definitely states that this will forever mark the limit. Further analysis, he says, will be based on a reasoning comparable with that involving a contradiction of the second law of thermodynamics, in which the aid of Maxwell's demons was invoked. In the kinetic theory of gases we are obliged to confine our investigations to the average effect of molecular impacts, and we must, in dealing with the present problem, consider only the average effects of the light disturbances, extending as they do over relatively long intervals of time.

As we shall see presently, the dispersion by prisms and gratings can be accounted for without assuming the presence in the light of any periodicity whatever. Up to the present time no experiment has been devised capable of proving or disproving the presence in white light of regular wave-trains.

Interference Experiments in the Light of the Pulse Hypothesis. — In the Chapter on Interference we have treated all of the problems by tacitly assuming the presence of regular wave-trains. We will now examine a number of cases and see whether the observed effects can be accounted for on the hypothesis that white light consists of irregular pulses. Can we, in other words, account for the colored fringes seen with Fresnel's mirror or two slits, assuming the incident light to consist of a single pulse?

Schuster shows, in a long and interesting paper (*Phil. Mag.*, June 1894), that the above question can be answered in the affirmative. It is easy to see how periodicity can be manufactured by a grating or prism, but there are a number of cases which at first sight may seem irreconcilable with our hypothesis. If a pulse falls upon a pair of Fresnel mirrors, we shall have at a specified point in space two pulses, the interval between them depending upon the position of the point. If we receive the double disturbance upon the slit of a spectroscope, the prism or grating draws out each pulse into trains of periodic waves, and maxima and minima due to the interference of the two sets of waves appear. But even without the spectroscope a number of colored fringes can be seen, and it may appear impossible to account for these on the hypothesis that we are dealing merely with a pair of pulses.

Schuster shows that the interference in this case is a physiological effect, due to a peculiarity of the eye. The retinal elements can be regarded as tuned to the three primary colors, and we are obliged to consider each element as containing some sort of a vibrating system, which responds to the light-waves. If the retinal vibration has a period of its own, which seems probable, since it responds to certain wave-lengths and not to others, it is not difficult to see how interference takes place when two pulses strike the element in succession. The effect on the vibrator will depend on the time elapsing between the two impacts. The first pulse starts the vibration, and the second increases or annuls it according to the state of the vibration when the second pulse arrives. The phenomenon is thus seen to depend upon the fact that a periodic disturbance is set up in the eye, which lasts until the second pulse arrives. Schuster explains the fact that the fringes can be photographed in the same way, the vibrators in this case being located in the molecules of the silver salts.

Schuster's explanation may seem a little fanciful at first sight, but the reasoning is perfectly logical, and the assumptions are not at all improbable. We have a perfectly analogous case in an experiment of Hertz with electrical waves, which may be cited in this connection. The analogy is obvious, though it does not appear to have been noticed.

Hertz found that when his electrical waves were reflected from a wall, the resonator sparked when placed at certain definite distances from the wall, while no sparks were *observed* in intermediate positions. He drew the erroneous conclusion that he was dealing with stationary waves, formed by the interference of the direct and reflected waves. His experiment appeared to indicate that there was a definite periodicity present in his electrical radiation, just as the interference fringes observed with Fresnel's mirrors appear to indicate a certain amount of periodicity in white light.

Other experimenters found, however, that the positions of maximum sparking depended not at all upon the dimensions of the vibrator or source of the radiation, but solely upon the size of the wire loop, which served as a resonator, and that the same effects would be observed if the radiation consisted of a single pulse only. The ex-

planation of the phenomenon is found in the fact that the vibrations of the resonator persist for some time, and the intensity of its sparking depends upon the state of its vibration at the moment when the reflected pulse meets it. At the given point in space the condition of the vibration of the resonator, when the reflected wave meets it, will obviously depend upon its period, *i.e.* upon the size of the wire loop. The only difference between this case and the optical one is that in the latter the pulses are travelling in the same direction, while in the former they are going in opposite directions.

The interference fringes observed by the eye or recorded by the photographic plate are thus seen to depend upon a resonance phenomenon. If we could explore the field over which the radiation from the two sources of light is spread, with some instrument not biassed by resonance, no trace of the fringes should appear. The smoked strip of the bolometer is such an instrument: it absorbs all wave-lengths equally well, and is free from resonance effects, roughly speaking at least. Now, it is found that when the bolometer is used to explore the region, no recurring maxima and minima are found, the curve having the form shown in Fig. 382. There is a central maximum bordered on each side by a minimum, beyond which points the curve is practically level. The occurrence of the two minima can be explained by the distribution of energy in the spectrum.

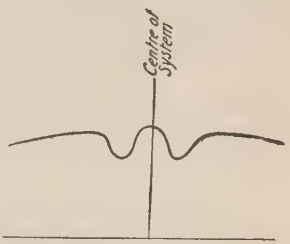


FIG. 382.

"The fact," says Schuster, "that white light shows any objective interference (as with bolometer) without the artificial introduction of regularity is due to the prevalence of certain wave-lengths over others. Whatever regularity there is in the light is intimately connected with the distribution of intensity in the spectrum.

"We cannot help speculating as to the ultimate cause which renders the regularity of vibration a function of the temperature only, and independent of the natural periods of the molecules. Perhaps the solution of the difficulty will be found in the fact that our observations tell us nothing directly as to the vibrations of the atoms or molecules. What we observe is the disturbance of the medium, and the distribution of energy in the spectrum of an incandescent black body, which is in thermal equilibrium, may indicate a property of the medium rather than that of matter. That is to say, the motion of the vibration in the molecule may be perfectly irregular, but the medium may take up and propagate some vibrations quicker than others. There are many signs tending to show that the time is not far distant when, in order to explain the connection between optical and electrical facts, we must recognize some structural properties of the medium, and the regularity in the radiation of a black body may be intimately connected with such structural properties."

Corbino has made the suggestion that the phenomenon of light-

beats obtained with white light by any of the methods originated by Righi or by himself is adverse to the hypothesis of Gouy. The different component rays (sinusoidal disturbances) into which a prism or grating decomposes the complex vibration which constitutes white light, having a common origin, ought to be capable of interfering with one another, producing beats, that is, beats should be obtained by uniting two streams of light taken from two adjacent points of a continuous spectrum. If the light from a narrow white source is divided into two streams, which are received upon the slit of a spectroscope, the spectrum is crossed by dark bands. If now the periods of one of the interfering pencils is altered, say by means of Righi's revolving Nicol arrangement, the spectrum of this pencil, according to Gouy's hypothesis, is merely slightly displaced with reference to the spectrum of the other pencil. Each sinusoidal train takes the place of its neighbor, so to speak, and should be capable of interfering with one of equal wave-length in the other set. The fringes should therefore appear exactly as before, that is, stationary. If, however, the sinusoidal components do not have a common origin, *i.e.* if they originate at the source independently of one another, each set in the modified pencil can only interfere with the set in the unmodified pencil *which had the same wave-length before the modification was introduced.* These two trains now have different wave-lengths, and should therefore produce beats, or moving fringes, which is the phenomenon actually observed.

Analysis of White Light by a Grating. — In considering the action of a diffraction grating when analyzing white light, it will help us

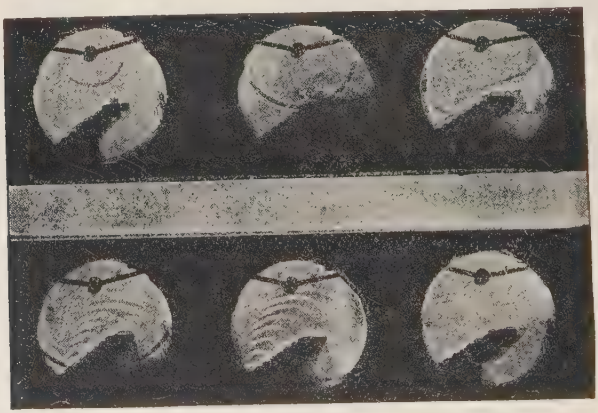


Fig. 383.

to get a clear idea if we bear in mind a well-known acoustical effect. If a sudden sharp noise such as is made by clapping the hands together is reflected from a high flight of steps, the sound comes back to us as a musical note; in other words, the steps impress the element of periodicity upon the reflected disturbance, each step throwing off an echo-wave. These reflected wavelets reach us in succes-

sion, owing to the fact that the distance from our ear to the successive steps increases in arithmetical progression. The formation of a train of waves by the reflection of the sound-wave from a spark from a flight of steps is shown in Fig. 383. This is a photograph of the actual phenomenon, made in the same manner as the photographs illustrating the reflection of waves from curved surfaces, which we have studied in the Chapter on Reflection. The grating acts in the same manner when analyzing white light. This explanation was offered by Young in 1813, but Lord Rayleigh appears to have been the first to make use of the conception in treating grating problems. The ruled lines which constitute the grating prevent the "shadow-producing interference," as imagined by Fresnel to account for regular reflection, and the secondary wavelets go off in all directions, instead of uniting to form a regularly reflected wave-front.

These wavelets or impulses will pass by any given point with a periodicity depending on the location of the point. In the last section of the Chapter on Diffraction this action of the grating has been explained.

Analysis by a Prism. Origin of Prismatic Colors.—The mechanism by which a prism converts an impulsive disturbance into a periodic one is not quite as obvious as it is in the case of the grating, where we have a periodic structure.

We can, however, get a clue as to the mode of its action in the following way:

As we know, the phase of the vibration is everywhere the same on a wave-front. If this condition holds, the wave will be propagated parallel to itself, and no lateral effects will be produced. If, however, certain portions are blocked off, as by diffracting screens or gratings, lateral effects are produced, or we have deviations of a portion of the energy, which no longer obeys the laws of rectilinear propagation. The same thing results if, instead of blocking off portions at regular intervals, we change the phase of the vibration periodically: this occurs in the case of laminary gratings in which the retardation effected by the strips alters the phase of the vibration. We shall now show that a prism is capable of impressing a somewhat similar condition on the front of an impulsive disturbance.

In the section on group velocity we have seen that in a dispersing medium the group is propagated with a velocity different from that of the component waves which form it. We will now prove in a very elementary manner that, as the group proceeds, it changes its form, becomes inverted, and eventually reappears in its original form. Take first the simple type of group previously considered, formed by two trains of waves of slightly different wave-lengths.

The two trains are shown in the lower part of Fig. 384, the resultant in the upper part. We will select as a given form of our group the shape which it has at the moment when it has its maximum amplitude above the line of equilibrium: this is the condition shown in the diagram, the maximum amplitude being at 1. The two sets of component waves travel in the direction of the arrow, the

shorter ones (solid lines) at the higher velocity. It is clear that the group will not be propagated without change of form, for at an instant later the two sets of waves will be nowhere exactly in step, and we shall have at no place an amplitude as great as that figured above. The waves will eventually get into step again at 2, and we shall again have our maximum amplitude above, and the original form of the group restored. Notice, however, that before this event occurs the waves will be exactly together at the trough immediately to the right of B, and we shall have the same maximum amplitude, only in this case it will be below the line. The form of the group will be the same as before, only it will be *inverted*. This inversion of the form of the group before its reëstablishment is of fundamental importance in considering the action of dispersing

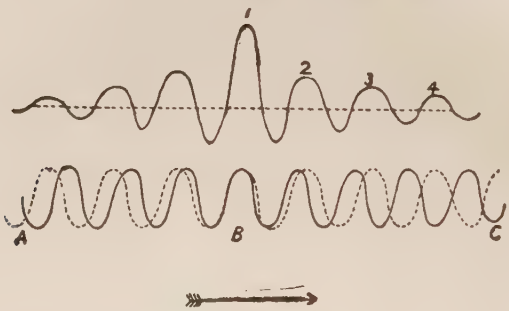


FIG. 384.

media upon white light. If we take less simple groups made up of a large number of component trains, the original form will reappear at stated intervals only in special cases.

If the dispersion is represented by the formula

$$V = a + b\lambda,$$

$\frac{dV}{d\lambda}$ is a constant, and the group-velocity is independent of the wave-length. Our group may have the form shown in Fig. 385, which is taken from Schuster's last paper on the subject (Boltzmann's *Festschrift*, p. 569, 1904). The form of this group is given by the equation

$$y = \frac{h^2}{h^2 + x^2},$$

and its successive appearances as it advances through a medium of dispersion $V = a + b\lambda$ are shown.

As a matter of fact the dispersion formula assumed is not possessed by any known medium, but the problem is simplified by the assumption of a medium of this nature.

It is clear that if the hypothetical medium is formed into a prism, the pulse will leave the second surface with a periodicity impressed upon it, that is to say, at certain points it will emerge in its original form, and in other places in its inverted form.

We will now go back to our original simple case of a group formed from two infinite trains of waves, and show that the periodicity impressed upon the group-front will produce a periodic disturbance at the focus of the telescope pointed in a direction parallel to that in which the group is advancing, identical with the periodicity of the

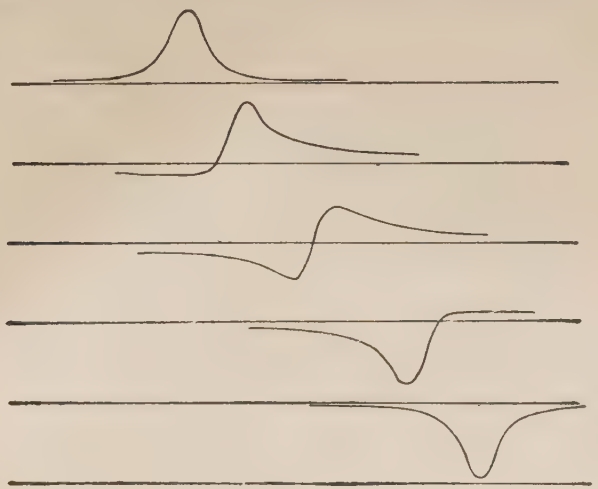


Fig. 385.

component trains. I am indebted to my friend Professor Ames for the following very simple and easily intelligible treatment:

" Let us consider the action of a prism upon such a group, and for the sake of simplicity let the group have a plane front and fall per-

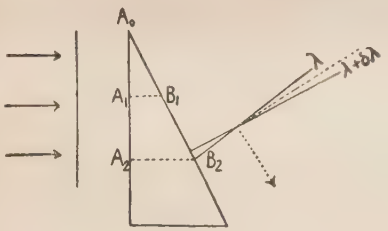


Fig. 386.

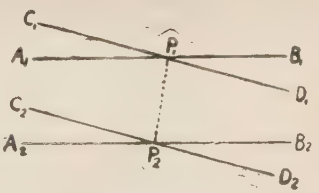


Fig. 387.

pendicularly upon the face of the prism (Fig. 386). We may choose any feature of the group by which to recognize it, and note its periodic recurrence, *e.g.* the condition marked by the sum of the two amplitudes of the component trains. As the group advances towards the prism, this ' crest ' moves forward with the velocity V_e, that of waves in the pure ether; when the group enters the prism, it changes its form, the ' crest ' recurring at intervals equal to X; consequently at certain points B_1, B_2, etc., on the second face of the prism, such that $A_1B_1 = X$, $A_2B_2 = 2X$, etc., the ' crest '

will emerge. Thus the vertex A_0, and the points B_1, B_2, etc., may serve as centres of secondary disturbances in a Huygens's construction, and a plane drawn tangent to these secondary spheres may be called the 'group-front.' It is apparent, however, that in the time T required for the 'crest' to reappear at B_1, after disappearance at A_1, the component *trains of waves* have advanced a greater distance than A_1B_1, and have emerged from the prism and passed on as two separate trains in slightly different directions, owing to their different indices of refraction.

"There is thus a periodicity in the group-front, due to the fact that at certain regularly spaced intervals there is the maximum amplitude. This is caused obviously by the superposition of the two crests of the component *trains of waves*, whose wave-fronts cross at a small angle. We can therefore study the direction of advance of any one 'crest' in the group-front, and at the same time calculate the periodicity produced when the group is received by a telescope, by plotting the traces of the two trains of waves. Let the lines A_1B_1 and C_1D_1 (Fig. 387) be the traces of the crests of the two trains of waves at any instant. P_1, their point of intersection, will then be a 'crest' of the group-front; at a certain time later, the two wave-crests will have advanced through equal distances to positions A_2B_2 and C_2D_2, and their point of intersection, P_2, will mark the new position of the 'crest' of the group-front. In other words, P_1P_2, a line perpendicular to the bisector of the angle between A_1B_1 and C_1D_1, may be called the direction of advance of the group; that is, the receiving telescope must have this direction.

"To deduce the periodicity observed by the telescope, one has but to draw the crests of the two trains of waves as they are at any

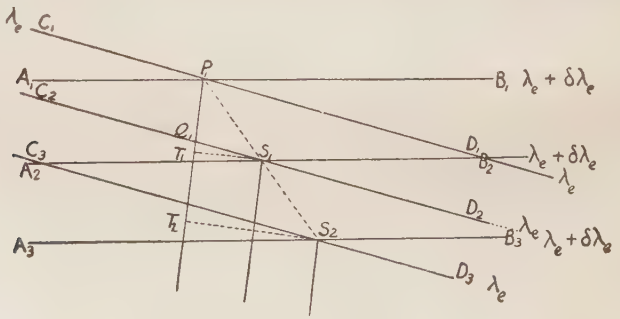


Fig. 388.

instant, for a distance of several wave-lengths. Thus let A_1B_1, A_2B_2, A_3B_3, etc., be the traces, *at any one instant*, of the wave-crests of the train whose wave-length in the free ether is $\lambda_e + d\lambda_e$; let C_1D_1, C_2D_2, C_3D_3, etc., be those of the train whose wave-length is λ_e, at the same instant; and let P_1, S_1, S_2, etc., be their points of intersection. (Fig. 388.)

"As the trains *advance*, the 'crest' P_1 moves, as has just been shown, in the direction $P_1Q_1T_1$; the 'crest' S_1 moves in a parallel

direction, etc.; consequently the periodicity observed by the telescope is given by the distance P_1T_1, where T_1 is the foot of the perpendicular dropped from S_1 upon $P_1Q_1T_2$. If the angle between A_1B_1 and C_1D_1 is called α, this periodic distance

$$P_1T_1 = \frac{1}{\cos\frac{\alpha}{2}}\left(\lambda + \frac{d\lambda_e}{2}\right),$$

and therefore in the limit equals λ_e.

"The case of a more complicated group or of a pulse is, to a certain extent, equally simple. Any group or pulse may be analyzed into a number of simple groups like those discussed above, each such group being 'associated' with a certain train of waves of wavelength λ. If such a complex group enters a dispersive medium, two things must be noted: (1) since the velocity of any simple group is $V - \lambda\frac{dV}{d\lambda}$, the different component groups will have different velocities, and so their group-fronts will be differently refracted, both on entering and on emerging; (2) since the distance required for a certain feature of a group to reappear, *i.e.* the length X, is different for the different groups, they will recur at different intervals, and therefore the complex group itself could not reappear. These complications might be avoided if a dispersive medium could be found for which

$$V - \lambda\frac{dV}{d\lambda} \text{ and } \left(V - \lambda\frac{dV}{d\lambda}\right)\frac{1}{\frac{dV}{d\lambda}}$$

are both constant. These conditions are satisfied if the dispersive formula for the medium obeys the relation $V = A + B\lambda$, where A and B are constants; for, in this case, the group velocity is A, and the periodic distance X is A/B; both of which are independent of λ, and therefore the same for all the component simple groups.

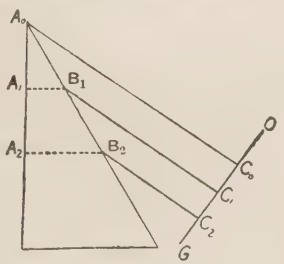


FIG. 389.

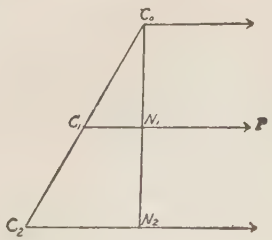


FIG. 390.

"To avoid any refraction of the *wave-fronts* of the ultimate trains of waves on entrance into a prism, we may, as before, consider normal incidence. Then, again, we will have what may be called a 'group-front' for the emerging light, by drawing a plane tangent to secondary spherical disturbances, having A_0, B_1, B_2, etc., as centres,

where $A_1B_1 = X$, $A_2B_2 = 2X$, etc. (Fig. 389). Let the trace of this plane be OG. It will contain periodicities, for the conditions are the same at C_0, C_1, C_2, etc. — the points of tangency. As is seen by considering the complex group made up of simple ones, the condition at these points is due to a superposition of trains of waves, and as these advance, the different component simple groups separate out and give rise to different periodicities proceeding in different directions. We may trace these in the following manner (Fig. 390): Let $C_0C_1C_2\ldots$ be the 'group-front,' then the effects propagated in the direction C_1P — which is taken at random — have the periodicity C_1N_1, where the line $C_0N_1N_2\ldots$ is drawn perpendicular to the direction C_1P; for $C_2N_2 = 2C_1N_1$, etc. We will prove that this periodic distance C_1N_1 is equal to λ_e, where this is the wave-length of the *train of waves* which, after normal incidence on the prism, would on emergence have the *wave-front* $C_0N_1N_2\ldots$. The difference in time required for the group-front and the train of waves to traverse the prism along the line A_1B_1 is

$$X\left(\frac{1}{u}-\frac{1}{V}\right) \text{ or } X\,\frac{V-u}{uV},$$

which, as proved above, equals $\frac{\lambda}{V}$, where λ is the wave-length of the train of waves while in the prism. Hence the distance of the wave-front in advance of the group-front, after emergence, along the line C_1P is $\frac{V_e\lambda}{V}$ or $\mu\lambda$, which equals λ_e. That is, the distance C_1N_1 equals λ_e.

"It is thus seen that if a telescope is pointed in different directions towards the prism, disturbances of different periodicities will be brought to a focus; and further, that the periodicity corresponding to any one direction is exactly that of the train of waves which would be brought to a focus if this train had been incident upon the prism instead of the group. In other words, a complex group gives rise, through the agency of the prism, to periodic effects advancing in different directions, which are identical — with an important limitation to be noted presently — with the effects which could have been produced if a complex train of waves had been incident upon the prism. Accordingly, the fact that a prism produces approximately homogeneous trains of waves when white light falls upon it, is not a proof of the existence in the white light of periodic component trains of waves.

"The 'resolving power' of the prism is evidently proportional to the number of periodicities which occur in the emergent 'group-front,' and if Δ is the thickness of the base of the prism, this number equals $\frac{\Delta}{X}$ or $\frac{\Delta}{u}\frac{dV}{d\lambda}$. This limits then the periodic nature of the resolved components.

"In thus explaining how an arbitrary group or pulse may, by means of a prism, produce what to a certain extent may be called trains of waves, a particular kind of dispersive medium has been considered. This is, however, no limitation upon the argument,

as Schuster notes, since for a series of simple groups associated with the wave-lengths not far removed from any definite value λ, the quantities u and X may be considered to have the same values, and so any arbitrary group may be treated as made up of these series of simple groups."

Lord Rayleigh's Treatment. — A very simple treatment of the origin of the prismatic colors has been given by Lord Rayleigh, which is so complete and satisfactory that it appears best to give it in the author's words:

"The fact that by the aid of a spectroscope interferences may be observed with light originally white used to be regarded as a proof of the existence of periodicities in the original radiation; but it seems now to be generally agreed that these periodicities are due to the spectroscope. When a *pulse* strikes a grating, it is obvious that the periodicity and its variation in different directions are the work of the grating. The assertion that Newton's experiments prove the colors to be already existent in white light, is usually made in too unqualified a form.

"When a prism, which has no periodicities of figure, is substituted for a grating, the *modus operandi* is much less obvious.

"I commence by remarking that, so far as I see, there is nothing faulty or specially obscure in the traditional treatment founded upon the consideration of simple, and accordingly infinite, trains of waves. By Fourier's theorem any arbitrary disturbance may be thus compounded; and the method suffices to answer any question that may be raised, so long at least as we are content to take for granted the character of the dispersive medium — the relation of velocity to wave-length — without inquiring further as to its constitution. For example, we find the resolving-power of a prism to be given by

$$\frac{\lambda}{d\lambda} = T\frac{d\mu}{d\lambda}, \quad \cdots\cdots\cdots \quad (1)$$

in which λ denotes the wave-length *in vacuo*, T the 'thickness' of the prism, μ the refractive index, and $d\lambda$ the smallest difference of wave-length that can be resolved. A comparison with the corresponding formula for a grating shows that (1) gives the number of waves (λ) which travel in the prescribed direction as the result of the action of the prism upon an incident pulse.

"But, although reasoning on the above lines may be quite conclusive, a desire is naturally felt for a better understanding of the genesis of the sequence of waves, which seems often to be regarded as paradoxical. Probably I have been less sensible of this difficulty from my familiarity with the analogous phenomena described by Scott Russel and Kelvin, of which I have given a calculation.[1] 'When a small obstacle, such as fishing-line, is moved forward slowly through still water, or (which, of course, comes to the same thing) is held stationary in moving water, the surface is covered with a beautiful wave-pattern, fixed relatively to the obstacle. On the up-stream side the wave-length is short, and, as Thomson has

[1] "The Form of Standing Waves on the Surface of Running Water," Proc. Lond. Math. Soc., xv, p. 69 (1883); *Scientific Papers*, ii. p. 258.

shown, the force governing the vibrations is principally cohesion. On the down-stream side the waves are longer and are governed principally by gravity. Both sets of waves move with the same velocity relatively to the water, namely, that required in order that they may maintain a fixed position relatively to the obstacle. The same condition governs the velocity, and therefore the wave-length, of those parts of the wave-pattern where the fronts are oblique to the direction of motion. If the angle between this direction and the normal to the wave-front be called θ, the velocity of propagation of the waves must be equal to $v_0 \cos\theta$, where v_0 represents the velocity of the water relatively to the (fixed) obstacle.' In the laboratory the experiment may be made upon water contained in a large sponge-bath and mounted upon a revolving turntable. The fishing-line is represented by the impact of a small jet of wind. In this phenomenon the action of a prism is somewhat closely imitated. Not only are there sequences of waves, unrepresented (as would appear) either in the structure of the medium or in the character of the force, but the wave-length and velocity are variable according to the direction considered.

" For the purposes of Scott Russel's phenomenon the localized pressure is regarded as permanent; but here it will be more instructive if we suppose it applied for a finite time only. Although the method is general, we may fix our ideas upon deep water, subject to gravity (cohesion neglected), upon which operates a pressure localized in a line and moving transversely with velocity V. In the general two-dimensional problem thus presented, the effect of the travelling pressure is insignificant unless V is a possible wave-velocity; but where this condition is satisfied, a corresponding train of waves is generated. In the case of deep water under gravity the condition is always satisfied, for the wave-velocities vary from zero to infinity.

" The limitation to a wave-train of velocity V is complete only when the time of application of the pressure is infinitely extended. Otherwise, besides the train of velocity V we have to deal with other trains, of velocities differing so little from V that during the time in question they remain sensibly in step with the first. As is known,[1] the behavior of such aggregates is largely a matter of the *group-velocity* U, whose value is given by

$$U = \frac{d(kV)}{dk}, \quad \cdots\cdots\cdots \quad (2)$$

k being proportional to the reciprocal of the wave-length *in the medium*. In the particular case of deep-water waves $U = \frac{1}{2}V$.

" From this point of view it is easy to recognize that the total length of the train of waves generated in time t' is $\pm(V-U)t'$. If τ be the periodic time of these waves, the wave-length in the medium is $V\tau$ and the number of waves is therefore

$$\pm \frac{V-U}{V}\frac{t'}{\tau}. \quad \cdots\cdots\cdots \quad (3)$$

[1] See, for example, *Nature*, xxv, p. 51 (1881); *Scientific Papers*, i, p. 540.

But for our present purpose of establishing an analogy with prisms and their resolving-power, what we are concerned with is not the number of waves at any time in the dispersive medium itself, but rather the number after emergence of the train into a medium which is non-dispersive; and here a curious modification ensues. During the emergence the relative motion of the waves and of the group still continues, and thus we have to introduce the factor V/U, obtaining for the number N of waves outside

$$N = \frac{V-U}{U}\frac{t'}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

"If X be the distance through which the pressure travels, $X = Vt'$; and if V_0 be the (constant) velocity outside and λ the wave-length outside, $\lambda = V_0\tau$. Thus

$$N = \left(\frac{V_0}{U} - \frac{V_0}{V}\right)\frac{X}{\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

"To introduce optical notation, let $\mu = V_0/V$, so that μ is the refractive index. In terms of μ

$$\frac{V_0}{U} = \mu - \lambda\frac{d\mu}{d\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

so that finally

$$N = -X\frac{d\mu}{d\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in close correspondence with (1). A very simple formula thus expresses the number of waves (after emergence) generated by the travel of the pressure over a distance X of a dispersive medium.

"The above calculation has the advantage of being clear of the complication due to obliquity; but a very little modification will adapt it to the case of a prism, especially if we suppose that the waves considered are emergent at the second face of the prism without refraction. In Fig. 391 let AC represent an incident plane pulse whose trace runs along the first face of the prism from A to B. AF, BE is the direction of propagation of the refracted waves under consideration, to which the second face of the prism is perpendicular. As before, if τ be the period, V the wave-velocity of the waves propagated in direction BE, U the corresponding group-velocity, t' the time of travel of the pulse from A to B, the number of waves within the medium is

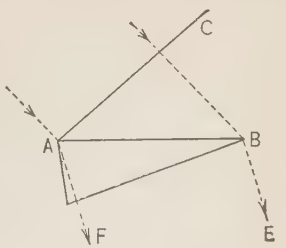


FIG. 391.

$$\frac{V-U}{V}\frac{t'}{\tau},$$

giving on emergence the number of waves expressed in (4). If V_0 be the velocity in vacuum, $\tau=\lambda/V_0$, and

$$t'=\frac{BC}{V_0}=\frac{AD}{V};$$

so that

$$\frac{t'}{\tau}=\frac{AD}{\lambda}\frac{V_0}{V}.$$

"Thus, as in (5), (6), (7),

$$N=\left(\frac{V_0}{U}-\frac{V_0}{V}\right)\frac{AD}{\lambda}=-AD\frac{d\mu}{d\lambda}, \quad \cdots \quad (8)$$

in agreement with (1).

"Although the process is less easy to follow, the construction of a train of waves from an incident pulse is as definite in the case of a prism as is that of a grating; and its essential features are presented to the eye in Scott Russel's phenomenon."

CHAPTER XXIV

THE RELATIVE MOTION OF ETHER AND MATTER

Aberration of Light. — The discovery was made by Bradley, in 1728, that the apparent direction of the stars was modified by the motion of the earth through space. To understand just how this results, let us take the case of a gun on shore which has sent its projectile through the hull of a ship. If the ship is at rest, the position of the gun could be determined by sighting through the shot-holes made by the entrance and exit of the ball. If, however, the ship is moving at high speed, it will have advanced a certain distance during the time occupied by the projectile in passing through the hull, and the point of exit will be further aft than in the previous case. A line drawn through the two holes will not, in the present instance, determine the true direction of the gun, as can easily be seen by constructing a diagram. The gun's position, as determined by this method, will appear to have shifted in the direction of the ship's motion, through an angle, the tangent of which is the ratio of the ship's velocity to that of the projectile. This angle is called the angle of aberration. Consider now the case of light-waves entering the object glass of a telescope. The lens transforms them into concave waves, and we will assume that the telescope is so pointed that they come to a focus on the cross-hairs of the eye-piece. If the earth were at rest, a line drawn from the point of intersection of the cross-hairs through the center of the lens would give the true direction of the star. But the earth and the telescope are in motion, and while the waves are travelling down the tube the tube is being carried forward. The focus point will in this case fall a little behind the point at which the rays would have met if the telescope had been at rest, and if the star image is now brought upon the intersection of the cross-hairs it is clear that the telescope is pointing a little ahead of the star's true position. The amount of the shift due to the earth's motion can of course be determined only by extending the observations over an entire year; the total change in the star's position will clearly be double the true angle of aberration, for the shift is in opposite directions when the earth is on opposite sides of its orbit around the sun. The case is analogous to that of a ship steaming around in a circle, the crew of which are endeavoring to locate the position of a gun on shore by sighting through the shot-holes.

Bradley found the total angle of aberration to be 40.89 seconds of arc, or that the actual shift due to the earth's motion in its orbit was 20.44 seconds. The velocity of light in space, which was given by dividing the earth's velocity by the tangent of this angle, agreed well

with the value found by Römer from observations of the eclipses of Jupiter's satellites.

The phenomenon of aberration clearly indicates that the medium which is transmitting the undulations must be at rest with respect to the telescope. If the ether in the tube were carried along with it, the point at which the waves came to a focus would be wholly uninfluenced by the motion of the tube, and there would be no aberration.

As we shall see presently, however, certain experiments appear to indicate that the earth carries the ether along with it, a condition which cannot well be reconciled with the phenomenon which we have just considered.

It is probable, however, that the trouble is to be sought for in the theory of the experiment rather than in the theory of aberration.

Airy's Experiment. — The angle of aberration being determined by the ratio of the earth's velocity to the velocity of light, we should expect a change if either one of these quantities could be altered. The velocity of light down the tube of the telescope can be diminished by filling the tube with water, and we should consequently expect the angle of aberration to be increased. This experiment was tried by Airy, who found, however, that the angle was the same as when the tube was filled with air.

To explain this we may assume that the water carries the contained ether along with it, not with its full velocity, for in this case there would be no aberration, but with a velocity sufficient to compensate for the change resulting from the diminished velocity of the light. That something analogous to this dragging along of the ether actually occurs, was proved experimentally by Fizeau, and subsequently by Michelson and Morley.

Fizeau's Experiment. — Fizeau arranged an apparatus in which two beams of light were caused to traverse a system of tubes through which water could be forced at a high velocity. A system of interference fringes was formed by the union of the two beams, and the effect of the motion of the fluid upon the position of the fringes was studied. The arrangement of the apparatus is shown in Fig. 392. Light from the slit at S after reflection from a plate of glass is made parallel by a collimating lens, and divided into two portions which traverse tubes containing running water. It is clear from the diagram that each interfering beam traverses the same thickness of ponderable medium, for each ray is obliged to pass through the entire tube system. This is accomplished by focussing the rays upon a plane mirror, the effect of which is to interchange the paths. Moreover, it will be seen that one ray is travelling always with the current, the other against it. On emerging from the apparatus the rays are brought to a focus at S' behind the plate (a portion at S also), where a system of interference fringes is formed. A shift of the fringes was observed when the water was put in motion, which could be doubled by reversing the direction of the current.

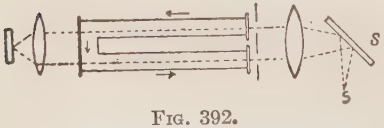


Fig. 392.

Let c be the velocity of light in vacuo, v the velocity in water, and V the velocity of the water. Assume that the ether is carried along by the water with a velocity $V\theta$, in which θ is a fraction. The velocity of the two interfering beams will be $v - V\theta$ and $v + V\theta$, and if l is the total length of the water path, the difference in time over the two paths will be:

$$\frac{l}{v - V\theta} - \frac{l}{v + V\theta}.$$

Fizeau observed a measurable displacement with a velocity of seven metres per second. The phase-difference can be determined by the shift of the fringes, from which the value of θ in the above equation can be determined. In the case of water, $V = .434$, that is, the motion of the water apparently gives to the contained ether a velocity very nearly half as great as its own. The general expression for θ, as developed by Fresnel, for any moving medium of refractive index μ is

$$\theta = \frac{\mu^2 - 1}{\mu^2} = 1 - \mu^{-2}.$$

This experiment was repeated in an improved form by Michelson and Morley (*Am. Journal of Sci.*, xxxi., p. 377 (1886)). In Fizeau's arrangement the distance between the slits which divide the beam into two portions is necessarily large, and the fringes are in consequence extremely close together and require very high magnification, with its accompanying loss of light. Michelson's arrangement permitted the use of an extended source of light such as a gas flame, and any desired distance between the tubes. Light from a source at S (Fig. 393) is divided at a half silvered surface at A, and sent around the water-tube system in opposite directions, as shown in the diagram. With tubes six metres long and a velocity of eight metres per second, the displacement observed on reversing the direction of the current amounted to less than the width of a single fringe. The results obtained were fairly concordant, however, the value .434 being found for θ. They also experimented with an air current moving with a velocity of 25 metres per second, but the effect in this case was too small to measure.

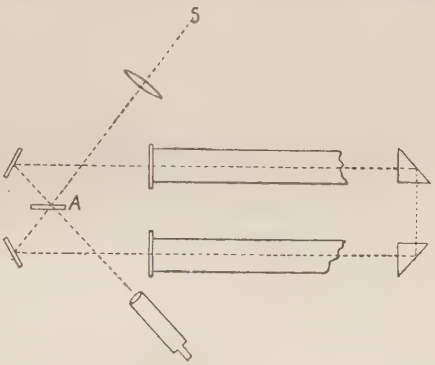


FIG. 393.

The expression for θ which has been given above was developed by Fresnel from the following considerations: He regarded the refractive index as the square root of the ratio of the ether density in the medium to the ether density in vacuo, the refraction being due

to the condensation of the ether within the pores of the medium. Consider a transparent plate, of ref. index μ, moving with velocity V, and let D_1 be the ether density within it, and D the density in vacuo. Then $D_1 = \mu^2 D$. If there is no flow of the ether around the edges of the plate, the same amount must enter the front surface in unit-time as leaves the back surface, or $DV = D_1(V - \theta V)$, which gives us, if we substitute $\mu^2 D$ for D_1,

$$\theta = 1 - \mu^{-2} = .438 \text{ for water.}$$

This amounts to saying that the condensed ether within the plate is carried forward with a velocity such that the *excess* of the ether in the body over that in the corresponding free space is carried along with the full velocity of the plate. We may, however, regard the condensed ether as a part of the medium, in which case we can say that the ether proper is entirely uninfluenced by the motion of the medium. If c is the velocity of light in a vacuum, and $\frac{c}{\mu}$ the velocity in a medium at rest, the absolute velocity of the light in a medium moving with a velocity V is

$$\frac{c}{\mu} \pm V(1 - \mu^{-2}),$$

the plus or minus sign being used according as the light travels in the same or opposite direction as that in which the medium moves. Commenting on the derivation of the above formula, Lord Rayleigh remarks: "Whatever may be thought of the means by which it is obtained, it is not a little remarkable that this formula and no other is consistent with the facts of terrestrial refraction if we once admit that the ether in the atmosphere is at absolute rest. It is not probable that the ether in moving refracting bodies can properly be regarded as itself in motion, but if we knew more about the matter we might come to see that the objection is verbal rather than real. Perhaps the following illustration may assist the imagination:

"Compare the ether in vacuum to a stretched string, the transverse vibrations of which represent light. If the string is loaded (say with beads) the velocity of propagation is diminished. This represents the passage of light through stationary refracting media. If now the loads be imagined to run along the string with a velocity not insensible in comparison with that of the waves, the velocity of the latter is modified. It appears that the suggested model would lead to a somewhat different law of velocity from that of Fresnel; but in bringing it forward the object is merely to show that we need not interpret Fresnel's language too literally."

Retardation by a Moving Plate. — Let us now investigate, following Lord Rayleigh, the effect of the motion of a plate upon the retardation which it exerts upon light-waves passing through in the same (or in the opposite) direction. Let the velocity of the plate be designated as before by V, its thickness by d, and its refractive index by μ. If the velocity of the ether within the plate is θV,

and the velocity of light in vacuo is c, we have for the absolute velocity of the wave in the plate:

$$\frac{c}{\mu}+\theta V.$$

The time t occupied by the wave in traversing the plate is not found by dividing d by the velocity as given above, for during the time t the anterior face of the plate, which the wave reaches last, is carried forward a distance Vt. The velocity of the wave in the plate multiplied by the time t is equal to the thickness of the plate plus the distance through which the plate moves in time t, or

$$\left(\frac{c}{\mu}+\theta V\right)t=d+Vt \text{ or } \frac{ct}{d}=\frac{\mu}{1+(\theta-1)\frac{\mu V}{c}}.$$

The time t_0 which would have been occupied in traversing the same distance $d+Vt$, had the plate been away, is given by

$$ct_0=d+Vt,$$

so that

$$\frac{ct_0}{d}=1+\frac{\frac{\mu V}{c}}{1+(\theta-1)\frac{\mu V}{c}} \text{ or } \frac{c(t-t_0)}{d}=\frac{\mu\left(1-\frac{V}{c}\right)}{1+(\theta-1)\frac{\mu V}{c}}-1$$

If we substitute in this Fresnel's value of θ, viz. $1-\mu^{-2}$, neglecting as insensible the square of $\frac{V}{c}$, we find

$$c(t-t_0)=(\mu-1)d\left(1-\frac{V}{c}\right),$$

an equation which gives us the relative retardation between a wave passing through the plate and one passing by its side. The retardation depending upon the sign of $\frac{V}{c}$, will be altered when the direction of the light is reversed, which can be done by a simple rotation of the apparatus through 180°. If, however, we employ a terrestrial source of light, such as a sodium flame, we must take into account the fact that the source is in motion, and that the waves are consequently shortened or lengthened by Doppler's principle.

If V is the velocity of the source, the wave-length is changed from λ to $\lambda\left(1-\frac{V}{c}\right)$ on the side of the source towards which it is moving and to $\lambda\left(1+\frac{V}{c}\right)$ on the opposite side. We thus see that if we measure the retardation in the above equation in wave-lengths, as we are obliged to do in all experiments, it is independent of V, that is, no displacement of the fringes is to be expected on rotating the apparatus through 180°. An experiment was devised

by Hock in 1869, in which the part of the retardation independent of V was eliminated. Two beams of light were passed, the one through a refracting plate, the other through the air: they were then brought to a focus on a mirror, as in Fizeau's experiment, by which the parts were interchanged. It would appear at first sight as if an effect of the motion of the plate should be observed in this case, but Lord Rayleigh shows that if the change in wave-length which occurs at reflection from a moving mirror is taken into account, no results are to be expected.

The Michelson-Morley Experiment. — Attempts have been made by Michelson and Morley to detect effects resulting from the relative motion of the earth and the ether. These effects depend upon the square of the ratio of the velocity of the earth in its orbit to the velocity of light, a term which can be neglected in all experiments involving such small velocities as occur in experiments such as that of Fizeau. The theory of these celebrated experiments, about which so much discussion has occurred, is as follows (Michelson and Morley, *Phil. Mag.*, xxiv., page 449):

Consider a system of interference fringes formed by a Michelson interferometer, the three mirrors of which occupy the positions A, B, C (Fig. 394) at the moment when the incident beam SA strikes the first plate. While the light is travelling from the mirror A to the mirrors B and C and back again to A, assume the whole apparatus carried forward by the earth in the direction of the incident light to the position $A'B'C'$. The ray reflected from B, which interferes with a given ray reflected from C, along the line $A'D'$, we must consider as travelling along $AB'A'$, the angle BAB' being equal to the angle of aberration. It must not be thought, however, that the path of the reflected ray is altered by the motion of the mirror. The change of path merely indicates that the ray which we are utilizing, and which strikes the mirror A in its second position at the point where the ray BA would have met it had the apparatus been at rest, is a ray reflected at the angle indicated. The mirror B, at the moment when reflection occurs at its surface, has moved only one-half of the distance between A and A', from which it follows that the angle BAB' is equal to the angle of aberration, the tangent of which is the ratio of $\frac{BB'}{2}$ to AB. The dotted lines in the figure are not quite correctly placed, as is obvious.

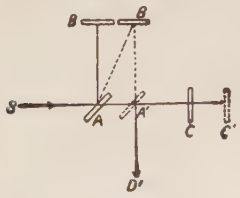


FIG. 394.

Suppose that the ether remains absolutely at rest, and let c = the velocity of light, and u = the velocity of the apparatus, *i.e.* of the earth in its orbit.

Further, let T = the time occupied by the ray in passing from A to C (located at the point to which it has been carried), and T^1 = the time in returning from C to A'. At the moment of reflection from C the mirror will occupy a position midway between C and C'.

Call D the distance AB or AC, then

$$cT = D + uT,$$
$$cT' = D - uT',$$

so that

$$T = \frac{D}{c-u}, \quad T' = \frac{D}{c+u},$$

the whole time being given by

$$T + T' = 2D\frac{c}{c^2 - u^2}.$$

If the system is at rest, $T = \frac{2D}{c}$, which is identical with the above equation when $u = 0$.

The actual distance traversed in the time $T + T'$ is obviously given by multiplying the time of transit by the velocity c, or

$$\text{Path} = 2D\frac{c^2}{c^2 - u^2} = 2D\left(1 + \frac{u^2}{c^2}\right),$$

which we obtain by simple division, neglecting $\frac{u^4}{c^4}$ and terms of higher order.

The length of the other path $AB'A'$ is

$$2D\left(1 + \frac{u^2}{c^2}\right)^{\frac{1}{2}} = 2D\left(1 + \frac{u^2}{2c^2}\right) \text{ (approx.), since } \frac{AA'}{AB} = \frac{2u}{c}.$$

It is thus seen that the effect of the motion of the apparatus is to slightly increase both paths, the increment being much greater, however, along the path parallel to the earth's motion. The path-difference which was originally zero is now

$$2D\left(1 + \frac{u^2}{c^2}\right) -$$
$$2D\left(1 + \frac{u^2}{2c^2}\right) = D\frac{u^2}{c^2}.$$

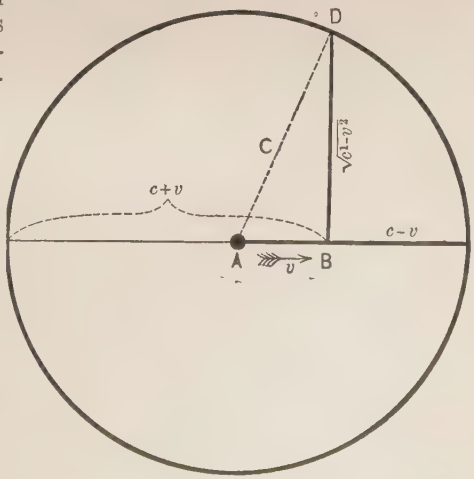


Fig. 395.

We may, if we choose, look at the thing from a slightly different point of view. Imagine an observer moving through space with a velocity v in the direction of the x axis, Fig. 395. Suppose that he sets off a flash of light which originates a spherical wave, which moves out from its origin with a constant velocity in all directions. As the wave spreads out, the observer is moving along towards the right,

drifting away from the point where the wave originated, and reaching the point B at the moment when the wave has the diameter c. It is clear that if he were unaware of his own motion, but had some means of following the motion of the wave, he would come to the conclusion that the velocity of the wave in the direction of $+x$ is $c-v$, and in the reverse direction $c+v$. In the direction perpendicular to the x axis he will also find the velocity altered, for he will, as he moves along, be dealing with the velocity with which that portion of the spherical wave which lies in the direction $\perp$ to x from his viewpoint is moving away from him. It is clear from the figure, that when he is at the point B he will regard the distance BD as the distance which the wave has travelled, and this distance is less than c, being in fact $\sqrt{c^2-v^2}$, as can be seen from the right triangle ADB. We can now imagine the observer furnished with an interferometer, and if he uses these new values for his light velocity in the two directions he can consider the two paths in his instrument as constant. If D is, as before, the distance between the mirrors, the time occupied by the light in making its to-and fro excursion between the half-silvered mirror (which can be regarded as the source of light) and the mirror lying in the direction of motion will be

$$\frac{D}{c-v}+\frac{D}{c+v}=\frac{2D}{c}\left(1+\frac{v^2}{c^2}+\cdots\right),$$

while the time over the other path will be

$$\frac{D}{\sqrt{c^2-v^2}}+\frac{D}{\sqrt{c^2-v^2}}=\frac{2D}{c}\left(1+\frac{v^2}{2c^2}+\cdots\right).$$

The time-difference over the two paths is obviously $\frac{D}{c}\frac{v^2}{c^2}$ and the path-difference $D\frac{v^2}{c^2}$ which is identical with our former expression.

This treatment of the problem is perhaps a little easier to follow than the other, and we shall have occasion to refer to it again in the following chapter.

If now we rotate the whole apparatus through 90°, the path $AB'A'$ will be the one which receives the larger increment, and a shift in the position of the fringes should result.

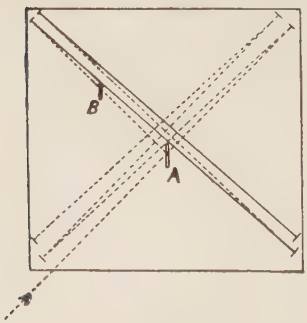


Fig. 396.

In the first experiments tried, the expected shift amounted to only about $\frac{1}{20}$ of the distance between the fringes; moreover, it was found impossible to rotate the apparatus without introducing strains, which caused slight changes in the position of the fringes. As a result no very definite conclusions could be drawn from the observations. The experiment was then repeated with improved apparatus. By means of multiple reflections the path D was in-

creased to 11 metres. The mirrors, 16 in number, were mounted on a heavy slab of stone which was floated on mercury. The apparatus was kept in slow rotation while the observations were taken, which did away with the strains which always occurred when it was brought to rest. A diagram of the apparatus is shown in Fig. 396, the number of mirrors having been reduced by one-half, however. The beam of light is divided at the half-silvered plate A. The compensating plate is located at B.

The value of $\frac{u^2}{c^2}$ is 10^{-8}, if the earth's orbital motion is alone considered, while D measured in wave-lengths of sodium light was 2×10^7. If the ether remains at rest relatively to the earth, we should expect a displacement of the fringes equal to

$$4\times10^7\times10^{-8}=.4 \text{ of a fringe width.}$$

The actual displacement observed was certainly less than $\frac{1}{20}$ of the expected, and probably less than $\frac{1}{40}$, from which the conclusion was drawn that since the displacement is proportional to the square of the velocity, the relative velocity of the earth and the ether is probably less than one-sixth of the earth's orbital velocity. This amounts to saying that the earth drags the ether in its vicinity along with it, a circumstance which cannot be reconciled with the phenomenon of stellar aberration, to account for which we must assume the ether at rest with respect to the earth.

An explanation of the absence of any fringe-shift in the experiment of Michelson and Morley was suggested simultaneously by Fitzgerald and Lorentz. This explanation was based upon an assumed change in the linear dimensions of matter resulting from its motion through the ether; a contraction of the base upon which the mirrors are supported occurring in the direction of motion would compensate for the increment of optical path due to the motion of the apparatus.

Attempts have been made to detect this hypothetical effect, but thus far all have been unsuccessful. If the effect occurs it might very well happen that its magnitude would vary with different materials. Morley and Miller therefore repeated the experiment under conditions such that the distance between the mirrors could be made to depend upon the length either of a metal rod or a pine stick. The result was the same in each case, however.

It occurred to Lord Rayleigh that the contraction in the direction of motion, if it existed, might give rise to double refraction, but he was unable to detect anything of the kind; and a subsequent experiment by Brace, performed with the greatest care, has established conclusively that no trace of double refraction occurs as a result of the motion of transparent media through the ether.

It is obvious that the failure of the earth's motion to influence in any way phenomena occurring in optical systems located wholly upon the earth are at once explained if we assume that the velocity of light is influenced by the motion of the source, *i.e.* if the velocities are additive, as would be the case on the corpuscular theory,

for in this case the moving observer would remain at the centre of the wave. This hypothesis has never even been seriously considered, for it is incompatible with the wave-theory. In the last chapter, we shall take up the whole matter from a quite new point of view, the theory of relativity, and it will be shown that from a certain point of view a moving source of light can be considered as remaining at the centre of waves which it originates, which is all that we require.

Lodge's Experiment. — The experiment of Michelson and Morley indicating that the earth drags the adjacent ether along with it, it occurred to Lodge [1] to investigate directly the effect of moving matter upon the ether. Two steel disks were mounted side by side and close together upon a common axle, and two interfering beams of light were passed in opposite directions around the annular space between the disks, by means of a system of mirrors. The disks could be rotated at a high speed, and if they dragged the ether wholly or in part the effect would be noticeable in a shift of the interference fringes, since one beam of light is travelling in the direction of rotation, the other in the opposite direction. No effect was observed at even the highest possible speeds. Thinking that perhaps the mass of the moving matter entered as a factor, Lodge substituted for the disks an immense spheroid of iron weighing half a ton, provided with a narrow circular crevasse along its equator, around which the luminous beams were reflected. The spheroid could be magnetized by means of a coil of wire, since it appeared possible that magnetization of the moving medium might have some effect. As in the previous case, the results were all negative, proving that at such speeds as can be handled in the laboratory the ether remains practically at rest. This is in agreement with all of the other experiments except the one performed by Michelson and Morley.

Influence of the Earth's Motion on Rotatory Polarization. — Lorentz developed a formula which apparently indicated that a change of one part in ten thousand in the rotation of the plane of polarization by active substances such as quartz was to be expected when the polarimeter, set parallel to the earth's orbit, was turned through 180°. Larmor, in his *Aether and Matter*, criticised this result, and concluded that no effect was to be expected. Lorentz replied, defending his position (*Proc. Amsterdam Acad.*, May 1902), and maintained that Larmor was in error. The subject was then attacked experimentally by Lord Rayleigh (*Phil. Mag.*, 4, p. 215, 1902), who found that the change, if it occurred, was less than $\frac{1}{10000}$ part of the total rotation. He used the five quartz blocks which had been prepared for Tait's rotation spectroscope, each block 5 cms. thick, the battery producing a rotation of over 4000 degrees for sodium light. As the difference in the rotations for D_1 and D_2 amounted to 11°, it was impossible to secure complete extinction with sodium light, and the helium tube was consequently employed, which gave an abundance of yellow monochromatic light. The apparatus was mounted on a horizontal stand, which could be

[1] Phil. Mag.

rotated on a pivot. No change whatever was observed, however, which was in agreement with the predictions of Larmor. Still more recently Bruce (*Phil. Mag.*, September 1905) has shown that any change must be less than $\frac{1}{10000000}$ of the whole.

Effect of the Earth's Motion upon the Intensity of Terrestrial Sources. — Fizeau came to the conclusion that if the ether was at rest with respect to the earth, the earth's orbital motion ought to affect the intensity of the light emitted by terrestrial sources, the light emitted in the direction of the earth's motion being less intense than that emitted in the opposite direction. This conclusion was reached by the following reasoning: Let A be a lamp (Fig. 397), B and C two screens, upon which the light falls, the whole apparatus moving with the earth through a stationary ether in the direction of the arrow, with a velocity v. Let the screens be at distance s from the lamp. To reach B the light has to traverse not the distance s, but the slightly smaller distance $\frac{sc}{c+v}$ (in which c is the velocity of light) since the screen is approaching the source during the passage of the light. To reach the screen C the distance traversed is $\frac{sc}{c-v}$. The intensity being inversely as the square of the distance, it follows that if J_0 is the intensity at each screen when the apparatus is at rest, or turned perpendicular to the direction of the earth's motion, the intensity when the apparatus is flying through the ether in the direction indicated will be

FIG. 397.

$$J = J_0\left(1 \pm \frac{2v}{c}\right) = J_0\left(1 \pm \frac{1}{5000}\right).$$

Fizeau proposed measuring the total intensity of the radiation at two points equidistant from a lamp by means of a pair of thermo elements opposed to each other, so as to secure compensation and no current. On rotating the apparatus through 180°, a feeble current would result from the slight change of temperature at the two points, due to the exchange of the slightly different intensities of radiation. Strangely enough, the experiment was not tried until 1903, when Nordmeyer (*Ann. der Physik*, 11, 284, 1903), working in Kayser's laboratory at Bonn, conducted a very careful series of experiments embodying the ideas of Fizeau.

The question was carefully considered from a theoretical standpoint by Bucherer, working in the same laboratory, his results appearing in a paper immediately preceding the one referred to above.

There are a number of points which must be carefully considered. A change of amplitude results from the difference of path, as shown above; in addition to this we have, however, a change in the wavelength of the emitted light due to Doppler's principle, and, as we have seen in the Chapter on Radiation Laws, a decrease of wave length, the amplitude remaining the same, means an increase in the

energy in unit volume of the medium. Moreover, if the intensity is measured by the thermopile, it is represented by the energy absorbed by the blackened surface of this instrument. If the instrument is moving against the light-waves, work must be done owing to the pressure of the radiation, the equivalent of this work appearing as heat in the body receiving the radiation. Bucherer considers all these questions carefully, and comes to the conclusion that Maxwell's theory leads to the conclusion that the intensity is increased by the amount $1+\frac{2v}{c}$, or in the same amount as found by the more elementary treatment given above. This is a first order effect, while all other experiments pertaining to the relative motion of the earth and the ether depend upon $\frac{v^2}{c^2}$, or on second order effects. Taking into account the pressure of radiation, however, and calculating the amount of energy absorbed by a moving screen, he finds that the common motion of the screen and source is without influence upon the intensity as measured by the heating of the absorbing surface. This result is obtained by first calculating the amount of energy absorbed by a screen moving in the direction in which the light is travelling. The relative velocity is then $c-v$, and the energy absorbed will be the $\frac{c-v}{c}$ part of that which would be absorbed if the body were at rest. From this is to be subtracted the heat equivalent of the work done by the pressure of the radiation, which is equal to the product of the pressure and the distance moved in unit time. This work can be done only at the expense of the absorbed energy.

Lorentz has also shown in a different manner that the earth's motion is without influence upon terrestrial sources of light.

The experimental investigation was conducted by Nordmeyer in the manner proposed originally by Fizeau, but with all the refinements employed at the present time. The results were purely negative, and it was established that the intensity was not changed by one part in 300,000 by the rotation of the apparatus, thus confirming the conclusions arrived at by Lorentz and Bucherer.

The experiment cannot, however, be regarded as proving that the ether is at rest with respect to the earth, since the same results are to be expected on the hypothesis of a moving ether. It is worthy of remark, however, that every experiment, with the exception of the one performed by Michelson and Morley, is in accord with the hypothesis of a stationary ether.

Optics of Moving Media: Lorentz's Treatment. — A very elegant treatment of the whole subject of the optical properties of moving media has been given by H. A. Lorentz. The assumption is made that in all cases the ether remains absolutely at rest, and all of the phenomena which we have had under discussion are explained by considering the system of electrons as having a motion of translation.

In the treatment of dispersion we have considered the compo-

nents of current density as made up of two parts, the displacement current in the ether and the convection current of the vibrating electrons, representing the components of current density by equations of the form

$$4\pi j_x = \frac{\partial X}{\partial t} + 4\pi eN\frac{\partial \xi}{\partial t}, \quad \ldots\ldots\ldots \quad (1)$$

in which ξ represents the displacement of the electron from its position of equilibrium within the body.

If now the whole system of electrons be set in motion of translation with a constant velocity, of which the components are v_x, v_y, v_z, the above equation must be modified by the addition of a term representing the convection current due to the motion of translation,

$$4\pi j_x = \frac{\partial X}{\partial t} + 4\pi eN\frac{d\xi}{dt} + 4\pi eNv_x. \quad \ldots\ldots\ldots \quad (2)$$

In this equation $\frac{\partial}{\partial t}$ indicates a change with respect to time *at a definite point in space*, while $\frac{d}{dt}$ indicates a change with time at a definite point *within the body*.

During the time element dt (in the formation of the differential quotient $\frac{d}{dt}$) the point in question moves a distance, the projections of which on the axes of coordinates are $v_x dt$, $v_y dt$, and $v_z dt$.

If x, y, z are referred to a stationary system of coordinates, we must write for a differentiation with respect to time as expressed by $\frac{d}{dt}$ (referred to a fixed point in the body),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}, \quad \ldots\ldots\ldots \quad (3)$$

since the quantity to be differentiated changes by the amount $v_x dt\frac{\partial}{\partial x}$, $v_y dt\frac{\partial}{\partial y}$, $v_z dt\frac{\partial}{\partial z}$ as a result of the motion of the body.

The components of the magnetic current density are represented as before by

$$4\pi s_x = \frac{\partial \alpha}{\partial t}, \text{ etc.} \quad \ldots\ldots\ldots \quad (4)$$

Our previous equation of motion of the electron was

$$m\frac{\partial^2 \xi}{\partial t^2} + re^2\frac{\partial \xi}{t} + \frac{4\pi e^2}{\theta}\xi = eX.$$

In the presen case the electrons are moving forward as a system, and can be regarded as constituting an electric current with components ev_x, ev_y, ev_z, on which the magnetic forces of the light-waves will act.

Our equation of motion must therefore be modified in a manner analogous to that employed in the treatment of magnetic rotation

by the hypothesis of the Hall effect, except that in that case the magnetic forces acting were those of the steady field due to the magnet.

Our equation now takes the form

$$m\frac{\partial^2\xi}{\partial t^2}+re^2\frac{\partial\xi}{\partial t}+4\pi\frac{e^2}{\theta}\xi=eX+\frac{e}{c}(v_y\gamma-v_z\beta). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For periodic changes we write as before

$$\frac{d\xi}{dt}=\frac{i}{\tau'}\xi,\ \frac{d^2\xi}{dt^2}=-\frac{1}{\tau'^2}\xi,$$

in which $\tau'=\frac{T'}{2\pi}$ refers to the period with respect to the moving body and τ with respect to a fixed point in space.

In the case of a moving body the periodicity of the vibration with respect to the body will be different from the periodicity with respect to a fixed point in space. If the medium is moving in the direction in which the light is travelling, it is obvious that the waves will pass more slowly by a given point in the medium than a given point in space; in other words, the periodic time will be greater.

Call T' and T the periodic times with respect to the moving medium and a fixed coordinate system. We have then $\tau'=\frac{T'}{2\pi}$ and $\tau=\frac{T}{2\pi}$.

Let us now determine the ratio of the two periodicities in terms of the velocity of the light and the velocity of the moving medium.

If ω is the velocity of the light in the moving body with reference to a fixed coordinate system, *i.e.* its absolute velocity, v_n the velocity of the body in the direction of the normal to the wave-front, and λ the wave-length in the moving medium, the periodicity T with respect to the fixed point in space is $T=\frac{\lambda}{\omega}$, while the periodicity with respect to a point in the medium is $T'=\frac{\lambda}{\omega-v_n}$, and the ratio

$$\frac{\tau'}{\tau}=\frac{T'}{T}=\frac{\omega}{\omega-v_n}=1+\frac{v_n}{\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

since v_n is small in comparison to ω, and can be neglected in the denominator of $\frac{v_n}{\omega-v_n}$.

Writing as before (see Chapter on Dispersion Theory, p. 404),

$$a=\frac{r\theta}{4\pi},\quad b=\frac{m\theta}{4\pi e^2},$$

we obtain from equation (5),

$$4\pi e\xi\left(1+i\frac{a}{\tau'}-\frac{b}{\tau'^2}\right)=\theta\left(X+\frac{v_y\gamma-v_z\beta}{c}\right),$$

an equation analogous to equation (3), p. 404.

From this equation is developed by a somewhat lengthy series of transformations (see Drude's *Optik*, pp. 423–427) the expression

$$\frac{n^2}{c^2}=\frac{\partial^2 X}{\partial t^2}+2\frac{n^2-1}{c^2}\frac{\partial}{\partial t}\left(v_x\frac{\partial X}{\partial x}+v_y\frac{\partial X}{\partial y}+v\frac{\partial X}{\partial z}\right)=\Delta x.$$

In the process it appears that $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$, which, in previous cases, has been shown to be equal to zero, is in the present case equal to $\frac{n^2-1}{c^2}\frac{\partial}{\partial t}(v_x X+v_y Y+v_z Z)$, which indicates that in a moving medium the electric force is no longer propagated in plane transverse waves; in other words, that the electric force is no longer in the wave-front.

It has seemed best to omit the steps by which the above equation is deduced, and simply state the final result. The two modifications which have been introduced into the equations applied to matter at rest, to make them conform to moving matter, should be clear in our minds, however.

First, the velocity of translation of the electrons is now sufficiently great to cause them to be acted upon by the magnetic field of the light-waves. Were this the case in matter at rest, we should have the refractive index a function of the intensity of the light, as we have seen in the Chapter on Magneto-Optics.

Secondly, the change in the period of the light with respect to the moving matter due to Doppler's principle is taken into account.

We can now deduce the velocity of light in a moving medium, such as the water in the tube system in Fizeau's apparatus.

Velocity of Light in Moving Medium. — Writing

$$X=Ae^{\frac{i}{\tau}\left(t-\frac{p_1x+p_2y+p_3z}{\omega}\right)},$$

and differentiating X twice with respect to t and once with respect to x, y, and z, and substituting in the final equation given above, gives us

$$\frac{n^2}{c^2}-\frac{2(n^2-1)}{c^2}\frac{p_1v+p_2v_n+p_3v_z}{\omega}=\frac{1}{\omega^2},\text{ or }\frac{n^2}{c^2}\left(1-\frac{2(n^2-1)}{n^2}\frac{v_n}{\omega}\right)=\frac{1}{\omega^2},$$

in which v_n is the velocity of the medium in the direction of the normal to the wave-front.

From the above equation we get

$$\omega^2=\frac{c^2n^2\omega}{n^4\omega-2n^4v_n+n^2v_n}=\frac{c^2}{n^2}\left(\frac{n^2\omega}{n^2\omega-2n^2v_n+2v_n}\right),$$

which by simple division of the last fraction gives

$$\omega^2=\frac{c^2}{n^2}\left(1+\frac{2(n^2-1)v_n}{n^2\omega}\right),\quad\ldots\ldots\quad(7)$$

if we neglect $2n^2v$ and v_n in the denominator of the remainder, which we can do since they are small in comparison to the first term $n^2\omega$.

This equation gives us for ω, the velocity in the moving medium, with reference to a fixed system of coordinates,

$$\omega = \frac{c}{n}\left(1+\frac{n^2-1}{n^2}\cdot\frac{v_n}{\omega}\right),$$

the second term of which can be freed from ω if we substitute for it its approximate value $\frac{c}{n}$, which is justifiable, since $\frac{v_n}{\omega}$ is a very small fraction. We find thus

$$\omega = \frac{c}{n}+\frac{n^2-1}{n^2}v_n, \quad \ldots\ldots\ldots \quad (8)$$

an expression identical with the one developed by Fresnel, with the exception that in the above expression n is not the refractive index for the absolute period T, but for the relative period T'. If we wish to get an expression for ω in terms of the refractive index of the medium at rest for the absolute period T, we proceed as follows:

Referring to equation (6), we see that

$$T' = T\left(1+\frac{v_n}{\omega}\right).$$

If now we designate the refractive index of the medium at rest for the absolute period T by ν, we have

$$n = \nu+\frac{\partial\nu}{\partial T}\cdot T\frac{v_n}{\omega} = \nu+\frac{\partial\nu}{\lambda\partial}\lambda\frac{v_n}{\omega},$$

in which $\lambda = cT$, the wave-length in vacuo.

Substituting in equation (8) gives us

$$\omega = \frac{c}{\nu}\left(1-\frac{\partial\nu}{\partial\lambda}\cdot\frac{\lambda}{\nu}\frac{v_n}{\omega}\right)+\frac{n^2-1}{n^2}v_n,$$

or if we write the approximate values $n=\nu$, $\omega=\frac{c}{\nu}$ in the members containing v_n,

$$\omega = \frac{c}{\nu}+v_n\left(\frac{\nu^2-1}{\nu^2}-\frac{\lambda}{\nu}\frac{\partial\nu}{\partial\lambda}\right).$$

In this expression $\frac{c}{\nu}$ is the velocity of light in the medium at rest for light of absolute period T, while the second term represents the change in velocity due to the motion of the medium.

This change is slightly larger than in the previous formula, since $\frac{\partial\nu}{\partial\lambda}$ is negative in normally dispersing media.

It is worthy of remark, however, that the results obtained by Michelson and Morley are better expressed by the simpler formula, which is identical with the original one of Fresnel. Experiment gave for water the value .434 for the term to be multiplied by v_n; the first formula gives .438, the second .451.

In conclusion, it may be remarked that all experimental evidence, with the exception of the Michelson-Morley experiment, is in favor of a stagnant ether. A test of the matter, about which there could be no question as to its interpretation, would appear to be to measure the velocity of light on the earth's surface, first in the direction of its motion through space, and then in the opposite direction, with the accuracy required to show the ether's motion if it exists.

Various methods for accomplishing this have been suggested, but all have been shown to be impracticable, and as we shall see in the next chapter it is impossible to reach any conclusion in this way.

CHAPTER XXV

THE PRINCIPLE OF RELATIVITY

We have seen in the last chapter that, so far as the testimony of all experiments which have been tried up to the present time goes, the motion of the earth does not affect optical phenomena exhibited with terrestrial sources of light. So far as we have been able to find, the measured velocities of light parallel and perpendicular to the direction of the earth's motion are the same. It is true that we have never made independent observations in both directions, and we shall see presently that even the results of such determinations would be open to question, but the Michelson-Morley experiment has shown, beyond any doubt, that we must remodel our views regarding the stationary ether assumed by Lorentz. The Theory of Relativity is an attempt in this direction. It was first definitely formulated in its entirety by Einstein, though the work of Lorentz may be regarded as having paved the way for it. It has been placed on a substantial mathematical basis by Minkowski, and may be regarded as partially proved by the experiments of Kaufmann and of Bucherer upon the mass of the moving electron, and the negative results of all ether-drift experiments. The Fitzgerald-Lorentz hypothesis of a contraction of matter in the direction of its motion, which was formulated as a sort of loop-hole of escape from the difficulty arising from the Michelson-Morley experiment, will be seen to be the logical outcome of the much more general postulates established by Einstein, which denies at the outset the possibility of ever measuring or even detecting absolute motion through space.

The relativity of motion as applied to ordinary mechanics dates back to the times of Newton and Galileo. Whether or not a mass has kinetic energy depends upon our point of view, and our view will be the same whether the mass is moving by us, or we are moving by the mass. If we are moving with the mass, it has no kinetic energy as far as we are concerned.

With regard to the ether, however, a different view has been held, for it has been regarded as a sort of fixed frame of reference, with respect to which bodies could be considered at rest or in motion. Einstein's postulate denies the possibility of absolute velocity (which must be considered possible if we have a substantial medium or ether) and consequently abolishes the ether at the outset. With the disappearance of the ether we are forced to remodel our views concerning light and electro-magnetic waves, which now become a type of energy which propagates itself in an absolute vacuum, with a velocity which depends, not upon the physical properties of the space through which it is moving, but upon the properties

possessed by the electro-magnetic energy. Light thus seems to be in the nature of something expelled by the source, and Einstein is of the opinion that the next phase of the development of theoretical physics will be the formulation of a new theory of light which will be in the nature of a fusion of the wave-theory with the old corpuscular theory. The Principle of Relativity has been worked out in two different ways. Einstein has determined the effects which relative motion between two observers has upon the units of time and space used by the observers; and Minkowski has ascertained the nature of the transformations necessary to pass from a stationary to a moving system of coordinates, *which shall leave* Maxwell's equations unaltered. Similar conclusions are arrived at by the two methods. These conclusions completely upset our old ideas of time and space. The unit of time, which we have regarded as absolute, becomes by the principle of relativity dependent upon the state of the observer. The units of time in a moving system are longer when considered by a stationary observer. If, in a system which is in motion relative to us, there are two synchronous clocks, placed at a distance apart, and in the line of the motion, they will not be synchronous for us, and what is more, the distance between them will be less for us than for an observer moving with them. Some writers hold that these changes are only apparent, that is, they are psychological in character, arguing, for example, that an electron appears shortened in a different direction to each of a number of observers moving in different directions with respect to it. The physical condition of the electron does not, however, depend upon the state of mind of the observers. This argument is open to criticism, however. It presupposes that an electron or a meter bar has a *real* length or shape. On the relativity theory *there is no such thing as real length, either of time or space.*

As an illustration we may recall that, what to a *stationary* observer is an electro-static field, is an electro-magnetic field to a *moving* observer. What is the *real* nature of the field? It is one thing to one observer, and quite a different thing to the other. Both observers are right, each from his own point of view, and the term " real nature of the field " becomes meaningless. It is very difficult to form a conception of how the units of time and space are affected by relative motion, and of the relations between time and space.

Planck considers that this new conception of our notions of time surpasses in boldness anything that has appeared up to the present time in speculative science, in comparison to which non-Euclidean geometry is child's play.

The Theory of Relativity starts out with two postulates.

The first of these states that the uniform motion of translation cannot be measured or even detected by an observer stationed on the moving system from observations confined to the system. This amounts to saying that motion through the ether (if the ether exists at all) will be wholly without influence upon all optical experiments made with terrestrial sources of light.

The second postulate is that the velocity of light in space is a

constant, independent of the relative velocity of the source and the observer.

Combined with the first postulate this leads us to the extraordinary conclusion that the *measured* velocity with which light passes a moving observer is the same as for a stationary observer. Going back now to the conception which we made use of in the treatment of the Michelson-Morley experiment, of a moving observer who sets off a flash of light, and then drifts away from the center of the wave, we see at once that, on the theory of relativity, so far as any measurements which he could make, even with imaginary apparatus, are concerned, he will find himself always at the centre of the wave. Before discussing how this seeming impossibility can be explained, we must first define what we mean by time, and what we mean by the coincidence in time of two events.

Einstein defines time and isochronism (Gleichzeitigkeit) in the following way.

Suppose we have a clock at a point A in space. An observer at A can establish a time value for an event at A by observing the position of the hand of a clock, at the moment at which the event occurs. This event it must be noticed is referred to the clock at A, the time of which we will call the "A time." An observer at a distant point B refers an event happening there to a clock at B; that is, he records it in "B time." It is, however, impossible to compare the times of happening of two events, one at A and the other at B, without establishing a further point, for we have not yet defined a "common time" for the points A and B. We must clearly understand what we mean when we say the clocks are together in time. Einstein's definition of the isochronism amounts to saying that two clocks are together when the time of transit of a flash of light from one clock to the other, as measured by the clocks, is the same in each direction.

As an illustration we will suppose that an imaginary inhabitant of the sun has established heliographic communication with the earth, and that we have decided to synchronize a clock on the earth with one on the sun. Light requires about 8 minutes to cover the distance, so we must allow for this in setting the clocks. We will begin by assuming that the clocks ar together, and establish a method by which an observer on the sun and one on the earth could ascertain this fact. Suppose we send a signal (a flash of light) to the sun at the moment when the hands of our clock indicate noon. We will call this 0 time for convenience. The signal reaches the sun at 8 minutes past, as indicated by the solar clock, and a signal is instantly flashed back to the earth that the arrival time of the earth signal was 8. This signal reaches the earth at time 16, and we have the equal times of transit 16–8 and 8–0. This will be true only if the clocks are perfectly synchronized. If the solar clock was a minute ahead of ours, the observer there would signal 9 as the time of arrival and we should receive this signal at 16 as before, and have

$$16-9>9-0.$$

As we shall see presently, we must define what we mean when we say that the clocks are together. The above definition, based on the assumed equality of the times of transit of a light signal to and from, is as good a definition as any we can make, but as we shall now see, it holds only for observers on the sun-earth system.

Suppose now that the sun and earth are moving together through space, in the direction of the line joining them, the earth being in advance and the sun following. From what we know of the motion of the solar system, this would occur when the earth is between the sun and the constellation Hercules. Suppose the clocks to be together as before, and our signal sent off at time zero. Since the sun is moving towards the point occupied by the earth at the moment when the flash started, the signal will reach the sun in less time, say at 7 minutes past 0. (We are of course assuming a very high velocity of the solar system.) The solar observer flashes back the signal "7," which requires more than 8 minutes to reach us as we are moving away from it. Let us suppose that it reaches us in 9 minutes, our clock indicating 16 as before, which will be a sufficient approximation to the truth to answer our present purposes. We find that 16–7 is not equal to 7–0, and conclude that the clocks are not together, the solar clock appearing to be one minute behind ours. We now signal this circumstance to the solar observer and he sets his clock ahead 1 minute. We shall now find the times of transit equal. The signal actually reaches the sun at 7 minutes past by our time and 8 minutes past by solar time. It returns to us in 9 minutes, and our clock indicates 16 minutes of elapsed time. We have received the signal "8" from the sun and have

$$16-8=8-0.$$

The clocks therefore appear together, though the solar observer has actually advanced his one minute. The clocks will not be together for a stationary observer outside of the sun-earth system. To him the forward (earth) clock in the moving system would be one minute behind the following (solar) one.

We brought about this state of affairs by "considering the system in motion. The same condition will obtain, however, if we set the observer, situated outside of the sun-earth system, in motion in the opposite direction, for there is no such thing as absolute motion, by the principle of relativity, and there is absolutely nothing by which we can distinguish a motion of the sun-earth system past a fixed outside observer, from a motion of the outside observer past the sun-earth system.

All that we can say is that if we consider the sun and earth as moving relatively to some fixed system of coordinates, an observer fixed in this system will find that the clocks which we have set together are not together from his point of view.

In other words we can attach no *absolute* meaning to the coincidence of two events. Two events which coincide on a moving, system from the point of view of an observer moving with the system, do not coincide from the point of view of a stationary ob-

server, or as Einstein puts it, "Two events which are simultaneous when regarded from one coordinate system, are not to be regarded as simultaneous when considered from a coordinate system which is in motion with respect to the first.

The time at two widely separated points does not, however, come into determinations of the velocity of light as we make them. In fact we do not use clocks at all, though the revolving wheel used in the Fizeau method amounts to a clock. In the Michelson-Morley experiment no measurements of time are made at all, and it is not at once apparent how considerations of time have any bearing on this experiment.

If two widely separated clocks A and B are "together" or "in time" the condition which must be fulfilled, as defined by Einstein, is that the time required for light to pass from A to B as measured by the clocks shall equal the time required for light to pass from B to A, or if the flash leaves A at time T_A (indicated by clock A) and arrives at B at time T_B (indicated by clock B) and being reflected back to A where it arrives at T'_A (indicated by clock A) that we shall find

$$T'_A - T_B = T_B - T_A$$

Now suppose that we have a long rigid bar of length l with a clock at each end of it. This bar moves along the x axis with a velocity v. Two stationary clocks are placed at such a distance apart that the clocks on the moving bar pass them at the same moment. An observer in the stationary system synchronizes his clocks so that the above condition holds. Observers on the moving system set their clocks so that they read the same as the stationary clock *at the moment* at which they pass them. We should now imagine that all four clocks are together, yet if an observer on the moving bar applies the test for synchronism he will find that it does not hold, and will obtain

$$T'_A - T_B = \frac{l}{c+v} \text{ and } T_B - T_A = \frac{l}{c-v}.$$

The difference in time between two clocks synchronous in a moving system, when considered by a stationary observer, depends upon the distance between the clocks in the line of motion and their velocity. If there is no distance between them in the line of motion, *i.e.* if the line joining them is perpendicular to the line of motion, they will be synchronous for the stationary observer also. If one clock is revolving around the other, as the whole system moves forward, it will alternately be "ahead of" and "behind" the time indicated by the other, for the clock which is in advance is always "slow." The difference in time between two clocks separated by a distance l, and moving with a velocity v parallel to the line joining them, can be shown to be

$$\frac{vl}{c^2\sqrt{1-\frac{v^2}{c^2}}} \text{ or approximately } \frac{vl}{c^2}.$$

If we examine the question a little more carefully we shall see that setting the forward clock behind in time is not sufficient in itself to make the times of transit in each direction appear equal to each other, and equal to the time of transit when the system is at rest. Suppose that we are on a stationary system, and measure the time required for light to travel to a distant mirror and return to us. Now consider our system in motion $\perp$ to the line joining us with our mirror and repeat the experiment. The time of transit ought to be found longer, as was shown in the discussion of the Michelson-Morley experiment. By the principle of relativity no difference can be detected. In this case we are dealing with a single clock only, *i.e.* we have eliminated the complication of two widely separated clocks, and are therefore driven to the conclusion that a clock in a moving system runs slower, *i.e.* the second registered by a moving clock appears longer to a stationary observer. The ratio will obviously be that of the distances traversed by the light in the two cases, or if we construct a right triangle one leg of which is equal to the distance of our mirror, and the other equal to the distance we move while the light is going to the mirror, the required ratio will be that of the hypothenuse or the longer leg, or,

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

which is the ratio of the second registered by a moving clock to that registered by a stationary one. This change in the unit of time makes the velocity of light measured in a direction perpendicular to the line of motion come out the same as the velocity measured in the stationary system.

Suppose, however, that we swing our distant mirror around until it is in the line of motion. In this case the increment of path due to the motion of the system is twice as great as in the previous case. The changed rate of our clock can only account for one-half of the discrepancy, and we conclude that the distance of the mirror has been reduced in the same ratio as that in which the second has been increased, *i.e.* that the unit of length in a moving system has been shortened in the ratio

$$\frac{\sqrt{1-\frac{v^2}{c^2}}}{1},$$

which is the ratio of a moving meter to a stationary one.

By the principle of relativity we have thus established three circumstances. The unit of time of a moving system appears longer to a stationary observer, the unit of length appears shorter, and the leading clock is behind in time.

Let us now apply these results to an imaginary experiment upon the velocity of light. Instead of a clock we will use the Fizeau toothed wheel. Going back to the theory of the Michelson-Morley experiment, we must form a clear picture in our mind of how the

velocities along the different paths ought to appear affected by the earth's motion. The distance along the path which is parallel to the direction of motion is *increased* for the light going, and *reduced* for the light returning, the increment, however, being greater than the reduction, which makes the whole path greater. This increment is double the increment along the other path ($\perp$ to direction of motion), so that in the experiment it was the difference between these two increments which was expected to make itself felt.

Let us consider that we have a toothed wheel and a distant mirror on a stationary system, and send a beam of light between two of the teeth to our distant mirror which reflects it back in such a direction that it passes through the same pair of teeth. We now set the wheel in rotation with such a speed that the returning beam falls upon the tooth which is adjacent to the aperture. We now set the system in motion in a direction from the wheel to the mirror. The total path traversed becomes greater as a result of this motion, so that, on the return of the light, the wheel should have turned through a greater angle and carried the tooth past the point on which the beam falls. This would give us a means of measuring absolute motion, and since this is impossible our wheel must turn slower by an amount just sufficient to make the tooth cover the returning beam. But, as we have seen, distances are shortened in the moving system, so that we can regard one half of the discrepancy taken care of in this way, the other half by the slower speed of rotation. In this hypothetical experiment we have considered the system first as at rest and then in motion. In practice we cannot bring our system to rest, but we can turn it so that it is first parallel and then perpendicular to the line of the earth's motion. In this case the wheel (or clock) will be running at a constant rate, so that the change in unit of length is the only factor left. This is all that we need, however, for in this case we have only to balance the difference between the increments of the two paths resulting from the motion, which is only half as great as the increment of the path parallel to the direction in which the system moves.

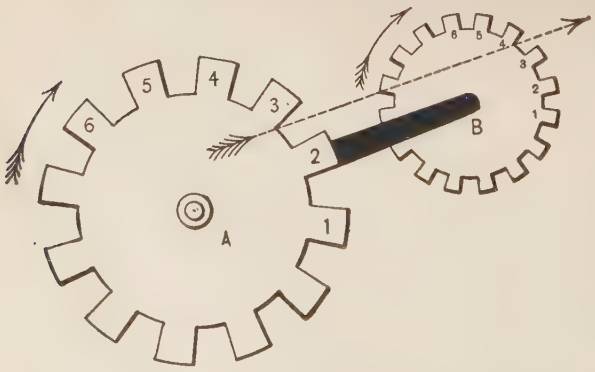


Fig. 398.

The greatest change in the time we should expect to find in measuring the time of transit, in one direction only. We need two wheels for this, serving as clocks at the sending and receiving stations. These wheels we can consider as mounted on a single axle and turning together (Fig. 398). Suppose

the wheels at rest as a system, but turning with such a speed that the light which passes between the 2d and 3d tooth of the first wheel A, passes between the 3d and 4th tooth of the second wheel B. If we set the system in motion in the same direction as that of the light ray, the wheels will have turned through a greater angle when the light reaches the second wheel, and the light should be arrested say by the 4th tooth. If we set the system in motion in the opposite direction, the light should be arrested by the third tooth. In order to insure that this shall not happen, and it cannot happen according to the principle of relativity, it will be necessary to establish a phase-difference between the wheels, by turning the forward wheel back a little, in other words, make the leading clock slow. This implies that a revolving rigid body in motion parallel to its direction is, from the point of view of a stationary observer, not only shortened, but twisted as by torsion.

The reader is advised to make a pasteboard model of the two wheels, and assume convenient magnitudes for c, v, l, etc. It is easy to see that a twist of the axis only, will not make the velocity of propagation appear the same in both directions, for the moving and stationary states, and that we must consider the reduced speed of rotation and the shortening of the axis when the system is in motion. It will be found that when the system is travelling with the light, the shortening and speed reduction act in the same sense as the setting back of the forward clock, but that when the system is travelling in the opposite direction they act in a sense opposed to the setting back of the forward wheel. For example, if we let $l=100$, $c=25$, and $v=2$, we have for the time when the system is at rest $\frac{100}{25}=4$. When moving in the same direction as the light the time is increased to $\frac{100}{25-2}=4.35$, when in the opposite direction it is diminished to $\frac{100}{25+2}=3.7$.

The increment is .35, the decrement .3, and the difference, or the time increase for to-and-fro passage, is .05.

It is clear that the increment and the decrement would have to be equal if equality in the times of transit was to be secured by setting one wheel or the other ahead by a fixed amount. The difference in time between the two wheels in either case is $\frac{vl}{c^2}=\frac{200}{625}$ or about .3. This amount of the time difference is compensated for by the difference in the setting of the clocks, the rest, or .05, is taken care of by the reduction in the speed of rotation and the shortening of the axis.

We must next consider the case of the Michelson-Morley experiment in the light of the principle of relativity. Here no clock, wheel, or other mechanical time-measuring instrument is used. The sodium atom which furnishes the light in this experiment is to be sure a clock of far greater precision than any mechanical clock, and by the principle of relativity it must " run slow " in a moving

system. This change in the speed of the atomic clock will not produce any effect, however, as it remains constant throughout the experiment.

Obviously the difference in the actual time at the two ends of the path does not come into play, and the only thing remaining is the change in the unit of length. This is sufficient to take care of the discrepancy which in the present case is the *difference* between the increments of the two paths, which is only about one-half of the increment of the path parallel to the motion. If the other path were shortened to zero, the interferometer could be considered as analogous to the toothed wheel. In this experiment we are of course moving with the instrument, and the length of the path, from our viewpoint, is not affected by turning it into the line of the earth's motion. The change of length by the relativity principle, is only from the viewpoint of the stationary observer, and a question may occur as to where the stationary observer is in this case. It must be remembered, however, that we think of the wave of light as spreading out with reference to a fixed frame of reference, and that the instrument is moving when considered from the viewpoint of an observer fixed in this frame of reference. This means that the path must be considered as changed so far as the wave is concerned, or from the viewpoint of the wave, so to speak.

One interesting consequence of the theory of relativity is that the frequency of the light emitted by a moving source must appear changed to a stationary observer, even when there is no component of the motion in the line of sight. This means that there should be a small effect superposed on the Doppler effect, which is wholly independent of the direction of the motion. The wave-lengths emitted by the canal rays in a direction perpendicular to their motion ought then to be slightly longer than those emitted in the case of the same gas in an ordinary discharge tube. There appears to be little hope of detecting the effect, however, since the canal rays always show the undisplaced line as well as the displaced, due to the circumstance that there are stationary centres of emission as well as moving.

In the foregoing treatment an attempt has been made to give an idea of how the principle of relativity is applied in the case of measurements of the velocity of light. In spite of what has been said about setting systems in motion, it must be clearly understood that absolute motion plays no part at all in the theory. There is nothing to distinguish the motion of a system past an observer, from his motion past the system, and the physical phenomena will appear the same in each case. There are applications of the theory much wider than any that have been taken up in this very elementary treatment. Especially interesting are the conclusions regarding the apparent " mass " of radiation. If we have a body, isolated in space, which is emitting radiation in a single direction, the " recoil pressure " of radiation will exert a force which we should imagine would give to the body an absolute motion. If E is the amount of energy emitted per second, this force will be $\frac{E}{c}$. On the prin-

ciple of relativity there can be no absolute motion, and we are forced to the conclusion that the radiation must possess mass and carry away a portion of the total mass with it. If we call M the mass of the radiation, its momentum is Mc, which must be equal to the force exerted on the system. We have then $\frac{E}{c} = Mc$, or $M = \frac{E}{c^2}$ from which it follows that we must regard a portion of the mass as represented by the energy contained in a substance. The liberation of energy should be accompanied by a reduction in the mass (as measured by its inertia). Whether a change in weight would result is uncertain, for we cannot be sure that inertia and weight are proportional under all circumstances. Einstein suggests that the vast liberation of energy which accompanies radio-active changes may enable us to detect the change in the mass. It is possible that the theory of relativity will reawaken an interest in the old question of whether a change of weight results from chemical action, or any change of state which involves the liberation or absorption of a large amount of energy. Einstein even goes so far as to suggest that the radiation is something that is expelled as definite units which move out without loss of energy. This means, of course, that the energy is not equally distributed over what we have been accustomed to speak of as the wave-front, but is localized at points. Planck has come to the conclusion, from a different line of reasoning, that energy is liberated or absorbed only in units (Lichtquanten), but does not believe in the discontinuities in the energy distribution over the wave-front. He draws attention to the circumstance that interference with a large path-difference shows that the units of radiation must be regarded as having lengths equal to many thousand wave-lengths.

The principle of relativity has been subjected to experimental tests, with good results, by Kaufmann and Bucherer, who have studied the laws which govern the motions of the electrons which constitute the β rays of radium and travel at velocities of the same order of magnitude as that of light. The reader is referred to the original papers of Einstein (*Annalen der Physik*, 1905, 1907) and Minkowski (*Phys. Zeitschr.*, 10, p. 104, 1909) for the complete mathematical treatment. Minkowski adopts a novel method of showing the space and time relations. To the three axes of coordinates, x, y, and z, he adds a fourth, the time axis. Time thus becomes the fourth dimension, and the transformation equations correspond to a rotation of the four-dimensional reference system (x, y, z, ict through the imaginary angle arc $tg\left(i\frac{v}{c}\right)$. Minkowski starts out with the statement that we only observe a place at a given time, and a time at a given place. Space and time are thus bound together and a time axis is added to the coordinate system. This brings us at once into non-Euclidean geometry, for the time axis runs in a direction perpendicular to all three of the directions x, y, and z, of our space coordinate system. We cannot of course conceive of this direction at all, except perhaps in a vague way by the

methods given in the popular articles in the fourth dimension. A point with coordinates x, y, z, t is called by Minkowski a "Welt-punkt." A point moving parallel to the t axis traces out a "Welt-linie," and represents the existence in time of a stationary point. If the welt-linie is inclined to the t axis, it represents a point moving with respect to x, y, and z. If the point moves in the direction of the x axis, the line lies in the plane xt. A line inclined at $\angle\,\alpha$ to the t axis represents a constant velocity $v = \tan\alpha$. If we lay off on the t axis, instead of t, the value ct, then $\tan\alpha = \frac{v}{c}$, the velocity of the point, with the velocity of light as a unit.

Minkowski represents his new coordinate system diagrammatically in the following way. He imagines an abstraction represented by the equation $c^2t^2 - x^2 - y^2 - z^2 = 1$, which is analogous to the hyperboloid. Abolishing one of the space coordinates, we have room for the time coordinate, and can represent the thing in

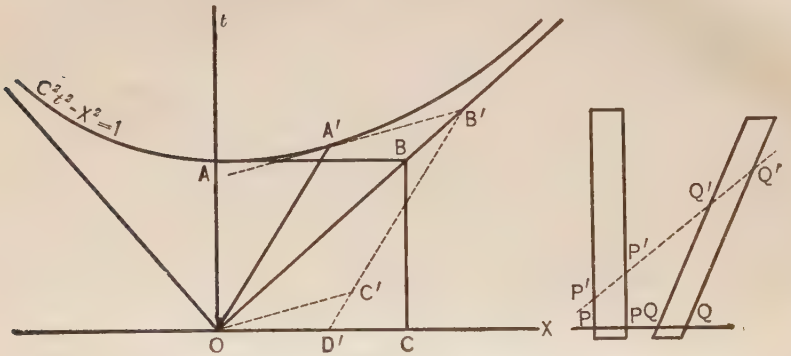


FIG. 399.

perspective diagrammatically. It is as well to abolish two of the space dimensions, and content ourselves with a cross-section parallel to the xt plane. We now have the hyperbola $c^2t^2 = x^2 - 1$ shown in Fig. 399 with its asymptote OB'. The line OA' (a radius vector), inclined to the t axis, represents a velocity v parallel to x. Minkowski now draws a tangent to the hyperbola at A' which cuts the asymptote at B', and constructs OC' and $B'D'$ parallel to $A'B'$ and OA', obtaining in this way an oblique system of coordinates, in which the hyperbola is represented by an equation of the same form as before. The transformation from one system of coordinates to the other represents the transformations required in the principle of relativity, for the oblique system represents a motion with velocity v in relation to the other system. The coordinates of the complete oblique system are designated by x', y', z', t'.

The reader should refer to the original paper, and we have room for but a brief mention of one way in which this construction is used. Suppose we wish to show the relativity shortening. On our diagram, in which we are dealing with space of one dimension,

we construct a vertical strip of width $l \cdot OC$ represented by PP, parallel to t, and an oblique strip parallel to OA'. (The quantity OC is the unit of length on the x axis.) In the case of the oblique strip, if we put t' and x' in place of x and t, it is at rest and the vertical strip is considered in motion. The width PP of the vertical strip represents the length which we are investigating. The oblique strip represents equal length $Q'Q' = l \cdot oc'$ when considered at rest. These strips represent two Lorentz electrons, and holding to our original coordinates xt, the extension of the moving electron is QQ, taken parallel to x, since $Q'Q' = l \cdot OC'$ and $QQ = l \cdot OD'$. If now $\frac{dx}{dt}$ = velocity of the second electron $OD' = OC\sqrt{1-\frac{v^2}{c^2}}$ and

$$PP : QQ = 1 : \sqrt{1-\frac{v^2}{c^2}},$$

which shows the shortening resulting from the motion.

Next consider the second electron (represented by the oblique strip) at rest, adopting now the coordinates X' and t'. The length represented by the first strip is $P'P'$ parallel to OC', and we find the first shortened with respect to the second, for

$$P'P' : Q'Q' = OD : OC' = OO' : OC = QQ : PP.$$

All of the other relations can be deduced in analogous ways, and the method is a most beautiful one. The reader is strongly advised to read the original memoir carefully, as it is very difficult to get much of an idea of it from such a brief synopsis as the one just given.

INDEX

Abbé, diffraction theory of microscope, 223.
Aberration of light, 667.
Abraham and Lemoine, method of, 549.
Abramczyk, emission and absorption of rock salt, 601.
Absent spectra, 209.
Absorbing media and dispersion formula, 395.
Absorbing particles, polarization by, 628.
Absorbing solutions, 15.
Absorption, by flames, 593.
  by gases and vapors, 443.
  by hydrogen gas, 433.
  by luminous hydrogen, 446.
  by metals, 449, 464.
  by porous surfaces, 449.
  by single resonator, 454.
  electro-magnetic theory, 457.
  of sodium vapor, 443.
  theories of, 450.
Absorption bands, 437.
  and emission, 601.
  calculation of position, 389.
  effect of density on, 444.
  form of, 448.
  influence of solvent, 442.
  laws of, 437.
  temperature influence, 441.
Achromatic interference fringes, 139, 166.
Achromatic lenses, 106.
Achromatic prisms, 104.
Achromatism, 104.
Airy, diffraction by circular aperture, 236.
  experiment on aberration, 668.
  rotatory polarization, 482.
  theory of rainbow, 345.
Ames, origin of prismatic colors, 661.
Amici principle, 79.
Anderson, emission of rare earths, 603.
Ångström, effect of density on absorption, 444.
Anomalous dispersion, 113.
  and solar phenomena, 115.
  of sodium vapor, 422.
Anomalous propagation of waves near foci, 261.
Anthelion, 347.
Aperture, diffraction by circular, 236.
  diffraction by rectangular, 196.
Aplanatic lens, 78.
Aplanatic points, 79.
Aplanatic surfaces, 48.
Arago, interference of polarized light, 148.
Astronomical phenomena and interference, 137.
Astronomical refraction, 82.
Axis of single wave velocity, 316.
  ray velocity, 318.

Babinet, velocity of circular components, 481.
Babinet's principle, 238.
Balmer series of sodium, 443.
Bartholinus, double refraction, 303.
Bartolli, radiation pressure, 609.
Beats, light, 138, 340.
Becquerel, E., phosphorescent spectra, 567.
Becquerel, J., magnetic rotation, 543.
  phosphorescence at low temperature, 567.
Beer's law of absorption, 438.
Bélopolsky, Doppler principle, 23.
Biaxal crystals, 306.
  isochromatic surfaces, 323.
Bichat and Blondlot, time for Faraday effect, 500.
Billet, split lens, 147.
Bi-prism for interference, 130.
  variable angle, 131.
Black-radiator, perfect, 607.
Body color, 439.
Boltzmann, emission as a function of temperature, 614.
Boundary conditions, 356.
Brace, circular components in magnetic rotation, 497.
Bradley, aberration, 667.
Braun, F., polarization by metal gratings, 635.
Brewster's law, 289.

Calorescence, 556.
Canal rays, 24.
Carvallo, nature of white light, 652.
Cauchy, dispersion formula, 375.
Caustics, 53, 57, 71.
  interference fringes bordering, 148.
Chemical changes effected by light, 590.
Christiansen's experiment, 110.
Circular components in rotatory polarization, 480, 486, 497.
Circularly polarized light, by total reflection, 330.
  direction of revolution, 337.
  intensity of, 328.
  production of, 328.
Circular polarization, 326.
Circular polarized light, convergent, 324.
Coblentz optical properties of stibnite, 400.

Color, and wave-length, 10.
body, and surface, 439.
of metals, 456.
Color photographs of Lippmann, 176.
Color screens, 14.
Colors, of frilled films on metal, 172.
of iridescent crystals and opals, 160.
of mixed plates, 252.
of thin plates, 151.
subjective, 10.
Compensator, Jamin, 334.
Concave gratings, 231.
theory of, 233.
Concentration of light in grating spectra, 226.
Conical refraction, 317, 318.
Convergent polarized light, 318.
Corbino, magnetic rotation:
nature of white light, 655.
of absorbing flames, 530, 540.
Cornu; per cent of polarization, 298.
quartz prism, 480.
Cornu's spiral, 192, 248.
Corresponding points of sources in interference, 131.
Cotton, absorption of circular components, 494.
elliptical polarization by absorbing active media, 495.
Kirchhoff's law, 593.
polarization of light by magnetic field, 521.
rotatory dispersion, 495.
Zeeman effect, 515.
Cotton and Mouton, magnetic double refraction, 547.
Critical angle, 66.
Crystal plates in convergent polarized light, 320.
Curved rays, 90.
Cyanine prisms, 114.
Czapski, refraction by curved surface, 73.

Descartes, theory of rainbow, 342.
Deviation by prism, 74.
minimum, 75.
Dewar, phosphorescence at low temperatures, 566.
Dichromatism, 438.
Diffraction by absorbing screens, 634.
circular disk and aperture, 185, 236.
large number of apertures, 240.
narrow slit, 186, 193, 201.
narrow wire, 193.
particles on mirror, 242.
rectangular aperture, 196.
straight edge, 183, 192, 249.
thin lamina, 250.
two circular apertures, 239.
two parallel slits, 189, 195, 202.
Diffraction grating, 203.
intensity distribution, 209.
number of spectra and, 209.
Diffraction of light, 181.
Diffraction phenomena, two classes, 186, 244.
Diffraction problems, introduction of angular measure, 200.
Direct vision prisms, 105.
Dispersion, anomalous, 113.
curves, 102–103.
electro-magnetic theory, 401.
Newton's experiments, 102.
nitroso-dimethyl aniline, 399.
of powders, 112.
of rock salt, fluorite, etc., 394.
of solutions of aniline dyes, 398.
selective (anomalous), 379.
selective, of absorbing gas, 418.
theory, Helmholtz, 380.
Dispersion formula, of Cauchy, 375.
of Maxwell, Sellmeier, 376.
Displacement currents in ether, 353.
Doppler effect, in star spectra, 25.
in canal rays, 24.
Doppler-Fizeau principle, 23.
Double refraction, 303.
Fresnel's theory, 309.
in non-crystalline media, 324.
magnetic, 547.
of vibrating plate, 325.
Drude, dispersion theory, 401.
surface films, 370.
theory of magnetic rotation, 524.
Drude and Nernst, stationary light-waves, 176.
Du Bois, polarization by gratings, 632.
Du Bois and Elias, magnetic rotation, 543.
Dufour, abnormal Zeeman effect, 521.

Earth's motion, effect on intensity of light, 677.
effect on rotary polarization, 676.
Echelette grating, 227.
Echelon, mica for demonstration, 278.
resolving power of, 276, 286.
theory of, 275.
Echelon grating, 274.
Egoroff and Georgiewski, polarization by magnetic field, 520.
Einstein, "Licht-quanten hypothesis," 554.
principle of relativity, 684.
Electro-magnetic theory, 350.
Electro-optic effect, 548.
Electro-optics of benzol, 550.
Ellipsoidal mirrors, 48.
Elliptically polarized light, by reflection, 335.
constants of, 334.
direction of revolution, 336.
position of axes, 335.
production and properties of, 332.
theory of, 369.

Elliptical polarization, 326.
Elster and Geitel, photo-electric effect, 552.
Emission, of polarized light, 604.
of substances with absorption bands, 601.
polarization by, 300,
Emission and absorption, 592.
of incandescent salts, 603.
of rare earths, 603.
Energy, flow in interference fringes, 128.
average kinetic, of vibrating particle, 120.
Eriometer, 241.
Errors of gratings, ghosts, 220.
Ether, contractile, 3.
Exner, pseudo-lenses, 87.
Extinction coefficient, 397.

Fabry and Perot, interferometer, 278.
Faraday effect, 496.
duration of, 500.
Fermat's law, 72.
Fermat's principle, 63.
Féry, optical pyrometer, 616.
Feussner, polarizing prisms, 297.
Films, for showing interference colors, 170.
frilled, on metal plates, 172.
liquid, thickness of, 268.
Fish-eye views, 67.
Fizeau, effect of earth's motion, 677.
in moving media, 668.
interference methods in astronomy, 137.
velocity of light, 17.
Flames, absorption by, 593.
Flash-spectrum, 117.
Fluorescence, 557.
absorption, 564.
angle of emission and intensity, 561.
destruction of gas fluorescence by other gases, 583.
effect of solvent, 569.
methods of investigating, 558.
of iodine vapor, 582.
of mercury vapor, 579.
of sodium vapor, 571.
of vapors, 570.
theories of, 568.
Fluorescence, polarized, of crystals, 560.
of gases, 585.
Focal isolation of heat-waves, 414.
Focal length of lens, 80.
Focal lines, 60.
Foucault, velocity of light, 18.
Foucault polarizing prism, 297.
Franck and Wood, destruction of fluorescence by electro-negative gas, 584.
Fraunhofer class of diffraction phenomena, 186, 195.
Frequency and wave-length, 10.
Fresnel, bi-prism of, 130.
explanation of rectilinear propagation, 29.
explanation of rotatory polarization, 479.
interference of polarized light, 148.
mirrors for showing interference, 125.
optics of moving media, 669.
rhomb, 330.
theory of double refraction, 309.
Fresnel class of diffraction phenomena, 189, 244.
Fresnel integrals, 247.
Fringes, achromatic interference, 139, 166.
distribution of illumination in interference, 122.
interference with white light, 134.
limit to number of, 132.
shift by transparent plate, 133, 140.

Garbasso, electrical resonance, 630.
nature of white light, 651.
Garnett, colors of metal-glasses, 641.
of finely divided metals, 643, 645.
Gases, temperature radiation of, 594.
Gehrcke and Lummer, interferometer, 282.
Gehrcke, interference points, 284.
"Ghosts" of gratings, 220.
Goldstein, phosphorescence at low temperature, 567.
Gouy, acceleration of phase, 35.
anomalous propagation of waves, 261.
nature of white light, 648.
Graphical methods in diffraction, 190.
Grating, achromatization of prism by, 105.
analysis of white light, 656.
diffraction, 203.
echelette, 227.
echelon, 274.
laminary, 211.
plane diffraction, 213.
Grating spectra, intensity of, 218.
Gratings, concave, 231.
focal properties of, 217.
groove form and energy distribution of, 230.
intensifying glass, 219.
polarizing action of, 632.
predominant spectra, 226.
resolving power, 215.
Grimaldi, discovery of diffraction, 182.
experiment on interference, 124.
Group-velocity, 20.

Hackett, F. E., and Wood, R. W., magneto-optics of sodium, 537.
Hagenbach, fluorescence, 559.
Haidinger's brush, 300.
Haidinger's fringes, 158.
Hale, Zeeman effect on the sun, 519.

Halos, 240, 347.
Hamilton, Sir Wm., conical refraction, 317.
Heat waves, 409, 411.
  focal isolation of longest, 414.
Helmholtz, dispersion theory, 380.
  Haidinger's brush, 300.
  opthalmometer, 69.
Hull, radiation pressure, 612.
Huygens, discovery of polarization, 288.
Huygens's construction for crystals, 305.
Huygens's principle, 28.
  and reflection, 41.
  verification of, 37.
Hydraulic analogy of radiator, 600.
Hydrogen, absorption by, 433, 446.
Hyperboloid mirrors, 50.

Iceland spar, double refraction, 303.
Illuminators, monochromatic, 12.
Images, nature of optical, 222.
  formation of, by reflection, 60.
Incidence, principal, 464.
Infra-red photographs of landscapes, 625.
Ingersoll, magnetic rotation, 502.
Intensification of glass gratings, 219.
Interference, application to astronomical phenomena, 137.
  by thick plate, 159.
  colors of chlorate of potash crystals, 160.
  distribution of illumination, 122.
  multiple reflections and, 157.
  of light, 119.
  of polarized light, 148.
  of waves of different lengths, 138.
  of white light, 134.
  when sources are in line, 146.
Interference fringes, bordering caustics, 148.
  consonance and dissonance, 152.
  energy flow in, 128.
  polarized, 168.
Interference points, 284.
Interference refractometer, 136.
Interferometer:
  adjustments, 266.
  investigation of long heat waves, 411.
  Michelson's, 265.
  of Fabry and Perot, 278.
  of Lummer and Gehrcke, 282.
  refractive index by, 268.
  resolving power of, 285.
  silvering of plates, 281.
  visibility curves, 272.
Invisibility of objects, 98.
Iodine vapor, fluorescence of, 582.
  temperature emission, 597.
Iridescent crystals, 160.
Isochromatic surfaces, 322.
Ives, study of Lippmann process, 177.

Jamin, compensator, 334.
Johonnott, thickness of liquid films, 268
Julius, solar anomalous dispersion, 115.

Kayser, "black-body" radiator, 608.
  emission by gases, 596.
Kelvin, Lord, contractile-ether, 3.
Kerr, electro-optic effect, 548.
  magneto-optic effect, 501.
Kirchhoff's law, 593, 605.
Kirchner, optical resonance, 640.
Kundt, anomalous dispersion, 113.
  magnetic rotation of iron films, 501.
Kurlbaum, determination of radiation constant, 615.

Ladenburg, emission and absorption of gases, 595.
Ladenburg, E., photo-electric effect, 552.
Ladenburg and Loria, absorption by hydrogen, 434, 446.
Lamb, absorption by resonator, 454.
Lambert's law of absorption, 437.
Laminary diffraction phenomena, 250.
Laminary grating, 211.
Landolt's fringe, 301.
Larmor, radiation pressure, 610.
Lebedew, radiation pressure, 611, 613.
Lenard, emission of fused salts, 603.
Lenard and Klatt, phosphorescent spectra, 568.
Lens, aplanatic, 78.
  focal length, 80.
  refraction by, 77.
  spherical, 78.
Lenses, achromatic, 106.
Light, aberration of, 667.
  sources for experimental purposes, 11.
  velocity of, 17.
Light beats, 138, 340.
Light-waves, as standards of length, 269.
  possibility of structure in, 555.
Lippich, on Landolt's fringe, 301.
Lippman, color photographs, 176.
Lippman plates, colors of, and optical resonance, 640.
Lippman process, preparation of plates for, 177.
Lloyd, conical refraction, 317.
Lloyd's single mirror, 130.
Lodge experiment, with moving media, 676.
Lommel, diffraction by particles on mirror, 242.
  Stokes's law, 559.
Lorentz, absorption and Zeeman effect, 521.
  optics of moving media, 678.
  theory of Zeeman effect, 523.
  Zeeman effect and spectral series, 512.
Lummer, black body, 607.
Lummer and Gehrcke, interferometer, 282.

Lummer and Pringsheim, energy curves, 622.
Stefan's law, 615.
Lummer and Sorge, elliptical polarization, 370.

Macaluso and Corbino, magnetic rotation, 530, 540.
Mach, experiment on rotatory dispersion, 479.
Magnetic double refraction, 547.
Magnetic rotation, and field strength, 500.
and second law of thermodynamics, 500.
at low temperatures, 543.
circular components, 497.
direction of, 499.
explanation of, 497.
of mercury vapor, 539.
of plane of polarization, 496.
of sodium vapor, 531–536.
positive and negative of sodium, 537.
theory of, 524, 544.
within absorption band, 542.
Magnetic rotatory dispersion, 501.
Magnification by an atmosphere, 84.
Magnifying power of prisms, 76.
Malus, law of, 293.
polarization by reflection, 288.
Martens, selective reflection, 417.
Maxwell, radiation pressure, 609.
Maxwell's equations, 350.
Mercury lamp, 13.
Mercury vapor, fluorescence, 579.
selective reflection, 430.
Meslin, circular interference fringes, 147.
Metal-glasses, colors of, 641.
Metallic reflection, theory, 460.
Metal particles, colors shown by fine deposits, 635, 637, 643–645.
Metal powders, transmission of long heat-waves by, 632.
Metals, absorption by films of varying thickness, 466.
absorption of, 449, 456, 464.
color of, 456.
emission coefficients and temperature, 475.
optical constants of, 469.
relation between electrical and optical properties, 472.
theory of dispersion of, 469.
Michelson, astronomical phenomena and interference, 137.
diffraction gratings, 287.
echelon grating, 274.
interferometer, 265.
light-waves as standards of length, 269.
velocity of light, 19.
visibility curves, 272.
Michelson and Morley, experiment, 672.
velocity of light in moving media, 669.
Microscope, diffraction theory of, 223.
objective, 79.
Mills, velocity of circular components, 499.
Minimum deviation of prism, 75.
Minkowski, relativity, 693.
Mirage, 81, 85.
Mirrors, concave and convex, 54.
interference of Fresnel, 125.
Mock suns, 347.
Molecular changes effected by light, 589.
Molecular collisions, effect upon spectra, 588.
Morse, phosphorescent spectra, 568.
Moving media, optics of, 678.
Moving plate, retardation of, 670.

Newcomb, velocity of light, 20.
Newton, experiments on dispersion, 102.
interference colors of thin plates, 152.
Newton's rings, 154.
effect of prism on, 166.
Nichols, E. F., transmission of quartz, 392.
Nichols and Hull, radiation pressure, 611.
Nichols and Merritt, fluorescence, 560.
Nicol prism, 296.
Nitroso-dimethyl aniline, dispersion of, 399.
polarization by, 290.
Non-homogeneous media, 81.
Normal velocity surface, 313.
Norrenberg, polariscope, 291.

Opals, colors of, 160.
Ophthalmometer, 69.
Optical images, nature of, 222.
Optical resonance, 630, 635.
Optic axis, 316.

Parabolic mirrors, 49.
Parhelia, 347.
Parhelic circle, 347.
Paschen, emission and absorption of gases, 595.
Periodic motion, 5.
Pernter, theory of rainbow, 346.
Perot, Fabry and, interferometer, 278.
Pflüger, absorption by hydrogen, 434, 446.
dispersion of absorbing media, 395.
energy distribution in spark spectrum, 623.
test of Kirchhoff's law, 604.
Pfund, simple mercury lamp, 13.
Phase change at polarizing angle, 368.
Phase distribution over illuminated area, 145.

Phosphorescence, 564.
and temperature, 566.
duration of, 565.
Phosphorescent spectra, 567.
Phosphoroscope, 565.
Photo-electric effect, in gases, 553.
selective, 553.
theory of, 554.
Photo-electric effects, 550.
and wave-length, 551.
Planck, "licht-quanten," 554.
radiation formula, 621.
theory of absorption, 452.
Plates, table of transmission of oblique, 366.
Pohl, theories of photo-electric effect, 555.
Pohl and Pringsheim, photo-electric effect, 551.
selective photo-electric effect, 553.
Poisson's problem, 38.
Polariscope, Norrenberg's, 291.
Polarization, abnormal, of light scattered by absorbing particles, 628.
angle of, 289.
by double refraction, 295.
by nitroso-dimethyl-aniline, 290.
by oblique emission, 300.
by reflection, 288.
by refraction, 292.
circular and elliptical (*see* under circular and elliptcial).
discovery of, 288.
of scattered light, 626.
partial, 337.
plane of, 291.
rotatory (*see* rotatory).
Polarized fluorescence of crystals, 560.
of gases, 585.
Polarized light, convergent, 318.
detection of, 298.
determination of per cent, 298.
interference of, 148.
Polarized waves, stationary, 301.
Polarizing action of gratings, 632.
Polarizing power of oblique plates, 364.
Polarizing prisms, aperture of, 297.
Porous surfaces, absorption by, 449.
Porter, A. B., nature of optical images, 222.
diffraction theory of microscope, 224.
Powders, determination of dispersion, 112.
residual-rays from, 410, 632.
Poynting, radiation pressure, 612.
Pressure of radiation, 609–613.
Principal incidence and azimuth, 464.
Pringsheim, deduction of Kirchhoff's law, 605.
Prism, achromatization by grating, 105.
Foucault's polarizing, 297.
minimum deviation, 75.
Nicol, 296.
quartz dispersion of, 103.
refraction by, 74.
Prismatic colors, origin of, 657.
Prisms, achromatic, 104.
calculation of achromatic and direct vision, 106.
cyanine, 114.
direct vision, 105.
magnifying power of, 76.
resolving power of, 108.
Pseudo-lenses, 87.
Pulse, nature of, constituting white light, 653.
Pyrometer, Féry optical, 616.

Quarter wave plate, 328.
Quartz, dispersion of, 391, 393.
residual rays from, 410.
rotation of polarization plane, 477.
table of rotatory power, 478.
transmission and absorption, 392.
Quartz prism, dispersion of the two rays, 103.
Quincke, laminary grating, 211.

Radiating body, hydraulic analogy of, 600.
Radiation, and surface condition, 598
as function of temperature, 614.
complete formulae, 621.
equilibrium, 608.
of solids and gases, 591, 594.
of solids and liquids, 597.
of transparent substances, 598.
pressure of, 609.
pressure on absorbing gas, 613.
pressure tangential component, 612.
spectral distribution, 616.
Rainbow, 342.
Rare earths, absorption by, 450.
Ray filters, 14.
Rayleigh, Lord:
group-velocity, 21.
iridescent crystals, 161.
nature of white-light, 650, 653.
optics of moving media, 670, 675.
origin of prismatic colors, 663.
residual blue, 627.
resolving power of prisms, 108.
scattering of light, 625.
shift of fringes by sodium vapor, 143.
surface films and reflection, 370.
theory of gratings, 210, 216.
visibility of transparent objects, 99.
Rays, curved, 90.
Rectilinear propagation, 29.
Reflection, at perpendicular incidence, 368.
by Huygens's principle, 41.
by metals, 466.
change of phase by, 464.
from plane and curved surfaces, Chapter 3.
from rough surfaces, 42.
metallic, theory of, 460, 465.

Reflection, of heat waves, 45.
selective, in ultra-violet, 417.
total, 65.
total, theory of, 371.
total penetration of disturbance into second medium, 373.
Reflection theory, 359.
Refraction, astronomical, 82.
at plane surface, 64.
by curved surfaces, 73.
by lens, 77.
by plane-parallel plate, 68.
by sphere, 78.
conical, 317, 318.
double, 303.
effect on width of beam, 67.
in non-homogeneous media, 81.
of sound waves, 97.
polarization by, 292.
theory of, 359.
Refractive index, determination with microscope, 71.
interferometer determination, 268.
of prism, 75.
of thin film, 157.
Refractometer interference, 136.
Relativity, principle of, 684.
Residual blue, 627.
Residual rays, 392, 409.
from absorbing gas, 429.
from powders, 410, 632.
longest, 411.
Resolving power of gratings, 215.
of interferometers, 285.
of prisms, 108.
of telescope, 238.
Resonance, for heat waves, 631.
optical, 630, 635.
radiation, 576.
spectra of iodine, 582.
spectra of sodium vapor, 575.
spectrum, transformation of, into band spectrum by molecular collisions, 588.
Respighi on scintillation, 93.
Resultant of disturbances of arbitrary phase, 124.
Rhomb of Fresnel, 330.
Righi, light-beats, 340.
Ritchie's experiment, 592.
Römer, velocity of light, 17.
Rosenthal, emission and reflection, 602.
Rotatory dispersion, 477, 492.
anomalous, 485, 492.
in absorbing media, 493.
Rotatory polarization, 476.
circular components, 480.
Fresnel's explanation, 479.
in convergent light, 482.
magnetic (*see* magnetic rotation).
of liquids, 482.
of quartz, 477, 478.
theory of, 486–492.
Rotatory polarization, unequal absorption of circular components, 481.
velocity of circular components, 481.
Rotatory power, influence of solvent, 484.
Rough surfaces, reflection and refraction by, 42.
Rowland, concave gratings, 231.
Rubens, investigation of dispersion, 390.
Rubens and Ashkinass, dispersion of quartz, 393.
Rubens and Du Bois, polarization by gratings, 633.
Rubens and Hagen, emission coefficients of metals and temperature, 475.
optical and electrical properties of metals, 472.
reflection by metals, 465.
Rubens and Hollnagel, interferometer study of long waves, 411.
Rubens and Kurlbaum, radiation and temperature, 621.
Rubens and Nichols, residual rays, 392.
resonance for heat-waves, 631.
Rubens and Trowbridge, dispersion of rock salt, fluorite, etc., 394.
Rubens and Wood, focal isolation of heat-waves, 414.
Runge, theory of concave grating, 233.
Runge and Paschen, Zeeman effect, 512.

Sagnac, anomalous propagation of waves, 261.
Scattering of light by small particles, 624.
metal spheres, 630.
sodium and potassium fogs, 639.
Schmidt, electro-optics of benzol, 550.
Schuster, double refraction, 311.
Huygens's principle, 34.
nature of white light, 654.
origin of prismatic colors, 658.
Talbot's bands, 260.
Scintillation, 91.
Secondary maxima of grating, 205.
Selective dispersion, of absorbing gas, 418.
of hydrogen, 433.
Selective reflection, 409.
by absorbing gas, 429.
of powders, 410.
Sellmeier, dispersion formula, 376.
Shadow-bands, 91.
Siedentopf and Szigmondy, ultra-microscope, 641.
polarization effects in ultra-microscope, 642.
Silvering of interferometer plates, 281.
Sodium, "flash-spectrum," 117.
Sodium vapor, absorption by, 443, 513.
magnetic rotation of, 531.
resonance spectra, 575.
selective dispersion of, 418.
shift of helium fringes by, 142.

Solar phenomena and anomalous dispersion, 115.
Solitary waves and Huygens's principle, 40.
Sound-waves, photographs of secondary, 30.
  reflection of, 48.
  refraction of, 97.
Spark-spectrum, energy distribution in, 623.
Spherical lens, 78.
Spherical surfaces, reflection from, 53.
Spherical wave, refraction at plane surface, 70.
Springsteen and Wood, magnetic rotation, 531.
Stark, the Doppler effect in canal rays, 24.
Stars, scintillation of, 91.
Stationary light-waves, 174.
Stefan-Boltzmann law, 614.
Stibnite, optical properties, 400.
Stöckl, dispersion of solutions of aniline dyes, 398.
Stokes, fluorescence, 557–559.
  polarization by oblique plates, 365.
Stokes's law, 559.
Striae, method of, 94.
Sun, temperature of, 616.
  Schmidt's theory of, 83.
Surface color, 439.
Surface films, 369.

Talbot's bands, 255.
Telescope, resolving power of, 238.
Telescopes, reflecting, 51.
Temperature, effect on axes of crystals, 324.
  emission of iodine vapor, 597.
  radiation of gases, 594.
Testing glass plates by interference methods, 159.
Theories, older, 1.
Theory, electro-magnetic, 4.
Thomson, Sir J. J., scattering of light by metal spheres, 630.
Toepler, "Schlieren-methode," 94.
Total reflection, 65.
Transmission, of heat waves, 412.
  coefficient, 438.
Tyndall, calorescence, 556.
  double refraction of vibrating plate, 325.
  residual blue, 627.
  scattering of light, 624.
Trowbridge, A., and Wood, R. W., energy distribution of gratings, 230.
  residual rays from quartz, 410.
Trowbridge, reflecting power of ice, 408.
  residual rays from powders, 410.
Trowbridge, Rubens and, dispersion, 394.
Ultra-microscope, 641.
  polarization effects in, 642.
Ultra-violet selective reflection, 417.
Urbain, absorption of europium, 450.

Velocity-group, 20.
Velocity of light, 17.
  in moving media, 668.
Vibrations, composition of, 121.
Visibility curves, 272.
Voigt, magnetic double refraction, 547.
  magnetic rotation, 542.
  total reflection, 374.

Water, vision under, 67.
Wave, absence of back, 8.
Wave-front, 8.
Wave-front motions in cinematograph, 57.
Wave-fronts, reflected, 56.
Wave-length, 10.
Wave-motion, 6.
Waves, light, as standards of length, 269.
  plane and spherical, 9.
  produced by obstacle moving in fluid, 663.
  secondary, 29.
  solitary, 40.
  stationary light, 174.
  stationary polarized, 301.
Wave surface in crystals, 304, 314.
Welsbach light, theory of, 600.
White light, analysis by grating, 656.
  interference of, 134, 156.
  nature of, 648.
Wiedemann and Schmidt, fluorescence of vapors, 570.
Wien, diffraction by absorbing screens, 634.
  radiation formula, 621.
Wiener, stationary light-waves, 175.
  stationary polarized waves, 301.
Wien's law, 616.
Wood, photographs of sound waves and cinematograph demonstration, 56.
  abnormal polarization of light scattered by absorbing particles, 628.
  absorption of sodium vapor, 443.
  Balmer series of sodium, 513.
  destruction of fluorescence by other gases, 583.
  dispersion of, 399.
  echelette grating, 227.
  fish-eye views, 67.
  flash-spectrum of sodium, 116.
  fluorescence and resonance spectra of iodine, 582.
  fluorescence of mercury vapor, 579.
  fluorescence of sodium vapor, 572.
  infra-red photographs, 625.
  intensity of fluorescence and emission angle, 562.

Wood, interference colors of crystals of chlorate of potash, 162.
interference colors of frilled films on metal, 172.
magnetic rotation of sodium vapor, 531–536.
magnetic rotation spectra, 536.
nitroso-dimethyl-aniline, polarization by, 290.
optical resonance, 635.
polarized fluorescence of gases, 586.
polarized interference fringes, 00.
positive and negative magnetic rotation of sodium vapor, 537.
radiation of transparent substances, 598.
residual rays from powders, 632.
resonance spectra, 575.
scattering of light by sodium and potassium fogs, 639.
selective dispersion of sodium, 420–430.
selective reflection of mercury vapor, 429.
shift of helium fringes by sodium vapor, 142.
Wood, Rubens and, focal isolation of heat-waves, 414.
Wood and Franck, destruction of fluorescence, 585.
resonance spectrum of iodine in helium, 588.
Wood and Trowbridge, energy distribution of gratings, 230
residual rays, 410.

Young, change of phase by reflection, 155.
colors of mixed plates, 252.
discovery of interference, 125.
eriometer, 241.
interference colors of thin plates, 152.

Zeeman, 509–512; 516–519.
Zeeman effect, 503.
abnormal in band spectra, 521.
and spectral series, 512.
asymmetrical triplet, 509.
complicated types of, 508.
elementary explanation, 504.
inverse, 516.
magnetic rotation in absorption band, 542.
magnitude of separation, 514.
on the sun, 519.
shift of middle component, 510.
study without a spectroscope, 515.
theory of, 523.
Zeeman and Geest, magnetic double refraction, 547.
Zone-plates, 38, 217.
Zones, Fresnel-Huygens, 36.

Printed in the United States of America.